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WEAK MONOTONICITY OF INTERVAL MATRICES*

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Abstract. This article concerns weak monotonicity of interval matrices, with specific emphasis on its relationship with a certain class of proper splittings.

Key words. Weak monotonicity; Moore-Penrose inverse; Interval matrix; range kernal regularity; proper splitting

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1. Introduction and Preliminaries. A real $n \times n$ matrix A is *monotone* if $Ax \geq 0$ implies $x \geq 0$, where by $x \geq 0$ we mean that all the components of x are nonnegative. It can be easily shown that A is monotone if and only if A is nonsingular and $A^{-1} \geq 0$, where for a matrix B, we denote $B \geq 0$, if all the entries of B are nonnegative. Due to this fact, monotone matrices are also referred to as *inverse positive* matrices. Monotone matrices were first introduced by Collatz in the context of solving systems of linear equations that emerge upon employing finite difference techniques for elliptic partial differential equations. For more details, we refer to the book [6]. The concept of monotonicity has since been extended in many ways. Mangasarian used the same implication as above while letting A to be a rectangular matrix. He then showed that A is (rectangular) monotone if and only if A has a nonnegative left inverse. Berman and Plemmons introduced a hierarchy of extended notions of monotonicity where usual inverses were replaced by various types of generalized inverses. We refer to the book [5] for the details. The most general among these extensions is the notion of weak monotonicity. The m by n matrix A is *weak monotone* if $Ax \geq 0$ implies $x \in N(A) + \mathbb{R}^n_+$, where $N(A)$ denotes the null space of A. Suppose that $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ is (entrywise) nonnegative such that the system $Ax = b$ has a solution. A well known result is that A is weak monotone if and only if each system $Ax = b$, where b is nonnegative, has a nonnegative solution. This statement underscores the importance of weak monotonicity. Specifically, for problems modeled by

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economic considerations, the vector b is an economic requirement, which is typically nonnegative. If the model represents the underlying problem correctly, then it follows that A is weak monotone, since usually the nonnegative solution being used in the present is a solution of the system $Ax = b$.

On the one hand, the notion of weak monotonicity is very general. On the other hand, in being so, it becomes rather difficult to derive results for such matrices. Hence, in many instances, additional assumptions are made to obtain results that could be applied meaningfully, in practice. (This statement also places in proper perspective, one of our results, viz., Theorem 2.2 that will be proved in this article).

In the literature, many authors have studied the problem of characterizing inverse positive matrices in terms of the so-called splitting of the matrix concerned. For a real $n \times n$ matrix A, a decomposition $A = U - V$ is a *splitting* if U is invertible. Any such splitting naturally leads to the iterative method

$$x^{k+1} = U^{-1}Vx^k + U^{-1}b, \quad k = 0, 1, 2, \dots$$

for numerically solving the linear system $Ax = b$, $b \in \mathbb{R}^n$. It is well known that this iterative scheme converges to a solution of $Ax = b$, for any initial vector x^0, if and only if the spectral radius of $U^{-1}V$ is strictly less than 1. Standard iterative methods like the Jacobi, Gauss-Seidel and successive over-relaxation methods arise from different choices of U and V. Here, it is pertinent to mention the notion of a weak regular splitting, proposed by Ortega and Rheinboldt [9]: $A = U - V$ is a *weak regular splitting* if U is invertible, $U^{-1} \geq 0$ and $U^{-1}V \geq 0$. Below, for easy reference, we state the result for weak regular splitting.

Theorem 1.1. *Let $A = U - V$ be a weak regular splitting of the matrix A. Then A is nonsingular and $A^{-1} \geq 0$ if and only if $\rho(U^{-1}V) < 1$.*

In the main section, first we present an analogue of this result (Theorem 2.2) to weak monotone matrices. In order to be able to do this, we extend the notion of a weak regular splitting to what we call as a *weak pseudo regular splitting of weak monotone type*. Theorem 2.2 turns out to be important in proving two other main results in this paper.

Next we turn our attention to interval matrices. Let us introduce a bilateral and a unilateral interval as follows. Let $A, B \in \mathbb{R}^{m \times n}$. A *bilateral interval* J is defined by $J = [A, B] = \{C : A \leq C \leq B\}$. A *unilateral interval* J is defined by $J = (-\infty, B] = \{C : C \leq B\}$. Let $J = [A, B]$ be a bilateral interval. J is said to be *regular* if C^{-1} exists for all $C \in J$ and is referred to as *inverse-positive* if $C^{-1} \geq 0$ for all $C \in J$.

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For unilateral intervals the following is known.

Theorem 1.2. *(Theorem 25.4, [8]) Let $B, C \in \mathbb{R}^{m\times n}$, $C \leq B$ and B be invertible with $B^{-1} \geq 0$. Then $C^{-1} \geq 0$ if and only if* $\mathrm{int}(\mathbb{R}^n_+) \cap C\mathbb{R}^n_+ \neq \phi$.

Our next main result (Theorem 2.3) is a generalization of the above result for weak monotone matrices.

Next, we turn to a result proved in [11], where a characterization of inverse positivity of bilateral intervals was given.

Theorem 1.3. *(Theorem 1, [11]) Let $J = [A, B]$. Then the following statements are equivalent:*

(a) *J is inverse positive.*
(b) *$A^{-1} \geq 0$ and $B^{-1} \geq 0$.*
(c) *$B^{-1} \geq 0$ and $\rho(B^{-1}(B - A)) < 1$.*
(d) *$B^{-1} \geq 0$ and J is regular.*

Our third main result (Theorem 2.4) presents an extension of this result, once again for weak monotone matrices.

We close this section by presenting some preliminary results that will be used in the rest of the discussion. The last section presents proofs of the three results mentioned as above and also considers two problems of independent interest, viz., weak monotonicity of singular Z-matrices (Theorem 2.6) and weak monotonicity of two matrices that are similar through certain specific invertible matrices (Theorem 2.7).

Throughout all matrices will have real entries. $\mathbb{R}^{m\times n}$ denotes the set of all $m \times n$ matrices over reals. For $A \in \mathbb{R}^{m\times n}$, we denote the transpose of A and the range space of A by A^t and $R(A)$, respectively. For complementary subspaces L and M of $\mathbb{R}^n$, the projection of $\mathbb{R}^n$ on L along M will be denoted by $P_{L,M}$. If in addition, L and M are orthogonal then we denote this by P_L.

For a given $A \in \mathbb{R}^{m\times n}$, the unique matrix X satisfying $AXA = A$, $XAX = X$, $(AX)^t = AX$ and $(XA)^t = XA$ is called the Moore- Penrose inverse of A and is denoted by $A^\dagger$. Some of the well known properties of $A^\dagger$ which will be frequently used are: $R(A^t) = R(A^\dagger)$; $N(A^t) = N(A^\dagger)$; $AA^\dagger = P_{R(A)}$; $A^\dagger A = P_{R(A^t)}$. In particular, if $x \in R(A^t)$ then $x = A^\dagger Ax$. For a detailed study of generalized inverses, we refer to [2].

Recall that the spectral radius $\rho(A)$ of a matrix $A \in \mathbb{R}^{n\times n}$ is defined to be the maximum of the moduli of all the eigenvalues of A.

Apart from weak monotonicity, we will also be making use of another extension

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of monotonicity of matrices, called row monotonicity [3].

Definition 1.1. *The matrix* $A \in \mathbb{R}^{m\times n}$ *is row monotone if* $Ax \geq 0$ *and* $x \in R(A^t)$ *imply that* $x \geq 0$.

To characterize row monotone matrices, the authors in [7] introduced and studied the notion of a B_{row}-splitting, as follows.

Definition 1.2. *A splitting* $A = U - V$ *of* $A \in \mathbb{R}^{m\times n}$ *is a proper splitting if* $R(A) = R(U)$ *and* $N(A) = N(U)$*. A proper splitting* $A = U - V$ *of* $A \in \mathbb{R}^{m\times n}$ *is a* B_{row}*-splitting if it satisfies the following conditions: (i)* $VU^\dagger \geq 0$ *and*
(ii) Ax, $Ux \geq 0$ *and* $x \in R(A^t)$ *imply* $x \geq 0$.

Another important result which will be used later is given next.

Theorem 1.4. *(Theorem 3.6, [1]) Let* $A \in \mathbb{R}^{m\times n}$ *be weak monotone and* $A^\dagger A \geq 0$*. Then* $A^\dagger$ *has a decomposition* $A^\dagger = K - L$ *where* $K \geq 0$ *and* $R(A) \subseteq N(L)$.

2. Main Results. In this section we prove the three main results which were briefly mentioned in the introduction. Central to the discussion is the notion of a proper splitting and the first result below collects some well known properties of such a splitting.

Theorem 2.1. *(Theorem 1, [4]) Let* $A = U - V$ *be a proper splitting of* $A \in \mathbb{R}^{m\times n}$*. Then*

(a) $A = U(I - U^\dagger V)$,
(b) $I - U^\dagger V$ *is non-singular,*
(c) $A^\dagger = (I - U^\dagger V)^{-1}U^\dagger$ *and*
(d) $A^\dagger b$ *is the unique solution to the system* $x = U^\dagger V x + U^\dagger b$ *for any* $b \in \mathbb{R}^m$.

Remark 2.1. *We observe that if* $A = U - V$ *is a proper splitting of* A *then* $R(V) \subseteq R(A)$ *and that* $A^t = U^t - V^t$ *is a proper splitting of* A^t*. In that case, we have* $R(V^t) \subseteq R(A^t)$*. Thus, if* $A = U - V$ *is a proper splitting of* A *then* $AA^\dagger = P_{R(A)} = P_{R(U)} = UU^\dagger$ *and* $A^\dagger A = P_{R(A^t)} = P_{R(U^\dagger)} = U^\dagger U$*. Thus,* $UU^\dagger V = P_{R(U)}V = P_{R(A)}V = V$*, since* $R(V) \subseteq R(A)$*. Also* $VU^\dagger U = V(U^\dagger U)^t = (U^\dagger UV^t)^t = (P_{R(U^t)}V^t)^t = (P_{R(A^t)}V^t)^t = (V^t)^t = V$*, since* $R(V^t) \subseteq R(A^t)$.

Analogous to the case of nonsingular linear systems, for the linear system $Ax = b$ defined by a singular or rectangular matrix A, a proper splitting leads to the iterative method of the form $x^{k+1} = Hx^k + c$, for $H \in \mathbb{R}^{n\times n}$ and $c \in \mathbb{R}^n$. H is called the iteration matrix of the method. Once again, the convergence of the sequence x^{k+1} to a solution of $Ax = b$ (for any initial vector x^0) is guaranteed by the spectral radius condition $\rho(H) < 1$, [4]. For a proper splitting given above, we have $H = U^\dagger V$ and $c = U^\dagger b$. The first main result (Theorem 2.2) gives a set of sufficient conditions under

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which $\rho(H) < 1$ can be guaranteed. We will use the following notion.

Definition 2.1. *A decomposition $A = U - V$ of $A \in \mathbb{R}^{m\times n}$ is a weak pseudo regular splitting of weak monotone type if it is a proper splitting such that $U \geq 0$, U is row monotone and $U^\dagger V \geq 0$.* For splittings as above, we have the following lemma.

Lemma 2.1. *Let $A \in \mathbb{R}^{m\times n}$ and $A = U - V$ be a weak pseudo regular splitting of weak monotone type. Then $A^\dagger A = U^\dagger U \geq 0$.*

Proof. We have $A = U - V$ with $U \geq 0$, U row monotone and $U^\dagger V \geq 0$. Let $x \geq 0$. Set $y = U^\dagger U x$. Then $y \in R(U^\dagger) = R(U^t)$ and $Uy = UU^\dagger Ux = Ux \geq 0$, since $U \geq 0$ and $x \geq 0$. Since U is row monotone, this implies that $y \geq 0$ i.e., $U^\dagger U x \geq 0$. Hence $U^\dagger U \geq 0$. The formula $A^\dagger A = U^\dagger U$ is proved in Remark 2.1. ▯

We are now in a position to prove the first main result. As mentioned in the introduction, this result presents a *version* of Theorem 1.1 for weak monotone matrices.

Theorem 2.2. *Let $A \in \mathbb{R}^{m\times n}$. Let $A = U - V$ be a weak pseudo regular splitting of weak monotone type. Then A is weak monotone if and only if $\rho(U^\dagger V) < 1$.*

Proof. Let $C = U^\dagger V$. Then $C \geq 0$. Also $CU^\dagger U = U^\dagger V U^\dagger U = U^\dagger V = C$, since (as was shown in Remark 2.1) we have, $VU^\dagger U = V$. In general, for $k \geq 1$ we have $C^{k+1}U^\dagger U = C^{k+1}$. From (a) and (c) of Theorem 2.1, we have $A = U(I - C)$ and $A^\dagger = (I - C)^{-1}U^\dagger$.

Necessity: Let A be weak monotone. Set $B_k = (I + C + C^2 + \cdots + C^k)U^\dagger$ where k is any positive integer. Then $B_k U = (I + C + C^2 + \cdots + C^k)U^\dagger U \geq 0$, since $U^\dagger U = A^\dagger A \geq 0$, by Lemma 2.1. Thus $B_k U \geq 0$ for all k. Also, $B_k \leq B_{k+1}$, since $C \geq 0$. Using $A^\dagger = (I - C)^{-1}U^\dagger$, we get $B_k U = (I + C + C^2 + \cdots + C^k)U^\dagger U = (I + C + C^2 + \cdots + C^k)(I - C)A^\dagger U = (I - C^{k+1})A^\dagger U = A^\dagger U - C^{k+1}A^\dagger U$. Thus $A^\dagger U - B_k U = C^{k+1}A^\dagger U \geq 0$, i.e., $B_k U \leq A^\dagger U$. Hence the sequence $\{B_k U\}$ is a monotonically increasing sequence which is bounded above. Hence $\{B_k U\}$ is convergent with respect to any matrix norm $\|.\|$. Also, $B_{k+1}U - B_k U = C^{k+1}U^\dagger U = C^{k+1}$. Therefore $\|C^{k+1}\| = \|B_{k+1}U - B_k U\| \to 0$ as $k \to \infty$, i.e., $\{C^k\}$ converges to zero. Thus $\rho(U^\dagger V) = \rho(C) < 1$.

Sufficiency: Let $\rho(C) < 1$ and $y = Ax \geq 0$. Then $x = A^\dagger y + w$, where $w \in N(A)$. So $x = (I - C)^{-1}U^\dagger y + w = (I - C)^{-1}z + w$, where $z = U^\dagger y$. We shall prove that $z \geq 0$. Now $Uz = UU^\dagger y$, i.e., $Uz = y$, since $y \in R(A) = R(U)$. This implies that $Uz \geq 0$ and $z \in R(A^\dagger) = R(A^t)$. Therefore $z \geq 0$, since U is row monotone. Hence $(I - C)^{-1}z \geq 0$ (by Theorem 3.16, [12]), so that $x \in \mathbb{R}^n_+ + N(A)$, proving that A is weak monotone. ▯

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We first prove a lemma that will be useful in our discussion.

LEMMA 2.2. *Let $A \in \mathbb{R}^{m\times n}$. If A is row monotone, then A is weak monotone. The converse is true if $A^\dagger A \geq 0$.*

Proof. Let A be row monotone. Let $Ax \geq 0$. Set $y = Ax$. Then $x = A^\dagger y + w = z + w$, where $w \in N(A)$ and $z = A^\dagger y$. We show that $z \geq 0$. We have $Az = AA^\dagger y = AA^\dagger Ax = Ax \geq 0$. Thus $Az \geq 0$ and $z \in R(A^t)$. This implies that $z \geq 0$, so that A is weak monotone.

Next, we assume that A is weak monotone and $A^\dagger A \geq 0$. We prove that A is row monotone. Let $Ax \geq 0$ and $x \in R(A^t)$. Since A is weak monotone, this implies that $x = x^0 + w$, where $x^0 \geq 0$ and $w \in N(A)$. So $x = A^\dagger Ax = A^\dagger Ax^0 \geq 0$, since $A^\dagger A \geq 0$ and $x^0 \geq 0$. So, A is row monotone. $\square$

REMARK 2.2. *In the above lemma, if we discard the assumption that $A^\dagger A \geq 0$, then A may not be row monotone, as the following example shows. Let $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. If $Ax \geq 0$, then $x = (x_1 + x_3, x_2 + x_3, 0)^t + (-x_3, -x_3, x_3)^t \in \mathbb{R}^3_+ + N(A)$. So, A is weak monotone. Also $A^\dagger = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \\ 1 & 1 \end{pmatrix}$, so that $A^\dagger A \ngeq 0$. Let $x = (4, -1)^t$. Then $y = A^t x = (4, -1, 3)^t \ngeq 0$. But $Ay = (7, 2)^t \geq 0$. Thus, A is not row monotone.*

For unilateral intervals, we have the following result. This is an extension of Theorem 1.2, given in the introduction.

THEOREM 2.3. *Let B, $C \in \mathbb{R}^{m\times n}$, $R(B) = R(C)$, $N(B) = N(C)$, $C \leq B$ and B be weak monotone. Further assume that $B \geq 0$, $R(C) \cap \operatorname{int}(\mathbb{R}^m_+) \neq \phi$ and $B^\dagger B \geq 0$. Then C is weak monotone if and only if $C\mathbb{R}^n_+ \cap \operatorname{int}(\mathbb{R}^m_+) \neq \phi$.*

Proof. Necessity: Let C be weak monotone. Choose $y \in R(C) \cap \operatorname{int}(\mathbb{R}^m_+)$; $y = Cx^0 > 0$. Then $x^0 = v + w$ where $v \in \mathbb{R}^n_+$ and $w \in N(C)$. So, $y = Cx^0 = Cv \in C\mathbb{R}^n_+$. Thus $y \in C\mathbb{R}^n_+ \cap \operatorname{int}(\mathbb{R}^m_+)$, proving that $C\mathbb{R}^n_+ \cap \operatorname{int}(\mathbb{R}^m_+) \neq \phi$.

Sufficiency: Suppose that $C\mathbb{R}^n_+ \cap \operatorname{int}(\mathbb{R}^m_+) \neq \phi$. Let $U = B$ and $V = B - C \geq 0$. Then $C = U - V$. Also $R(U) = R(B) = R(C)$, $N(U) = N(B) = N(C)$ and $U \geq 0$. We also have $U^\dagger U = B^\dagger B \geq 0$ and U is weak monotone. We have $B^\dagger = U^\dagger = K - L$ where $K \geq 0$ and $R(B) \subseteq N(L)$ (by Theorem 3.6, [1]). So $B^\dagger V = (K - L)(B - C) = K(B - C) \geq 0$, where we have used the fact that $R(B) = R(C)$ and $R(B) \subseteq N(L)$. Also B is row monotone, by Lemma 2.2. Thus $C = U - V$ is a weak pseudo regular splitting of weak monotone type. If we prove that $\rho(U^\dagger V) < 1$, it would then follow from Theorem 2.2 that C is weak monotone. Set $y = Cx^0 \in C\mathbb{R}^n_+ \cap \operatorname{int}(\mathbb{R}^m_+)$. Then

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$Cx^0 > 0$, so that $Bx^0 \geq Cx^0 > 0$. Therefore there exists $\epsilon > 0$ such that $\epsilon Bx^0 \leq Cx^0$. Then $(B-C)x^0 \leq (1-\epsilon)Bx^0$. Now, $VB^\dagger Bx^0 = (B-C)B^\dagger Bx^0 = (B-C)x^0 \leq (1-\epsilon)Bx^0 < Bx^0$ so that $\rho(VU^\dagger) < 1$ (by Theorem 16.1, [8]). $\square$

Remark 2.3. *In the result above, we have assumed that $B^\dagger B \geq 0$, $B \geq 0$ and $R(C) \cap \operatorname{int}(\mathbb{R}^n_+) \neq \phi$. We show that each of these conditions is indispensable.*

(i) *Let* $B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ *and* $C = \begin{pmatrix} 0 & -1 & -1 \\ 1 & -1 & 0 \\ 1 & -2 & -1 \end{pmatrix}$. *Then $B \geq 0$, $C \leq B$, $R(C) \cap \operatorname{int}(\mathbb{R}^n_+) \neq \phi$, $R(B) = R(C)$ and $N(B) = N(C)$. But $B^\dagger B \ngeq 0$. Here B is weak monotone. We have, $C(-1,-1,0)^t \geq 0$. If $(-1,-1,0)^t = (\alpha,\beta,\gamma)^t + (a,a,-a)^t$, where $(\alpha,\beta,\gamma) \in \mathbb{R}^3_+$ and $a \in \mathbb{R}$, then $\alpha + a = -1$, where $\alpha \geq 0$ and $a \geq 0$. This is impossible. So, C is not weak monotone.*

(ii) *Let* $B = \frac{1}{2}\begin{pmatrix} 2 & -1 \\ -1 & 1 \\ -\sqrt{3} & \sqrt{3} \end{pmatrix}$ *and* $C = \begin{pmatrix} -2 & -1 \\ -1 & -1 \\ -\sqrt{3} & \sqrt{3} \end{pmatrix}$. *Then $B^\dagger B \geq 0$, $R(B) = R(C)$, $N(B) = N(C)$ and $\operatorname{int}(\mathbb{R}^3_+) \cap R(C) \neq \phi$. But $B \ngeq 0$. Here, B is weak monotone. If $x = (-1,-1)^t$, then $Cx \geq 0$ but $x \notin \mathbb{R}^2_+ + N(C)$. So, C is not weak monotone.*

(iii) *Let* $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ *and* $C = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. *Here $C \leq B$, $R(B) = R(C)$, $N(B) = N(C)$, $B \geq 0$, $B^\dagger B \geq 0$, and $R(C) \cap \operatorname{int}(\mathbb{R}^3_+) = \phi$. As above, it can be shown that B is weak monotone, while C is not weak monotone.*

To prove our result for a bilateral interval, we need the notion of a range kernel regular interval, defined and studied in [10]. The curious reader is referred to the work reported in [10], where the authors have shown how the subset K defined below arises naturally in studying extensions of notions of monotonicity to intervals of matrices.

Definition 2.2. *A (bilateral) interval matrix $J = [A, B]$ is called* range kernel regular *if $R(A) = R(B)$ and $N(A) = N(B)$. Let J be range kernel regular. We define a subset of K of J as $K = \{C \in J : R(C) = R(A) = R(B), N(C) = N(A) = N(B)\}$.*

Now, we prove an extension of Theorem 1.3 to weak monotone matrices. In particular, this result shows how the weak monotonicity of any (every) matrix in K can be verified just by studying the matrices that define the interval J. Also, we would like to draw the attention of the reader to the fact that the definition of the set K defined as above leads naturally to the notion of a proper splitting, thereby enabling us to use the results on proper splitting, given earlier.

Theorem 2.4. *Let $J = [A, B]$ be range kernel regular. Assume that $B^\dagger B \geq 0, B \geq 0$ and $\operatorname{int}(\mathbb{R}^m_+) \cap R(A) \neq \phi$. Then the following conditions are equivalent:*

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(a) *C is weak monotone, whenever $C \in K$.*
(b) *A and B are weak monotone.*
(c) *B is weak monotone and $\rho(B^\dagger(B-A)) < 1$.*

Proof. (a) $\Rightarrow$ (b): Follows due to the fact that $A, B \in K$.

(b) $\Rightarrow$ (c): Consider $A = B - T$, where $T = B - A$. We note that $B \geq 0$, $R(A) = R(B)$, $N(A) = N(B)$, and $B^\dagger T \geq 0$ (as proved in the sufficiency part of Theorem 2.3), since B is row monotone (from Lemma 2.2). So $A = B - T$ is weak pseudo regular splitting of weak monotone type. Since A is weak monotone, it follows from Theorem 2.2 that $\rho(B^\dagger(B-A)) = \rho(B^\dagger T) < 1$.

(c) $\Rightarrow$ (a): As argued above, it follows that $C = B - (B - C)$ is a weak pseudo regular splitting of weak monotone type. As $0 \leq B - C \leq B - A$ we have

$$\rho(B^\dagger(B-C)) \leq \rho(B^\dagger(B-A) < 1.$$

Therefore C is weak monotone, by Theorem 2.2. $\square$

A square matrix A is a *Z-matrix* if all the off-diagonal entries of A are nonpositive. A *Z-matrix* A is an *M-matrix* if A can be written as $A = sI - B$, where $s \geq \rho(B)$ and $B \geq 0$. If $s > \rho(B)$, then A is invertible and $A^{-1} \geq 0$. When $s = \rho(B)$ one might expect that $A^\dagger \geq 0$. But this is not true in general, as the following result shows.

Theorem 2.5. *(Corollary 5, [3]) If $A = \rho(B)I - B$, where $B \geq 0$ and irreducible then $A^\dagger \ngeq 0$.*

Interestingly, under the assumptions of the above theorem, it follows that A is weak monotone, as we prove next. We need the notions of a cone and its dual cone, which we review briefly. Recall that a subset K of $\mathbb{R}^n$ is a cone if $x + y \in K$ for all $x, y \in K$ and $\alpha x \in K$ for all $\alpha \geq 0$ and $x \in K$. For a cone $K \subseteq \mathbb{R}^n$, let $K^* = \{y \in \mathbb{R}^n : \langle x, y\rangle \geq 0 \text{ for all } x \in K\}$. K^* is the dual cone of K. $(K)^{**} = (K^*)^*$ is the dual cone of K^*. If $K = \mathbb{R}^n_+$, then $K^{**} = K^* = \mathbb{R}^n_+$. If $K = \mathbb{R}^n_+ \cap M$ for some subspace M of $\mathbb{R}^n$, then $K^* = \mathbb{R}^n_+ + M^\perp$.

Theorem 2.6. *If $A = \rho(B)I - B$, where B is nonnegative and irreducible, then A is weak monotone.*

Proof. We have $A^t = \rho(B)I - B^t$. So, $N(A^t) = \{x / B^t x = \rho(B)x\}$. Since $B^t \geq 0$ and irreducible, $N(A^t) = \{\alpha x^0 : \alpha \in \mathbb{R}\}$, where $x^0 > 0$ satisfies the equation $B^t x^0 = \rho(B)x^0$ with $\sum_{i=1}^n x_i^0 = 1$, by the Perron Frobenius Theorem (Theorem 2.7 [12]). Let $x \in \mathbb{R}^n$. We show that a suitable $\alpha \in \mathbb{R}$ could be found such that $u = x - \alpha x^0 \in \mathbb{R}^n_+$. Set $I_+ = \{i : x_i \geq 0\}$ and $I_- = \{i : x_i < 0\}$. If $I_- = \emptyset$, we take $\alpha = 0$ so that $u = x \in \mathbb{R}^n_+$. If $I_- \neq \emptyset$ we take $\alpha = \min\{\frac{x_i}{x_i^0} : i = 1, 2, \ldots, n\}$. Then $\alpha \leq \frac{x_i}{x_i^0}$ for $i = 1, 2, \ldots, n$ so that $x_i - \alpha x_i^0 \geq 0$ for $i = 1, 2, \ldots, n$. It follows that,

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$\mathbb{R}^n_+ + N(A^t) = \mathbb{R}^n$. Now, taking the dual we get, $\mathbb{R}^n_+ \cap R(A) = \{0\}$. Let $Ax \geq 0$. Then $Ax \in R(A) \cap \mathbb{R}^n_+ = \{0\}$, i.e., $Ax = 0$. This implies that $x \in N(A) \subseteq \mathbb{R}^n_+ + N(A)$. So A is weak monotone. So, $\mathbb{R}^n_+ + N(A) = \mathbb{R}^n$. ☐

In the above Theorem, A may not be weak monotone, if B is reducible, as we show next. Let $A = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = s\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} s & 1 \\ 0 & s \end{pmatrix} = sI - B$, where $s > 0, B = \begin{pmatrix} s & 1 \\ 0 & s \end{pmatrix} \geq 0$. Clearly, B is reducible. We have $A(0,-1)^t = (1,0)^t \geq 0$. However, if $\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \in \mathbb{R}^2_+ + N(A)$, then $\beta = -1 \ngeq 0$.

Finally, we show that weak monotonicity is preserved under certain similarity transformations.

Theorem 2.7. *Let A, B, $C \in \mathbb{R}^{n\times n}$ be such that $A = C^{-1}BC$ with C and $C^{-1} \geq 0$. Then B is weak monotone if and only if A is weak monotone.*

Proof. Clearly it is enough to prove necessity. Let $Ax \geq 0$. Then

$$y = C^{-1}BCx \geq 0.$$

So $BCx = Cy \geq 0$. Since B is weak monotone, this gives $Cx = v + w$, where $v \in \mathbb{R}^n_+$ and $w \in N(B)$. So $x = C^{-1}v + C^{-1}w$. Observe that $C^{-1}v \geq 0$. Also, $AC^{-1}w = C^{-1}Bw = 0$ and hence $C^{-1}w \in N(A)$. Thus A is weak monotone. ☐

Remark 2.4. *A verbatim statement when the usual inverse is replaced by the Moore-Penrose inverse does not hold. This is shown as follows. Let*

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Then $C \geq 0$ and $C^\dagger = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \geq 0$. Here B is weak monotone. But

$$C^\dagger BC = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

is not weak monotone.

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