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CRITICAL EXPONENTS: OLD AND NEW*

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Abstract. For Hadamard and conventional powering, we survey past and current work on the existence and values of critical exponents (CEs) for such matrix classes as doubly nonnegative, totally positive, M- and inverse M-matrices. There are remarkable similarities and differences between Hadamard and conventional CEs for the various classes.

Key words. critical exponent, Hadamard powering, doubly nonnegative, totally positive.

AMS subject classifications. 15A16, 15A99.

1. Introduction. Let $\mathcal{C}$ be a class of m-by-n matrices, and suppose that for all $A \in \mathcal{C}$ a notion of continuous powering, $A^{\{t\}}$, is defined for all $t \geq 0$ (or, perhaps, all real t if appropriate). The *critical exponent* (CE) of $A \in \mathcal{C}$, with respect to $\mathcal{C}$ and this powering, is the least real number $g(A)$, such that $A^{\{t\}} \in \mathcal{C}$ for all $t > g(A)$, if such exists. Otherwise, $g(A) = \infty$. If each element of $\mathcal{C}$ has finite CE, the CE of $\mathcal{C}$ with respect to the powering is $M = g(\mathcal{C}) = \sup_{A \in \mathcal{C}}\{g(A)\}$. If M is finite, $\mathcal{C}$ is said to have CE ($g(\mathcal{C})$). We note that even if each element of $\mathcal{C}$ has finite CE, the class $\mathcal{C}$ may or may not have CE. We also note that if $\mathcal{C}$ is defined in several dimensions, the CE may vary with dimension. If submatrices of A in $\mathcal{C}$ are again in $\mathcal{C}$, the CE cannot decrease as a function of the dimension.

We are interested here in the CE's for a variety of natural classes and for two kinds of powering: a) that which (for $m = n$) is consistent with the conventional matrix product (with continuous conventional powers defined by continuous powers of the diagonal eigenvalue matrix of the spectral decomposition) and b) Hadamard (entry-wise) powering: $A^{(t)} = (a_{ij}^t)$ if $A = (a_{ij})$. The classes include: totally positive/totally nonnegative (TP/TN) matrices, doubly nonnegative (DN) matrices, M- and inverse M-matrices ($\mathcal{M}$, $\mathcal{IM}$), etc. In general, we might ask two questions: a) does a CE exist; and 2) if so, what is the value of the CE or what are upper and lower bounds for it. Our purpose here is to survey both past and current work on the existence and values of critical exponents.

Before continuing, we mention two simple examples. First, the Hadamard CE for

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Vandermonde matrices based upon an increasing positive sequence $x_1 < x_2 < \cdots < x_n$ is 0 as $V^{(t)}$ is the Vandermonde based upon $x_1^t < x_2^t < \cdots < x_n^t$ for all $t > 0$. Negative values of t reverse the order. Second, for 1-by-1 positive integer matrices, there is no Hadamard (nor conventional, which is the same) CE as transcendental powers of positive integers are not integers. In this case, we note that there is what might be called an *integer CE*, which is 0, as any nonnegative integer power of a positive integer is a positive integer. We will see other, more subtle, examples in which there is no CE, as well as classes that show that any positive integer may be a CE.

A concept related to the CE is that of *infinite divisibility*. A class $\mathcal{C}$ of matrices is called infinitely divisible (with respect to a given powering) (ID) if $A \in \mathcal{C}$ implies $A^{\{t\}} \in \mathcal{C}$ for any t, $0 < t < 1$. Though this terminology was not used, it is shown in [J, JS2011] that M-matrices and inverse M-matrices are ID with respect to conventional powering.

Finally, we note that in the case of some familiar classes, two versions might be studied: an open version (e.g. TP matrices) or a closed version (TN matrices). Typically, for reasons of continuity, there is little theoretical difference between two such classes, and we will, arbitrarily, discuss just one of them.

2. Exponential polynomials. An exponential polynomial is an expression of the form

$$\alpha_n\beta_n^t + \alpha_{n-1}\beta_{n-1}^t + \cdots + \alpha_1\beta_1^t$$

with bases $\beta_n > \beta_{n-1} > \cdots > \beta_1 > 0$ and coefficients α_i real and nonzero.

Lemma 2.1. *The number of zeros an exponential polynomial can experience (counting multiplicity) does not exceed the number of sign changes within the ordered sequence of coefficients* $\{\alpha_n, \alpha_{n-1}, \ldots, \alpha_1\}$.

This statement, an extension of Descartes' Rule of Signs by Laguerre [JG], is well-known and not proven here.

Exponential polynomials may be used to analyze a variety of CE problems. In the case of (diagonalizable) entry-wise nonnegative matrices with nonnegative eigenvalues (NND), continuous conventional powers of a NND matrix A can be defined by $A^t = VD^tV^{-1}$. Then, individual entries of A^t are exponential polynomials of a form equivalent to that used by most authors on matrix functions, e.g. [HJ 1994]: $(A^t)_{ij} = \lambda_n^t(v_nu_n)_{ij} + \cdots + \lambda_1^t(v_1u_1)_{ij}$, in which v_i are the columns of V, u_i are the rows of V^{-1}, and λ_i are the eigenvalues of A. As the minors of A^t are polynomials in the entries, each minor can be expressed as an exponential polynomial in which products of the eigenvalues are the bases and polynomials in the entries of V and V^{-1}

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form the coefficients.

Example 2.2. *For conventional powers of the 4-by-4 matrix*

$$B = \begin{bmatrix} 5 & 2 & 0 & 1 \\ 2 & 5 & 1 & 0 \\ 0 & 1 & 5 & 2 \\ 1 & 0 & 2 & 5 \end{bmatrix}, \quad \sigma(B) = \{2, 4, 6, 8\},$$

the entries can be written as

$$(B^t)_{ij} = \begin{cases} \frac{1}{4}8^t + \frac{1}{4}6^t + \frac{1}{4}4^t + \frac{1}{4}2^t & \text{for } i = j \\ \frac{1}{4}8^t + \frac{1}{4}6^t - \frac{1}{4}4^t - \frac{1}{4}2^t & (i\ j) \in \{(1\ 2), (2\ 1), (3\ 4), (4\ 3)\} \\ \frac{1}{4}8^t - \frac{1}{4}6^t - \frac{1}{4}4^t + \frac{1}{4}2^t & (i\ j) \in \{(1\ 3), (3\ 1), (2\ 4), (4\ 2)\} \\ \frac{1}{4}8^t - \frac{1}{4}6^t + \frac{1}{4}4^t - \frac{1}{4}2^t & (i\ j) \in \{(2\ 3), (3\ 2), (1\ 4), (4\ 1)\}. \end{cases}$$

Additionally, the determinant of the leading 3-by-3 principal submatrix equals

$$\frac{1}{4}6^t8^t2^t + \frac{1}{4}6^t4^t2^t + \frac{1}{4}8^t4^t2^t + \frac{1}{4}8^t6^t4^t = \frac{1}{4}96^t + \frac{1}{4}48^t + \frac{1}{4}64^t + \frac{1}{4}192^t$$

with other minors having similar forms.

The minors of nonnegative matrices raised to continuous *Hadamard* powers can also be written as exponential polynomials; however, the bases in this case are products of the entries and the coefficients are integers, as in the case below:

Example 2.3.

$$A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}, \quad A^{(t)} = \begin{bmatrix} 4^t & 1^t & 1^t \\ 1^t & 4^t & 1^t \\ 1^t & 1^t & 4^t \end{bmatrix}$$

The determinant of this matrix is $\det\left(A^{(t)}\right) = \left(4^3\right)^t - 3(4)^t + 2 = 64^t - 3(4)^t + 2$.

3. Conventional Powering.

3.1. Nonnegative, diagonalizable matrices with nonnegative eigenvalues.

Certain aspects of a class for conventional powering critical exponent analysis are natural. Diagonalizability is convenient, though it may not be essential. Nonnegative eigenvalues may be essential. If the eigenvalues are nonnegative, they will remain so in A^t but other defining characteristics may not be so clearly preserved. In

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contrast, for Hadamard powering, nonnegativity of entries is guaranteed, while the preservation of nonnegativity of eigenvalues (and, consequently, minors) may fail.

The following example illustrates this point:

Example 3.1.

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 5 & 4 \\ 4 & 7 & 10 \end{bmatrix}, \quad A^{(t)} = \begin{bmatrix} 2^t & 3^t & 4^t \\ 3^t & 5^t & 7^t \\ 4^t & 7^t & 10^t \end{bmatrix}.$$

Here, A has nonnegative entries and nonnegative eigenvalues, and $\det(A^{(t)}) = 100^t - 98^t + 2\,(84^t) - 90^t - 80^t$. For any $t \in (0,1)$, $\det(A^{(t)})$ is negative.

As noted above, an NND matrix can be decomposed into $A^t = VD^tV^{-1}$, with every entry given by

$$\left(A^t\right)_{ij} = \lambda_n^t (v_n u_n)_{ij} + \cdots + \lambda_1^t (v_1 u_1)_{ij}.$$

For integer values of t, A^t is clearly NND; the integer critical exponent is therefore 0.

Lemma 3.2. *Let $A \in NND$ and suppose that row i (column j) of A^t is (entry-wise) nonnegative for $t \in [s, s+1)$ and some $s > 0$. Then row i (column j) of A^t is nonnegative for all $t \geq s$.*

Proof. Suppose row i is nonnegative in A^t, for $t \in [k, k+1)$. Right multiplication of every matrix in this interval by A (a nonnegative matrix) gives every entry of row i in A^t as the inner product of two nonnegative vectors, for all $t \in [k+1, k+2)$. The process can be repeated for all successive intervals. ◻

An important implication of Lemma 3.2 is that if, for t' in an interval between two consecutive integers, $A^{t'}$ is not NND (that is, $A^{t'}$ has at least one negative entry), then A^t is not NND for some t in every earlier interval between consecutive integers. This reasoning leads to the next theorem, stated below. Here, and following, S^c denotes the complement of a set S with respect to an understood universe.

Lemma 3.3. *Let A be an n-by-n NND matrix. Then $A^t \in NND^c$ on at most finitely many intervals.*

Proof. The matrix A^t depends continuously on t and leaves the class NND only when some entry becomes negative. The entries are exponential polynomials, each with at most $n-1$ zeros, so each of the n^2 entries can become negative at most $\left\lfloor \frac{(n-1)}{2} \right\rfloor$ times. ◻

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Taken together, Lemmas 3.2 and 3.3 imply the existence of a CE and give an upper bound; a lower bound is proven in the next section.

THEOREM 3.4. *A CE for NND matrices, $g(NND)$, exists and satisfies*

$$n-2 \leq g(NND) \leq \left\lfloor \frac{(n-1)}{2} \right\rfloor n.$$

Proof. By Lemma 3.3, each of the n entries in a column can leave the class at most $\left\lfloor \frac{(n-1)}{2} \right\rfloor$ times. Because of the observation after Lemma 3.2, the maximum exponent at which a column can contain a negative entry is strictly less than $\left\lfloor \frac{(n-1)}{2} \right\rfloor n$. The lower bound of $n-2$ comes from consideration of an irreducible tridiagonal DN (and thus NND) matrix, discussed in the next section. □

An important observation is that, while Lemma 3.2 relied on the fact that integer powers of NND matrices are NND, the existence of a CE could have been proven with only the assumption that there exists a *fixed power* $l > 1$ such that A^l is in the class for all A. Instead of considering an interval of $t \in [k, k+1)$ over which A^t is nonnegative, such a proof would consider an interval $t \in (l^k, l^{k+1})$ (noting that A^{l^k} is in the class for all integer k), and proceed identically. Thus, two ingredients: 1) the existence of a fixed power that preserves membership in the class for all elements of the class and 2) a finite bound on the number of times continuous powers of an element can leave the class are enough to imply the existence of a CE in the NND case and, more generally, for any class $\mathcal{C}$. This result is summarized in Lemma 3.5 below.

LEMMA 3.5. *If a class $\mathcal{C}$ and powering $A^{\{\cdot\}}$ satisfy two properties; namely,*

1. *$\exists$ a power $l > 1$ such that for every element $A \in \mathcal{C}$, $A^{\{l\}} \in \mathcal{C}$, and*
2. *$A^{\{t\}}$ can be in $\mathcal{C}^c$ on only finitely many intervals of $t \in \mathbb{R}$,*

then a CE exists for that class and powering.

The proof for this statement is identical in structure to the proof in the NND case, in which the fixed power $l = 2$ and the finite limit on the number of times $A^{\{t\}}$ can be in $\mathcal{C}^c$ follows directly from the constraint on the number of real roots of exponential polynomials. This argument is helpful in establishing the existence of a CE in several cases but does not generally give the exact value of the CE.

3.2. Doubly Nonnegative Matrices. Doubly nonnegative (DN) matrices are positive semidefinite matrices with nonnegative entries. We conjecture that the conventional CE for DN is $n-2$ and we have proven this for $n < 6$ [JLW]. The symmetry

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of DN matrices allows decomposition of the form $A = UDU^T$, which in turn places certain restrictions on the signs of the eigenvectors and allows determination of better upper bounds than for NND matrices.

THEOREM 3.6. *The conventional critical exponent for DN matrices, $g(DN)$, satisfies*

$$g(DN) \leq \begin{cases} \frac{n^2-4n+5}{2} & \text{if } n \text{ is odd;} \\ \frac{n^2-5n+8}{2} & \text{if } n \text{ is even.} \end{cases}$$

A more thorough discussion of these bounds appears in [JLW], as does the following proof of the lower bound, $n-2$.

THEOREM 3.7. *The conventional CE for DN matrices is at least $n-2$.*

Proof. Let $A \in M_n$ be an irreducible, invertible, tridiagonal DN matrix. Then the $(1,n)$ and $(n,1)$ entries of A^t are zero for $t = 0,1,2,\ldots,n-2$. By Lemma 2.1, the exponential polynomials $(A^t)_{1n}$ and $(A^t)_{n1}$ each have at most $n-1$ roots; the integer zeros listed account for all of these. Thus, $(A^t)_{1n} \geq 0$ for all $t \geq n-2$ and $(A^t)_{1n} < 0$ for $t \in (n-3, n-2)$. ($(A^t)_{n1}$ behaves identically). So, $g(\mathrm{DN}) \geq n-2$. ∎

3.3. Totally Positive Matrices. A totally positive (nonnegative) matrix is one in which all minors are positive (nonnegative). Identifying a critical exponent for TP matrices requires considering not only the signs of the entries as the power is increased, but also the signs of the minors, which, like the entries, can be expressed as exponential polynomials. The conventional product of two TP matrices is TP [FJ]; thus, all positive integer conventional powers of a TP matrix are TP. Since TP matrices are diagonalizable with distinct positive eigenvalues [FJ], continuous conventional powers are well-defined.

THEOREM 3.8. *The conventional critical exponent for TP matrices exists and satisfies*

$$n-2 \leq g(TP) \leq \max_{k=1,\ldots,n} \left\{ (2(n-k)+1) \left\lfloor \frac{\binom{n+k-1}{k} - 1}{2} \right\rfloor \right\} + 1.$$

Proof. Existence is demonstrated by noting that conventional powering of TP matrices satisfies both the fixed power requirement (integer powers are TP) and the finite exit requirement (all entries and minors are exponential polynomials with a finite number of roots).

An upper bound can be found in a manner very similar to that for NND matrices; however, the required nonnegativity of the minors can lead to very high upper bounds

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very quickly. There are several useful observations that can simplify the problem, one of which is that a matrix is TP if and only if every initial minor of it is positive [FJ]. (The *initial minors* of a matrix are those both of whose index sets are consecutive and one of whose index sets begins with 1.) There are n^2 initial minors (each entry is the bottom right hand corner of an initial minor), $2(n-k)+1$ of which are size k. A calculation verifies that every k-by-k minor, when written as an exponential polynomial, has at most $\binom{n+k-1}{k}-1$ roots.

If all the k-by-k minors of a TP matrix R^t are nonnegative for some interval $t \in (k, k+1)$, then by Cauchy-Binet [FJ] those minors are nonnegative for all $t > k$. We can therefore think of each set of k-by-k minors as having its own CE upper bound, the maximum of which is an upper bound for TP matrices, and in doing so arrive at the upper bound given above. □

As before, a lower bound is easily established by example.

Theorem 3.9. *The critical exponent for conventional powers of TN (TP) matrices is at least $n-2$.*

Proof. Consider

$$R = \begin{pmatrix} 2 & 1 & 0 & \cdots & 0 \\ 1 & 2 & 1 & \ddots & 0 \\ 0 & 1 & 2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & 1 & 2 \end{pmatrix}$$

Proof: This matrix is TN for all n. By Theorem 3.7, we know that the $(1\ n)$-entry of R^t is negative for $t \in (n-3, n-2)$. Since any TN matrix may be arbitrarily little perturbed to become TP, the same statement holds for TP as well. □

3.4. M-matrices and Inverse M-matrices. Recall that an n-by-n real matrix with nonpositive off-diagonal entries is called an *M-matrix* if it is invertible and its inverse is entry-wise nonnegative. The nonnegative matrices that occur as inverses of M-matrices are called *inverse M-matrices.* We denote the two classes by $\mathcal{M}$ and $\mathcal{IM}$, respectively. Most of the known theory may be found in [J, JS2011, HJ1994, etc.]

Because matrices in $\mathcal{M}$ or $\mathcal{IM}$ may not be diagonalizable, the continuous powering that is consistent with integer powering is most easily defined by power series and limits for matrices in $\mathcal{M}$ [J, JS2011] and then transferred to $\mathcal{IM}$ via inversion, which is what we assume here.

It is clear that the conventional powering CE theory for $\mathcal{M}$ and $\mathcal{IM}$ are the

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same, but there is a noticeable dependence on n (in both cases). For $n = 1$ the two classes are the same and the conventional CE is $-\infty$. For $n = 2$, any (continuous) nonnegative power (which may be defined by diagonalization) of an M-matrix is an M-matrix, so that the conventional CE is 0, and likewise for $\mathcal{IM}$.

For $n \geq 3$, the situation is quite different. Since both classes are ID [J, JS 2011], either the CE is zero or it does not exist. This is because if any power, larger than 1, of an M-matrix is not an M-matrix then, by taking smaller and smaller roots, arbitrarily high powers of an M-matrix are not M-matrices. For $n \geq 3$, it is quite easy to construct an example of an M-matrix whose square has a positive off-diagonal entry and is, thus, not in $\mathcal{M}$, e.g.

$$A = \begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$A^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \text{ while } A^2 = \begin{pmatrix} 1 & -4 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}.$$

By direct summation with I of appropriate size, we conclude that for any $n \geq 3$, there is no conventional CE for either $\mathcal{M}$ or $\mathcal{IM}$. This example also shows that individual $\mathcal{M}$ ($\mathcal{IM}$) matrices may fail to have a CE and that there is no integer CE.

4. Hadamard Powering. We now turn to critical exponents for classes under Hadamard powering. We restrict our attention to matrix classes whose definition requires that the entries be nonnegative as this feature is clearly retained for any continuous Hadamard power.

4.1. Doubly Nonnegative Matrices. In contrast to the conventional powering case, all Hadamard powers of a DN matrix remain symmetric and nonnegative. The question of a CE then hinges on whether the quadratic form (or, equivalently, the eigenvalues) remains nonnegative (also in contrast to the conventional case). Both the history and the answer to the CE question here are interesting. At the time, the first author was a Ph.D student, a manuscript, by a very famous mathematician, circulated purporting to show that the CE for Hadamard powering of a DN matrix is 1. Some readers, including the first author, recognized that the proof given was flawed. Then in [FH] (see also [HJ 1994]), it was shown, in pursuit of the Bieberbach conjecture that the CE is actually $n - 2$, perhaps a surprising answer. The proof is more ingenious than lengthy. A nice integral representation of $A^{(t)}$ quickly gives sufficiency, and DN matrices of the form $uu^T + vv^T$ with u and v positive vectors in

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$\mathcal{R}^n$ are constructed to show necessity. It is curious that the two DN CE's, one for conventional and one for Hadamard, may well have the same curious answer (and do for $n < 6$) when the problems appear so different and the methodology that has been used in so different. Though it is likely too much to ask, a common proof would be most welcome.

4.2. Inverse M-matrices. Because $\mathcal{IM}$ matrices are entry-wise nonnegative, their Hadamard powers are nonnegative as well. For a Hadamard power to be $\mathcal{IM}$, the signs of its minors must be such that the inverse exists and has the requisite sign pattern for an M-matrix. For $n = 2$, a simple calculation shows that this happens whenever the Hadamard exponent is positive. Thus, the Hadamard CE for $\mathcal{IM}$, when $n = 2$, is 0. For $n = 3$, the calculation is not so simple, but the first author and R. Smith carried it out some time ago to find that the Hadamard CE for $\mathcal{IM}$, when $n = 3$, is also 0 (See [JS 2007a] for another view). Long before, the first author, and also M. Neuman, had raised the question of the Hadamard CE for $\mathcal{IM}$ for a given n. Bits and pieces gradually emerged.

In [C 2004], it was shown that the integer CE for $\mathcal{IM}$ is 1 for $n > 3$. (Johnson and Smith had ground out the case $n = 4$, independently). Interestingly at this point, using the ideas of exponential polynomials that we have outlined, it could have been easily shown that there is a (continuous) Hadamard CE for any n, just based on the fact that the Hadamard square of an $\mathcal{IM}$ matrix is $\mathcal{IM}$, but this was not recognized. In [JS 2007a and JS 2007b], it was shown that for any $A \in \mathcal{IM}$, from some Hadamard power on, all continuous Hadamard powers are $\mathcal{IM}$, i.e. that there is a finite CE for each individual $\mathcal{IM}$ matrix. Then in [C 2007] it was shown that for any $n \geq 3$, the continuous Hadamard CE of $\mathcal{IM}$ is at most 1. This, as well as [C 2004], used earlier works of Johnson and Smith and included a nice application of the Holder inequality. Since for $n > 3$, there are examples to show that $\mathcal{IM}$ is not Hadamard ID ([JS 2011]), it follows that the Hadamard CE for $\mathcal{IM}$ is 1 for all $n > 3$. This gives the CE for $\mathcal{IM}$ at each n as

n	Hadamard CE
$n = 1$	$-\infty$
$n = 2$	0
$n = 3$	0
$n \geq 4$	1

4.3. Totally Positive Matrices. The question of a Hadamard critical exponent for TP matrices is of particular interest for the following reason. Let A be TP_2 (that is, assume all entries and 2-by-2 minors are positive). Then there exists some power p, specific to A, such that $A^{(t)}$ is TP for all $t \geq p$ [FJ 2007]. Thus, there is a Hadamard critical exponent for each TP matrix A, which does not, by itself, imply that there is

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a critical exponent for the class TP, under Hadamard powering.

It is unknown whether or not a Hadamard critical exponent exists. While a critical exponent for TP matrices, if it exists, is at least $n-2$ (shown by adapting a proof for DN matrices from [FH]), higher lower bounds have been found for some n. For instance, a 4-by-4 example of a TP matrix with critical exponent between $t = 5.5$ and $t = 5.6$ is provided below:

Example 4.1.

$$\begin{pmatrix} 210 & 48547 & 80633 & 82930 \\ 71539 & 17126755 & 29592586 & 30438643 \\ 84121 & 21345134 & 39294848 & 42461372 \\ 73730 & 20385912 & 40697553 & 46818689 \end{pmatrix}$$

While a whole-class result remains unsettled, several specific subclasses can be shown to have CE, including Vandermonde and Hurwitz TP matrices [GT]. The 3-by-3 TP matrices do have Hadamard CE; no extension to higher dimensions has yet been established.

Theorem 4.2. *The Hadamard critical exponent for* 3*-by-*3 *TP matrices is* 1*.*

Proof. We begin by showing that the Hadamard square of a 3-by-3 TP matrix is TP. Let R be a TP matrix scaled so that both the first row and column are all ones. Without loss of generality, assume the (3,2)-entry to be greater than or equal to the (2,3) entry. Then R can be written as:

$$R = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1+\alpha & 1+\alpha+\beta \\ 1 & 1+\alpha+\beta+\gamma & 1+\alpha+\beta+\gamma+\delta \end{pmatrix}$$

in which α, β, γ, and δ are all positive.

Because R is TP, its determinant, $r = -\alpha\,\beta + \alpha\,\delta - \beta^2 - \beta\,\gamma > 0$. It can be shown by explicit calculation that the determinant of $R^{(2)}$, $r^{(2)}$, is expressable in the following way:

$$r^{(2)} = 4\,r + 2\,\alpha^2 r + 6\,\alpha\,r + \beta^2 r + \alpha\,\delta\,r + 2\,\gamma\,r + \beta\,\gamma\,r + 4\,\beta\,r + 3\,\alpha\,\beta\,r + 2\,\alpha\,\gamma\,r + 2\,\alpha\,\delta^2 + 2\,\alpha\,\gamma\,\delta$$

Its determinant being positive (and all other minors easily verifiable), $R^{(2)}$ is TP. It remains to show that R^t is TP for all other powers greater than 1. Rewrite R as follows:

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$$R = \begin{pmatrix} 1 & 1 & 1 \\ 1 & a & b \\ 1 & c & d \end{pmatrix}$$

The determinant of $R^{(t)}$, therefore, is $(ad)^t - d^t - (bc)^t + b^t + c^t - a^t$. The requirements upon the 2-by-2 minors of R that make it TP (for instance, $ad > bc$) define a partial ordering on a, b, c, and d. The terms with ambiguous order, d^t and $(bc)^t$, and b^t and c^t, can be rearranged in the determinant without changing the sign pattern of the ordered coefficients. Because the number of sign changes is $w = 3$, the maximum number of zeros that can be experienced by the determinant is also 3. These roots can occur either in the interval $t \in (0, 1)$, in which case there are not enough zeros remaining for the function to become negative again, or R^t can be TP for all $t \in (0, 1)$, which by repeated squaring of every matrix in the interval would imply the property of being TP for all $t > 0$. So the upper bound for the Hadamard critical exponent of a 3-by-3 TP matrix is 1. $\square$

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