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DETERMINANTS OF MULTIDIAGONAL MATRICES*

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Abstract. The formulas presented in [Molinari, L.G. Determinants of block tridiagonal matrices. Linear Algebra Appl., 2008; 429, 2221–2226] for evaluating the determinant of block tridiagonal matrices with (or without) corners are used to derive the determinant of any multidiagonal matrices with (or without) corners with some specified non-zero minors. Algorithms for calculation the determinant based on this method are given and properties of the determinants are studied. Some applications are presented.

Key words. Multidiagonal matrix with corners, Multidiagonal matrix without corners, Determinant.

AMS subject classifications. 15A15.

1. Introduction. The first purpose of this paper is to present and discuss algorithms for calculation of the determinant of multidiagonal matrices with (or without) corners. The second purpose is to compare them with some other algorithms in the case of pentadiagonal matrix with (or without) corners.

We consider the determinant of a multidiagonal matrix of order n with and without corners, i.e., matrices $A^{(p)} = (a_{i,j})$ and $B^{(p)} = (b_{i,j})$ with $a_{i,j} = 0$ if $\frac{p-1}{2} < |i-j| < n - \frac{p-1}{2}$ and $b_{i,j} = 0$ if $|i-j| > \frac{p-1}{2}$, where p (odd) denotes the number of diagonals. In particular, tridiagonal and pentadiagonal matrices with corners are of the form:

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$$A^{(3)} = \begin{pmatrix} a_{1,1} & a_{1,2} & & & & & & & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & & & & & & \\ & a_{3,2} & a_{3,3} & a_{3,4} & & & & & \\ & & a_{4,3} & a_{4,4} & a_{4,5} & & & & \\ & & \ddots & \ddots & \ddots & \ddots & & & \\ & & & \ddots & \ddots & \ddots & \ddots & & \\ & & & & a_{n-3,n-4} & a_{n-3,n-3} & a_{n-3,n-2} & & \\ & & & & & a_{n-2,n-3} & a_{n-2,n-2} & a_{n-2,n-1} & \\ & & & & & & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ a_{n,1} & & & & & & & a_{n,n-1} & a_{n,n} \end{pmatrix},$$

$$A^{(5)} = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & & & & & a_{1,n-1} & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & & & & & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} & & & & \\ & a_{4,2} & a_{4,3} & a_{4,4} & a_{4,5} & & & & \\ & & \ddots & \ddots & \ddots & \ddots & & & \\ & & & \ddots & \ddots & \ddots & \ddots & & \\ & & & & a_{n-2,n-4} & a_{n-2,n-3} & a_{n-2,n-2} & a_{n-2,n-1} & a_{n-2,n} \\ a_{n-1,1} & & & & & a_{n-1,n-3} & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ a_{n,1} & a_{n,2} & & & & & a_{n,n-2} & a_{n,n-1} & a_{n,n} \end{pmatrix}.$$

There are many papers about the determinants of tridiagonal matrices without corners, see e.g. El-Mikkawy [2]. The problem of pentadiagonal matrices without corners is also well studied in the literature too. Numerical algorithms for the determinant of pentadiagonal matrices are given e.g. by Sweet [10], Evans [3], Sogabe [8], [9]. Salkuyeh [7] and Molinari [6] give the formula for the determinant of block-tridiagonal matrices without corners, which in special case is a multidiagonal matrix without corners.

The problem of multidiagonal matrices with corners is not so widely studied in the literature. For example, Molinari [6] gives a formula to derive the determinant of tridiagonal matrix with corners. In the same paper the author gives also a formula for a determinant of block-tridiagonal matrix with (or without) corners, i.e., a matrix

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of the form

$$A = \begin{pmatrix} A_1 & B_1 & & & C_1 \\ C_2 & A_2 & B_2 & & \\ & \ddots & \ddots & \ddots & \\ & & C_{k-1} & A_{k-1} & B_{k-1} \\ B_k & & & C_k & A_k \end{pmatrix}, \tag{1.1}$$

where the matrices A_i, B_i, C_i are $m \times m$ and B_i, C_i are nonsingular for every $i = 1, 2, \ldots, k$. In case of block-tridiagonal matrix without corners, i.e. with $B_k = C_1 = 0$, the matrices B_i are nonsingular for every $i = 1, 2, \ldots, k-1$ while C_i are nonsingular for every $i = 2, \ldots, k$.

The aim of this paper is to show that the formulas presented by Molinari [6] may be used to find the determinant of any multidiagonal matrices with some specified non-zero minors. In the case when $n = \frac{p-1}{2}k$, Molinari's formulas can be applied directly, while in the remaining cases it is enough to use respective Schur complement of matrices and then apply Molinari's formulas.

We derive two algorithms for the calculation of determinant of pentadiagonal matrices with corners and study their computational complexity. We show that the algorithm based on the method presented in Sogabe [9] is more efficient in general then the algorithm based on Molinari's formula. However in some particular cases Molinari's formula gives an analytical solution independent of the size of the matrix (see Section 4).

For the problem of deriving the determinant of pentadiagonal matrices without corners, we compare the computational complexity of five algorithms. We show that Sogabe's [9] algorithm is much more efficient in general.

2. Determinant of multidiagonal matrix. Let $n = \frac{p-1}{2}k$, where p (odd) is the number of diagonals. Let $A^{(p)}$ (respectively, $B^{(p)}$) denote an $n \times n$ p-diagonal matrix with (respectively, without) corners. Then $A^{(p)}$ (respectively, $B^{(p)}$) can be presented as the block-tridiagonal matrix (1.1) with $m = \frac{p-1}{2}$. Molinari [6] shows that

$$\det A^{(p)} = (-1)^{m(k-1)} \cdot \det(T - I_m) \cdot \det(B_1 \cdots B_k) \tag{2.1}$$

with

$$T = \prod_{i=1}^{k} T_i, \qquad T_i = \begin{pmatrix} -B_i^{-1}A_i & -B_i^{-1}C_i \\ I_m & 0_m \end{pmatrix}, \tag{2.2}$$

where I_m (0_m) is the identity (zero) matrix of order m. Moreover,

$$\det B^{(p)} = (-1)^{m(k-1)} \cdot \det T_{11} \cdot \det(B_1 \cdots B_{k-1}), \tag{2.3}$$

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where T_{11} is the upper left block of size $m \times m$ of the transfer matrix $T = \prod_{i=1}^{k} T_i$ with

$$T_1 = \begin{pmatrix} -B_1^{-1}A_1 & -B_1^{-1} \\ I_2 & 0_2 \end{pmatrix}, \; T_k = \begin{pmatrix} -A_k & -C_k \\ I_2 & 0_2 \end{pmatrix},$$

and the remaining T_i $(i = 2, \ldots, k-1)$ as in (2.2).

Assume now $n \neq \frac{p-1}{2}k$. Then we may present matrices $A^{(p)}$ and $B^{(p)}$ as

$$A^{(p)} = \begin{pmatrix} U & X_A \\ Y_A & Z \end{pmatrix}, \qquad B^{(p)} = \begin{pmatrix} U & X_B \\ Y_B & Z \end{pmatrix},$$

where U is the upper left block of $A^{(p)}$ or $B^{(p)}$ of order $\frac{p-1}{2}k$, X_A (respectively, X_B) consists of $n - \frac{p-1}{2}k$ last columns of $A^{(p)}$ (respectively, $B^{(p)}$), Y_A (respectively, Y_B) consists of $n - \frac{p-1}{2}k$ last rows of $A^{(p)}$ (respectively, $B^{(p)}$), and Z is the submatrix of $A^{(p)}$ or $B^{(p)}$ of order $n - \frac{p-1}{2}k$.

Recalling the formula for the determinant of a partitioned matrix, we have

$$\det A^{(p)} = \det Z \cdot \det(U - X_A Z^{-1} Y_A) \tag{2.4}$$

and

$$\det B^{(p)} = \det Z \cdot \det(U - X_B Z^{-1} Y_B). \tag{2.5}$$

Observe that $U - X_A Z^{-1} Y_A$ and $U - X_B Z^{-1} Y_B$ are p-diagonal matrices with and without corners, respectively, and the formulas (2.1) and (2.3) can be used.

3. Determinant of pentadiagonal matrix.

3.1. Pentadiagonal matrix with corners. In this section first we describe an algorithm for computing the determinant of pentadiagonal matrices with corners, based on the method of Sogabe [9] and then we compare this new algorithm with the method based on block-tridiagonal matrices of Molinari [6]. We assume $n \geq 6$.

3.1.1. Algorithm. The algorithm consists of the following three steps:

Step 1. Transform $A^{(5)}$ into the matrix with $B_k = 0$.
Step 2. Transform into a Hessenberg matrix.
Step 3. Transform into an upper triangular matrix.

We show that the algorithm presented here takes $38n - 113$ operations for $n \geq 9$. For $n = 8$, $n = 7$, and $n = 6$ there are an additional 2, 4, and 7 operations needed, respectively.

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Step 1

We show that to transform $A^{(5)}$ into the matrix with $B_k = 0$ it is enough to multiply the original matrix $A^{(5)}$ by a series of suitable matrices $L^{(u)}$.

Let

$$L^{(u)} = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & \gamma^{(u)} & 1 & \\ & & \beta^{(u)} & & 1 \end{pmatrix}$$

with

$$\gamma^{(u)} = l^{(u)}_{n-1,u+2} = -\frac{a^{(u-1)}_{n-1,u}}{a^{(u-1)}_{u+2,u}}, \qquad \beta^{(u)} = l^{(u)}_{n,u+2} = -\frac{a^{(u-1)}_{n,u}}{a^{(u-1)}_{u+2,u}},$$

where $a^{(u)}_{i,j}$ denotes the (i,j)-entry of $L^{(u)} \cdot \ldots \cdot L^{(1)} A$ and $a^{(0)}_{i,j}$ is the (i,j)-entry of $A^{(5)}$, $u = 1, \ldots, n-4$.

Let $u = 1$. Then:

$$\begin{array}{llll} a^{(1)}_{n-1,1} = 0, & & & \\ a^{(1)}_{n-1,2} = a_{3,2}\gamma^{(1)}, & a^{(1)}_{n-1,3} = a_{3,3}\gamma^{(1)}, & a^{(1)}_{n-1,4} = a_{3,4}\gamma^{(1)}, & a^{(1)}_{n-1,5} = a_{3,5}\gamma^{(1)}, \\ a^{(1)}_{n,1} = 0, & & & \\ a^{(1)}_{n,2} = a_{n,2} + a_{3,2}\beta^{(1)}, & a^{(1)}_{n,3} = a_{3,3}\beta^{(1)}, & a^{(1)}_{n,4} = a_{3,4}\beta^{(1)}, & a^{(1)}_{n,5} = a_{3,5}\beta^{(1)}, \end{array}$$

and the remaining entries are the same as in the original matrix. This step costs 13 operations. Observe that for $n = 8$ $a_{n-1,5} \neq 0$ and we have 1 additional operation. For $n = 6, 7$ we have similar situation and hence additional 3 and 5 operations (entries $a_{n-1,4}$, $a_{n,5}$, and $a_{n-1,3}$, $a_{n,4}$ are nonzero).

Let $u = 2, \ldots, n-4$. Then:

$$\begin{array}{ll} a^{(u)}_{n-1,u} = 0, & \\ a^{(u)}_{n-1,u+1} = a^{(u-1)}_{n-1,u+1} + a_{u+2,u+1}\gamma^{(u)}, & a^{(u)}_{n-1,u+2} = a^{(u-1)}_{n-1,u+2} + a_{u+2,u+2}\gamma^{(u)}, \\ a^{(u)}_{n-1,u+3} = a^{(u-1)}_{n-1,u+3} + a_{u+2,u+3}\gamma^{(u)}, & a^{(u)}_{n-1,u+4} = a^{(u-1)}_{n-1,u+4} + a_{u+2,u+4}\gamma^{(u)}, \\ a^{(u)}_{n,u} = 0, & \\ a^{(u)}_{n,u+1} = a^{(u-1)}_{n,u+1} + a_{u+2,u+1}\beta^{(u)}, & a^{(u)}_{n,u+2} = a^{(u-1)}_{n,u+2} + a_{u+2,u+2}\beta^{(u)}, \\ a^{(u)}_{n,u+3} = a^{(u-1)}_{n,u+3} + a_{u+2,u+3}\beta^{(u)}, & a^{(u)}_{n,u+4} = a^{(u-1)}_{n,u+4} + a_{u+2,u+4}\beta^{(u)}, \end{array}$$

and the remaining entries are the same as in the original matrix. This step costs $(n-5)(4+16) = 20n - 100$ operations.

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Now, let

$$L^{(n-3)} = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & \beta^{(n-3)} & 1 \end{pmatrix}$$

with $\beta^{(n-3)} = -\frac{a_{n,n-3}^{(n-4)}}{a_{n-1,n-3}^{(n-4)}}$. Then

$$\begin{array}{ll} a_{n,n-3}^{(n-3)} = 0, & a_{n,n-2}^{(n-3)} = a_{n,n-2}^{(n-4)} + a_{n-1,n-2}^{(n-4)}\beta^{(n-3)}, \\ a_{n,n-1}^{(n-3)} = a_{n,n-1}^{(n-4)} + a_{n-1,n-1}^{(n-4)}\beta^{(n-3)}, & a_{n,n}^{(n-3)} = a_{n,n}^{(n-4)} + a_{n-1,n}^{(n-4)}\beta^{(n-3)}, \end{array}$$

which costs 8 operations. We denote the matrix $L^{(n-3)} \cdots L^{(1)} A^{(5)}$ by L.

It is easy to calculate that the first step of the algorithm costs $20n-79$ operations.

Step 2

This step is similar to the Step 1 in Sogabe [8]. Observe however, that the number of operations increases since the last column of the obtained matrix is nonzero.

Let

$$\Phi^{(u)} = \begin{pmatrix} I_u & & & \\ & 1 & & \\ & \phi_{u+2,u+1} & 1 & \\ & & & I_{n-u-2} \end{pmatrix}$$

with

$$\phi_{u+2,u+1} = -\frac{f_{u+2,u}^{(u-1)}}{f_{u+1,u}^{(u-1)}},$$

where $f_{i,j}^{(u)}$ denotes the (i,j)-entry of $\Phi^{(u)} \cdots \Phi^{(1)} L$ and $f_{i,j}^{(0)}$ is the (i,j)-entry of L, $u = 1, \ldots, n-2$.

Let $u = 1, \ldots, n-4$. Then

$$\begin{array}{ll} f_{u+2,u}^{(u)} = 0, & \\ f_{u+2,u+1}^{(u)} = f_{u+2,u+1}^{(u-1)} + f_{u+1,u+1}^{(u-1)}\phi_{u+2,u+1}, & \\ f_{u+2,u+2}^{(u)} = f_{u+2,u+2}^{(u-1)} + f_{u+1,u+2}^{(u-1)}\phi_{u+2,u+1}, & \\ f_{u+2,u+3}^{(u)} = f_{u+2,u+3}^{(u-1)} + f_{u+1,u+3}^{(u-1)}\phi_{u+2,u+1}, & f_{u+2,n}^{(u)} = f_{u+1,n}^{(u-1)}\phi_{u+2,u+1}, \end{array}$$

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and the remaining entries are the same as in $\Phi^{(u-1)}\cdots\Phi^{(1)}L$. This step costs $(n-4)(7+2)=9n-36$ operations. Observe that for a similar reason as in Step 1 with $u=1$, we have additional 1 and 2 operations for $n=7,8$ and $n=6$, respectively.

Let $u=n-3$. Then

$$\begin{aligned}
&f^{(n-3)}_{n-1,n-3}=0,\\
&f^{(n-3)}_{n-1,n-2}=f^{(n-4)}_{n-1,n-2}+f^{(n-4)}_{n-2,n-2}\phi_{n-1,n-2},\\
&f^{(n-3)}_{n-1,n-1}=f^{(n-4)}_{n-1,n-1}+f^{(n-4)}_{n-2,n-1}\phi_{n-1,n-2},\quad f^{(n-3)}_{n-1,n}=f^{(n-4)}_{n-1,n}+f^{(n-4)}_{n-2,n}\phi_{n-1,n-2},
\end{aligned}$$

and the remaining entries are the same as in $\Phi^{(n-4)}\cdots\Phi^{(1)}L$. This step costs 8 operations.

Let $u=n-2$. Then

$$\begin{aligned}
&f^{(n-2)}_{n,n-2}=0, && f^{(n-2)}_{n,n-1}=f^{(n-3)}_{n,n-1}+f^{(n-3)}_{n-1,n-1}\phi_{n,n-1},\\
&f^{(n-2)}_{n,n}=f^{(n-3)}_{n,n}+f^{(n-3)}_{n-1,n}\phi_{n,n-1},
\end{aligned}$$

and the remaining entries are the same as in $\Phi^{(n-3)}\cdots\Phi^{(1)}L$. This step costs 6 operations.

Thus, to obtain the Hessenberg matrix

$$\Omega=\begin{pmatrix}
\omega_{1,1} & \omega_{1,2} & \omega_{1,3} & & & & & & \omega_{1,n-1} & \omega_{1,n}\\
\omega_{2,1} & \omega_{2,2} & \omega_{2,3} & \omega_{2,4} & & & & & & \omega_{2,n}\\
 & \omega_{3,2} & \omega_{3,3} & \omega_{3,4} & \omega_{3,5} & & & & & \omega_{3,n}\\
 & & \omega_{4,3} & \omega_{4,4} & \omega_{4,5} & \omega_{4,6} & & & & \omega_{4,n}\\
 & & & \ddots & \ddots & \ddots & & & & \vdots\\
 & & & & \ddots & \ddots & \ddots & & & \vdots\\
 & & & & & \omega_{n-3,n-4} & \omega_{n-3,n-3} & \omega_{n-3,n-2} & \omega_{n-3,n-1} & \omega_{n-3,n}\\
 & & & & & & \omega_{n-2,n-3} & \omega_{n-2,n-2} & \omega_{n-2,n-1} & \omega_{n-2,n}\\
 & & & & & & & \omega_{n-1,n-2} & \omega_{n-1,n-1} & \omega_{n-1,n}\\
 & & & & & & & & \omega_{n,n-1} & \omega_{n,n}
\end{pmatrix},$$

with $\omega_{i,j}=0$ if $2<j-i<n-2$ and $i-j<1$ it is necessary to make $9n-22$ operations.

Step 3

This step is based on the method presented by Sogabe [9]. The difference is that this step takes more operations since the entries $\omega_{1,n-1}$, $\omega_{1,n}$ and $\omega_{2,n}$ are nonzero.

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Let

$$\Lambda^{(u)} = \begin{pmatrix} I_{u-1} & & & \\ & 1 & & \\ & \alpha^{(u)} & 1 & \\ & & & I_{n-u-1} \end{pmatrix}$$

with

$$\alpha^{(u)} = \lambda^{(u)}_{u+1,u} = -\frac{\omega^{(u-1)}_{u+1,u}}{\omega^{(u-1)}_{u,u}},$$

where $\omega^{(u)}_{i,j}$ denotes the (i,j)-entry of $\Lambda^{(u)}\cdots\Lambda^{(1)}\Omega$ and $\omega^{(0)}_{i,j}$ is the (i,j)-entry of Ω, $u = 1,\ldots,n-1$.

Let $u = 1,\ldots,n-5$. Then

$$\begin{aligned} &\omega^{(u)}_{u+1,u} = 0, \\ &\omega^{(u)}_{u+1,u+1} = \omega^{(u-1)}_{u+1,u+1} + \omega^{(u-1)}_{u,u+1}\alpha^{(u)}, \quad && \omega^{(u)}_{u+1,u+2} = \omega^{(u-1)}_{u+1,u+2} + \omega^{(u-1)}_{u,u+2}\alpha^{(u)}, \\ &\omega^{(u)}_{u+1,n-1} = \omega^{(u-1)}_{u,n-1}\alpha^{(u)}, && \omega^{(u)}_{u+1,n} = \omega^{(u-1)}_{u+1,n} + \omega^{(u-1)}_{u,n}\alpha^{(u)}, \end{aligned}$$

and the remaining entries are the same as in $\Lambda^{(u-1)}\cdots\Lambda^{(1)}\Omega$. This step costs $9(n-5)$ operations.

Let $u = n-4$. Then

$$\begin{aligned} &\omega^{(n-4)}_{n-3,n-4} = 0, \\ &\omega^{(n-4)}_{n-3,n-3} = \omega^{(n-5)}_{n-3,n-3} + \omega^{(n-5)}_{n-4,n-3}\alpha^{(n-4)}, \\ &\omega^{(n-4)}_{n-3,n-2} = \omega^{(n-5)}_{n-3,n-2} + \omega^{(n-5)}_{n-4,n-2}\alpha^{(n-4)}, \\ &\omega^{(n-4)}_{n-3,n-1} = \omega^{(n-5)}_{n-3,n-1} + \omega^{(n-5)}_{n-4,n-1}\alpha^{(n-4)}, \quad \omega^{(n-4)}_{n-3,n} = \omega^{(n-5)}_{n-3,n} + \omega^{(n-5)}_{n-4,n}\alpha^{(n-4)}, \end{aligned}$$

and the remaining entries are the same as in $\Lambda^{(n-5)}\cdots\Lambda^{(1)}\Omega$. This step costs 10 operations.

Let $u = n-3$. Then

$$\begin{aligned} &\omega^{(n-3)}_{n-2,n-3} = 0, \\ &\omega^{(n-3)}_{n-2,n-2} = \omega^{(n-4)}_{n-2,n-2} + \omega^{(n-4)}_{n-3,n-2}\alpha^{(n-3)}, \\ &\omega^{(n-3)}_{n-2,n-1} = \omega^{(n-4)}_{n-2,n-1} + \omega^{(n-4)}_{n-3,n-1}\alpha^{(n-3)}, \quad \omega^{(n-3)}_{n-2,n} = \omega^{(n-4)}_{n-2,n} + \omega^{(n-4)}_{n-3,n}\alpha^{(n-3)}, \end{aligned}$$

and the remaining entries are the same as in $\Lambda^{(n-4)}\cdots\Lambda^{(1)}\Omega$. This step costs 8 operations.

Let $u = n-2$. Then

$$\begin{aligned} &\omega^{(n-2)}_{n-1,n-2} = 0, \quad && \omega^{(n-2)}_{n-1,n-1} = \omega^{(n-3)}_{n-1,n-1} + \omega^{(n-3)}_{n-2,n-1}\alpha^{(n-2)}, \\ &\omega^{(n-2)}_{n-1,n} = \omega^{(n-3)}_{n-1,n} + \omega^{(n-3)}_{n-2,n}\alpha^{(n-2)}, \end{aligned}$$

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and the remaining entries are the same as in $\Lambda^{(n-3)}\cdots\Lambda^{(1)}\Omega$. This step costs 6 operations.

Let $u = n-1$. Then

$$\omega_{n,n-1}^{(n-1)} = 0, \quad \omega_{n,n}^{(n-1)} = \omega_{n,n}^{(n-2)} + \omega_{n-1,n}^{(n-2)}\alpha^{(n-3)},$$

and the remaining entries are the same as in $\Lambda^{(n-2)}\cdots\Lambda^{(1)}\Omega$. This step costs 4 operations.

Summing-up, Step 3 of this algorithm takes $9n-12$ operations.

3.1.2. Molinari's [6] method. Let $n = 2k$, $k \geq 3$ is integer. From (2.1)

$$\det A = \det(T - I_4) \cdot \prod_{i=1}^{k} \det B_i.$$

It is easy to check that for $i = 1, \ldots, k-1$

$$B_i^{-1} = \begin{pmatrix} a_{2i-1,2i+1} & 0 \\ a_{2i,2i+1} & a_{2i,2i+2} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{a_{2i-1,2i+1}} & 0 \\ -\frac{a_{2i,2i+1}}{d_i} & \frac{1}{a_{2i,2i+2}} \end{pmatrix}$$

with $d_i = \det B_i = a_{2i-1,2i+1}a_{2i,2i+2}$ and

$$B_k^{-1} = \begin{pmatrix} a_{n-1,1} & 0 \\ a_{n,1} & a_{n,2} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{a_{n-1,1}} & 0 \\ -\frac{a_{n,1}}{d_k} & \frac{1}{a_{n,2}} \end{pmatrix}$$

with $d_k = \det B_k = a_{n-1,1}a_{n,2}$. Calculating all d_i $(i = 1, \ldots, k)$ can be done in k operations. To obtain $-B_i^{-1}A_i$ and $-B_i^{-1}C_i$, $i = 2, \ldots, k$, i.e.,

$$-B_i^{-1}A_i = \begin{pmatrix} \frac{a_{2i-1,2i-1}}{e_i} & \frac{a_{2i-1,2i}}{e_i} \\ a_{2i-1,2i-1}f_i - \frac{a_{2i,2i-1}}{a_{2i,2i+2}} & a_{2i-1,2i}f_i - \frac{a_{2i,2i}}{a_{2i,2i+2}} \end{pmatrix}, \quad i = 1, \ldots, k-1,$$

$$-B_k^{-1}A_k = \begin{pmatrix} \frac{a_{n-1,n-1}}{e_k} & \frac{a_{n-1,n}}{e_k} \\ a_{n-1,n-1}f_k - \frac{a_{n,n-1}}{a_{n,2}} & a_{n-1,n}f_k - \frac{a_{n,n}}{a_{n,2}} \end{pmatrix}$$

and

$$-B_1^{-1}C_1 = \begin{pmatrix} \frac{a_{1,n-1}}{e_1} & \frac{a_{1,n}}{e_1} \\ a_{1,n-1}f_1 & a_{1,n}f_1 - \frac{a_{2,n}}{a_{2,4}} \end{pmatrix},$$

$$-B_i^{-1}C_i = \begin{pmatrix} \frac{a_{2i-1,2i-3}}{e_i} & \frac{a_{2i-1,2i-2}}{e_i} \\ a_{2i-1,2i-3}f_i & a_{2i-1,2i-2}f_i - \frac{a_{2i,2i-2}}{a_{2i,2i+2}} \end{pmatrix}, \quad i = 2, \ldots, k-1,$$

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$$-B_k^{-1}C_k = \begin{pmatrix} \frac{a_{n-1,n-3}}{e_k} & \frac{a_{n-1,n-2}}{e_k} \\ a_{n-1,n-3}f_k & a_{n-1,n-2}f_i - \frac{a_{n,n-2}}{a_{n,2}} \end{pmatrix},$$

with $e_i = -a_{2i-1,2i+1}$ and $f_i = \frac{a_{2i,2i+1}}{d_i}$, $i = 1, \ldots, k-1$, $e_k = -a_{n-1,1}$, and $f_k = \frac{a_{n,1}}{d_k}$, 10 and 6 operations for every $i = 1, \ldots, k$ are performed, respectively. Thus, determination of a single T_i requires 17 operations $(1 + 10 + 6)$.

Now multiply T_i and T_{i+1}, $i = 1, 3, \ldots, k-1$ if k is even and $i = 1, 3, \ldots, k-2$ if k is odd.

Consider the case that k is even. Let us denote by G_l the products $T_{2l-1} \cdot T_{2l}$, $l = 1, 2, \ldots, \frac{k}{2}$. Determination of each G_l costs 28 operations. We have $\frac{k}{2}$ such matrices, that are in fact 4-by-4 matrices with no zeros. Now, every product of two 4-by-4 matrices with no zeros costs $7 \cdot 16$ operations (we have $\frac{k}{2} - 1$ such products in $\prod_{l=1}^{\frac{k}{2}} G_l$), and we obtain that the determination of the matrix T can be performed in $173k + 28 \cdot \frac{k}{2} + 112 \cdot (\frac{k}{2} - 1) = 87k - 112$ operations for even k.

Now consider the case that k is odd. As in previous case, every $G_l = T_{2l-1} \cdot T_{2l}$, $l = 1, 2, \ldots, \frac{k-1}{2}$, costs 28 operations. Now, every product of two 4-by-4 matrices with no zeros costs $7 \cdot 16$ operations (we have $\frac{k-1}{2} - 1$ such products in $\prod_{l=1}^{\frac{k-1}{2}} G_l$) and the last multiplication, $G_{(k-1)/2} \cdot T_k$ costs 56 operations. We obtain that the determination of the matrix T can be performed in $17k + 28 \cdot \frac{k-1}{2} + 112(\frac{k-1}{2} - 1) + 56 = 87k - 126$ operations for odd k.

Finally, determination of $T - I_4$ costs $87k - 108$ operations for even k and $87k - 122$ for odd k.

The next step is to calculate the determinant of $T - I_4$. Since this is the 4-by-4 matrix, it is enough to transform it into the triangular matrix, costing 40 operations and additional 3 for product of diagonal entries. We obtain the following operation counts: $87k - 65$ for even k and $87k - 79$ for odd k.

Observe that according to (2.1), we have to calculate also the determinant of the product of matrices B_i. It is equivalent to calculate the product of determinants of 2-by-2 lower triangular matrices, that we in fact have already derived and denoted by d_i. Thus to obtain the final cost of this algorithm it is enough to perform $k - 1$ additional multiplications. By (2.1) we have one additional multiplication.

Summing-up, using the method of Molinari [6], deriving the determinant of pentadiagonal matrix with corners can be performed in $88k - 65$ operations for even k, and in $88k - 79$ for k odd.

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Let $n = 2k+1$, $k \geq 3$ is integer. From (2.4)

$$\det A = a_{n,n} \cdot \det(Z - \frac{1}{a_{n,n}} X_A Y_A)$$

with

$$X_A^T = \begin{pmatrix} a_{1,n} & a_{2,n} & 0 & \dots & 0 & a_{n-2,n} & a_{n-1,n} \end{pmatrix},$$
$$Y_A = \begin{pmatrix} a_{n,1} & a_{n,2} & 0 & \dots & 0 & a_{n,n-2} & a_{n,n-1} \end{pmatrix}.$$

Observe that the matrix $U - \frac{1}{a_{n,n}} X_A Y_A$ is a tridiagonal block matrix with corners, with A_i, B_i, C_i the same as in (1.1), except

$$A_1 = \begin{pmatrix} a_{1,1} - a_{n,1}e_1 & a_{1,2} - a_{n,2}e_1 \\ a_{2,1} - a_{n,1}e_2 & a_{2,2} - a_{n,2}e_2 \end{pmatrix},$$

$$A_k = \begin{pmatrix} a_{n-2,n-2} - a_{n,n-2}e_{n-2} & a_{n-2,n-1} - a_{n,n-1}e_{n-2} \\ a_{n-1,n-2} - a_{n,n-2}e_{n-1} & a_{n-1,n-1} - a_{n,n-1}e_{n-1} \end{pmatrix},$$

$$C_1 = \begin{pmatrix} -a_{n,n-2}e_1 & a_{1,n-1} - a_{n,n-1}e_1 \\ -a_{n,n-2}e_2 & -a_{n,n-1}e_2 \end{pmatrix},$$

$$B_k = \begin{pmatrix} -a_{n,1}e_{n-1} & -a_{n,2}e_{n-2} \\ a_{n-1,1} - a_{n,1}e_{n-1} & -a_{n,2}e_{n-1} \end{pmatrix}$$

with $e_u = \frac{a_{u,n}}{a_{n,n}}$, $u = 1, 2, n-2, n-1$. Determination of $U - \frac{1}{a_{n,n}} X_A Y_A$ can be performed in 36 operations. Now, we can use again the formula (2.1) for $n = 2k$. Thus, we need the following number of operations:

- determination of d_i, $i = 1, \dots, k-1$, and d_k: $(k-1) + 3 = k+2$ operations,
- determination of $-B_i^{-1}A_i$, $i = 1, \dots, k-1$, and $-B_k^{-1}A_k$: $10(k-1) + 16 = 10k + 6$ operations,
- determination of $-B_i^{-1}C_i$, $i = 2, \dots, k$, $-B_1^{-1}C_1$ and $-B_k^{-1}C_k$: $6(k-2) + 9 + 16 = 6k + 13$ operations.

Summing-up, the determination of every T_i can be achieved in $17k + 21$ operations.

The remaining part follows exactly as in the case of even n with respective k. Recall that for deriving the determinant of $A^{(5)}$ it is necessary to multiply the determinant of $U - \frac{1}{a_{n,n}} X_A Y_A$ by $a_{n,n}$, which gives us one additional operation. Finally we obtain $88k - 43$ for even k and $88k - 57$ for odd k.

For comparison, we have the following computational counts for the computation of the determinant of an $n \times n$ ($n \geq 9$) pentadiagonal matrix with corners:

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	Algorithm	Molinari's [6] formula	
$n = 2k$	$38n - 113$	$44n - 65$	for even k
		$44n - 79$	for odd k
$n = 2k+1$	$38n - 113$	$44n - 87$	for even k
		$44n - 101$	for odd k

Note that the method based on Molinari's [6] formula for block-tridiagonal matrix needs more elementary operations than the proposed algorithm. However, in some particular cases using the method of Molinari we can write the analytical form of the determinant; see Section 4.

3.2. Pentadiagonal matrix without corners. We start with comparison of the methods

1. Sweets ([10]) algorithm: $24n - 59$
2. Evans' ([3]) algorithm: $22n - 50$
3. Sogabe's ([8]) algorithm (1): $14n - 28$
4. Sogabe's ([9]) algorithm (2): $13n - 24$
5. Molinari's ([6]) formula:

$n = 2k$:	$44n - 126$	for even k
	$44n - 258$	for odd k
$n = 2k + 1$	$44n - 159$	for even k
	$44n - 291$	for odd k

Now we describe Molinari's [6] method.

Let $n = 2k$, $k \geq 3$ is integer. It is easy to see that $m = 2$. From (2.3)

$$\det B^{(5)} = \det T_{11} \cdot \prod_{i=1}^{k-1} \det B_i,$$

where T_{11} is the upper left block of size 2×2 of the transfer matrix T given in (2.2) and B_is (C_is) are lower (upper) triangular matrices of order 2.

It is easy to calculate that for $i = 1, \ldots, k-1$

$$B_i^{-1} = \begin{pmatrix} b_{2i-1,2i+1} & 0 \\ b_{2i,2i+1} & b_{2i,2i+2} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{b_{2i-1,2i+1}} & 0 \\ -\frac{b_{2i,2i+1}}{d_i} & \frac{1}{b_{2i,2i+2}} \end{pmatrix}$$

with $d_i = \det B_i = b_{2i-1,2i+1} b_{2i,2i+2}$. Calculating all d_is can be done in $k-1$ operations. To determine the products $-B_i^{-1} A_i$ and $-B_i^{-1} C_i$, $i = 2, \ldots, k-1$, i.e.,

$$-B_i^{-1} A_i = \begin{pmatrix} \frac{b_{2i-11,2i-1}}{e_i} & \frac{b_{2i-1,2i}}{e_i} \\ b_{2i-1,2i-1} f_i - \frac{b_{2i,2i-1}}{b_{2i,2i+2}} & b_{2i-1,2i} f_i - \frac{b_{2i,2i}}{b_{2i,2i+2}} \end{pmatrix}$$

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and

$$-B_i^{-1}C_i = \begin{pmatrix} \frac{b_{2i-1,2i-3}}{e_i} & \frac{b_{2i-1,2i-2}}{e_i} \\ b_{2i-1,2i-3}f_i & b_{2i-1,2i-2}f_i - \frac{b_{2i,2i-2}}{b_{2i,2i+2}} \end{pmatrix}$$

with $e_i = -b_{2i-1,2i+1}$, $f_i = \frac{b_{2i,2i+1}}{d_i}$, $10(k-2)$ and $6(k-2)$ operations are performed, respectively. Moreover, T_k and T_1 can be obtained in 7 and 13 operations, respectively. Thus, T_is can be achieved in $17k-13$ operations ($(k-1)+(10+6)(k-2)+7+13$).

Let now multiply T_i and T_{i+1}, $i = 1,3,\dots,k-1$ if k is even and $i = 1,3,\dots,k-2$ if k is odd.

First consider the case that k is even. Let us denote by G_l the products $T_{2l-1}\cdot T_{2l}$, $l = 1,2,\dots,\frac{k}{2}$. Determination of G_1 can be done in 26 operations, each G_l, $l = 2,\dots,\frac{k}{2}-1$, in 28 operations, and $G_{\frac{k}{2}}$ in 27 operations. All matrices G_l are 4-by-4 matrices with no zeros, and every product of two 4-by-4 matrices with no zeros can be performed in $7\cdot 16$ operations (we have $\frac{k}{2}-1$ such products in $\prod_{l=1}^{\frac{k}{2}} G_l$). Thus, to obtain the matrix T we need $(17k-13)+26+28(\frac{k}{2}-2)+27+112\cdot(\frac{k}{2}-1) = 87k-128$ operations for even k.

Next consider the case that k is odd. Similarly as in previous case, the determination of every $G_l = T_{2l-1}\cdot T_{2l}$, $l = 2,\dots,\frac{k-1}{2}-1$ can be achieved in $28(\frac{k-1}{2}-1)$ operations and the determination of $G_{\frac{k-1}{2}}$ costs additional 27 operations. Now, every product of two 4-by-4 matrices with no zeros can be performed in $7\cdot 16$ operations (we have $\frac{k-1}{2}-2$ such products in $\prod_{l=2}^{\frac{k-1}{2}} G_l$) and the last multiplication, $T_1^T\cdot\prod_{l=2}^{\frac{k-1}{2}} G_l^T$, in 48 operations. We obtain that the determination of the matrix T can be obtained in $(17k-13)+27+28\cdot(\frac{k-1}{2}-1)+112(\frac{k-1}{2}-2)+48 = 87k-260$ operations for odd k.

Finally, $\det T_{11}$ can be obtained in 3 operations. Multiplying all necessary determinants according to formula (2.3) we get additional $k-1$ operations.

Summing-up, the determination of $\det B^{(5)}$ can be achieved in $88k-126$ operations for even k and in $88k-256$ for odd k.

Let $n = 2k+1$, $k \geq 3$ is integer. From (2.5)

$$\det B = b_{n,n}\cdot\det(U - \frac{1}{b_{n,n}}X_BY_B)$$

with

$$X_B^T = \begin{pmatrix} 0 & \cdots & 0 & b_{n-2,n} & b_{n-1,n} \end{pmatrix},$$
$$Y_B = \begin{pmatrix} 0 & \cdots & 0 & b_{n,n-2} & b_{n,n-1} \end{pmatrix}.$$

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The calculation of $U - \frac{1}{b_{n,n}} X_B Y_B$ can be done in 10 operations. As a result we obtain pentadiagonal matrix of the form (1.1) with A_k replaced by

$$\begin{pmatrix} b_{n-2,n-2} - b_{n,n-2}e_{n-2} & b_{n-2,n-1} - b_{n,n-1}e_{n-2} \\ b_{n-1,n-2} - b_{n,n-2}e_{n-1} & b_{n-1,n-1} - b_{n,n-1}e_{n-1} \end{pmatrix}$$

with $e_u = \frac{b_{u,n}}{b_{n,n}}$, $u = n-2, n-1$. Observe that it does not change the number of operations to calculate the determinant in comparison with even n, i.e., this step costs $88k - 126$ operations for even k and $88k - 258$ for odd k. Thus, multiplying the above determinant by $b_{n,n}$ (one additional operation) we get that it can be done in $88k - 115$ operations for even k and in $88k - 247$ operations for odd k.

4. Applications. Tridiagonal matrices with corners arise, for example, in theory of experiments in the characterization of D-optimal design of experiment (Filipiak et al., [4]). One of the step is to derive the determinant of the matrix

$$W_3 = \alpha I_n - (H_n + H_n^T)$$

$\alpha > 2$, with H_n - the cyclic permutation matrix given by

$$H_n = \begin{pmatrix} 0^T & 1 \\ I_{n-1} & 0 \end{pmatrix}.$$

Using the respective formula from Molinari [6], we obtain

$$\det W_3 = -2 + \operatorname{tr}\left[\begin{pmatrix} \alpha & -1 \\ 1 & 0 \end{pmatrix}^n\right] = -2 + x^n + x^{-n}$$

where x is the eigenvalue of $\begin{pmatrix} \alpha & -1 \\ 1 & 0 \end{pmatrix}$, i.e., $x = \frac{\alpha + \sqrt{\alpha^2 - 4}}{2}$.

Similarly we may construct pentadiagonal matrix with corners

$$W_5 = \alpha I_n - (H_n + H_n^T) - (H_n^2 + H_n^{2^T}).$$

with $n = 2k$. Using (2.1) we obtain

$$\det W_5 = (-1)^{2(k-1)} \det(S^k - I_4) = \prod_{i=1}^{4} (\lambda_i^k - 1),$$

where λ_i is the ith eigenvalue of

$$S = \begin{pmatrix} \alpha & -1 & -1 & -1 \\ -\alpha - 1 & \alpha + 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

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This implies that the λ_is are the solutions of the characteristic polynomial of the form

$$\left[1-\tfrac{1}{2}(2\alpha+1-\sqrt{4\alpha+9})\lambda+\lambda^2\right]\cdot\left[1-\tfrac{1}{2}(2\alpha+1+\sqrt{4\alpha+9})\lambda+\lambda^2\right]=0,$$

i.e.,

$$\begin{aligned}
\lambda_1 &= \tfrac{1}{4}\left[1+2\alpha-\sqrt{4\alpha+9}-\sqrt{2}\sqrt{2\alpha^2-3-\sqrt{4\alpha+9}-2\alpha(\sqrt{4\alpha^2+9}-2)}\right]\\
\lambda_2 &= \tfrac{1}{4}\left[1+2\alpha-\sqrt{4\alpha+9}+\sqrt{2}\sqrt{2\alpha^2-3-\sqrt{4\alpha+9}-2\alpha(\sqrt{4\alpha^2+9}-2)}\right]\\
\lambda_3 &= \tfrac{1}{4}\left[1+2\alpha+\sqrt{4\alpha+9}-\sqrt{2}\sqrt{2\alpha^2-3+\sqrt{4\alpha+9}+2\alpha(\sqrt{4\alpha^2+9}+2)}\right]\\
\lambda_4 &= \tfrac{1}{4}\left[1+2\alpha+\sqrt{4\alpha+9}+\sqrt{2}\sqrt{2\alpha^2-3+\sqrt{4\alpha+9}+2\alpha(\sqrt{4\alpha^2+9}+2)}\right].
\end{aligned}$$

Now consider the following matrix:

$$W=(I_n+H_n+H_n^T)\otimes I_r$$

where $\otimes$ denotes the Kronecker product. Then

$$\det W=\det(I_n+H_n+H_n^T),$$

which is tridiagonal matrix with corners. From Molinari [6] it can be seen that

$$\det W=(-1)^{n+1}\cdot 2+\operatorname{tr}\left[\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}^n\right].$$

Since the matrix $\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}$ is a 6-involutory matrix, i.e.,

$$\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}^6=I_2,$$

and moreover,

$$\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}^3=-I_2,$$

it is easy to calculate that

$$\det W=\begin{cases}0 & \text{for } n=3u\\ -3 & \text{for } n\neq 3u \text{ and } n \text{ even}\\ 3 & \text{for } n\neq 3u \text{ and } n \text{ odd.}\end{cases}$$

The generalization to block matrices is interesting for the study of transport in discrete structures such as nanotubes or molecules (see e.g. Kostyrko et al. [5], Compernolle et al. [1], Yamada [11]).

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5. Discussion. It can be seen that the method based on Molinari's [6] formula for block-tridiagonal matrices needs more elementary operations than the algorithm proposed in Section 2 and Sogabe's [9] algorithm, respectively. Moreover, the formula for the evaluation of transition matrix T requires $k\ (k-1)$ inversions of matrices B_i. However, in some particular cases Molinari's method may be more useful. Consider an example based on Example 2 of Sogabe [8]. Assume $A^{(5)}$ is $10^3 \times 10^3$ matrix of the form

$$A^{(5)} = \begin{pmatrix} 0.2 & -1.3 & 1.2 & & & 0.1 & 0.3 \\ 0.3 & 0.2 & -1.3 & 1.2 & & & 0.1 \\ 0.1 & 0.3 & 0.2 & -1.3 & 1.2 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & \ddots & 1.2 \\ 1.2 & & & 0.1 & 0.3 & 0.2 & -1.3 \\ -1.3 & 1.2 & & & 0.1 & 0.3 & 0.2 \end{pmatrix}.$$

The proposed algorithm gives $\det A^{(5)} = \infty$, while Molinari's algorithm $\det A^{(5)} = 1.5179 * 10^{79}$, and this is the same result as determinant obtained directly using Mathematica and R. For the same matrix of order 100, we have $9.15866 * 10^{64}$ for proposed algorithm and $8.28367 * 10^7$ for Molinari's algorithm and using direct calculations of Mathematica and R.

It is worth noting that for the example of pentadiagonal matrix with corners based on Example 1 of Sogabe [8] we get the same value of determinant for both methods.

It can be seen that if the determinants of B_i are close to zero, Molinari's [6] method may be not stable.

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