Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 25, pp. 60-71, August 2012

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BIDIAGONAL DECOMPOSITIONS, MINORS AND APPLICATIONS*

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Abstract. Matrices, called ε-BD matrices, that have a bidiagonal decomposition satisfying some sign constraints are analyzed. The ε-BD matrices include all nonsingular totally positive matrices, as well as their matrices opposite in sign and their inverses. The signs of minors of ε-BD matrices are analyzed. The zero patterns of ε-BD matrices and their triangular factors are studied and applied to prove the backward stability of Gaussian elimination without pivoting for the associated linear systems.

Key words. Bidiagonal decomposition, totally positive matrices, Gaussian elimination, backward stability.

AMS subject classifications. 15A18, 15A15, 65F05, 65F15.

1. Introduction. In this paper we consider matrices, called ε-BD matrices, that admit a bidiagonal decomposition with some sign constraints. These matrices include the important class of nonsingular totally positive matrices, their inverses and the class of matrices presented in [2]. Let us recall that a matrix is *totally positive* (TP) if all its minors are nonnegative. TP matrices are also called totally nonnegative matrices and present applications in many fields (see the surveys [1] and [6] and the recent books [7] and [14]). The relationship between TP matrices and bidiagonal decompositions has been deeply analyzed (cf. [15], [8], [10]). Matrices whose inverses are totally positive arise in the discretization of partial differential equations. More examples of ε-BD matrices appear in Computer Aided Geometric Design (cf. [13]).

Section 2 studies the sign restriction of the bidiagonal decomposition of ε-BD matrices and analyzes the zero pattern of the entries of these matrices. The relationship between the zero pattern of ε-BD matrices and their triangular factors is analyzed in Section 3, and it is applied to obtain very small backward error bounds of Gaussian elimination without row exchanges for the associated linear systems. This application extends to ε-BD matrices the backward stability for totally positive linear systems proved in [4]. Section 4 includes numerical experiments showing, for linear systems

*Received by the editors on October 15, 2011. Accepted for publication on June 17, 2012. Handling Editor: Ljiljana Cvetkovic.

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Research Partially Supported by the Spanish Research Grant MTM2009-07315, by Gobierno de Aragón and Fondo Social Europeo.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 25, pp. 60-71, August 2012

whose coefficient matrices are ε-BD, higher accuracy of Gaussian elimination without row exchanges than that of Gaussian elimination with partial pivoting.

2. Definitions and some basic properties. Let us start by introducing basic notation. Given $k \in \{1, 2, \dots, n\}$ let $Q_{k,n}$ be the set of increasing sequences of k positive integers less than or equal to n. If $\alpha, \beta \in Q_{k,n}$, we denote by $A[\alpha|\beta]$ the $k \times k$ submatrix of A containing rows numbered by α and columns numbered by β. Also, let $A[\alpha] := A[\alpha|\alpha]$. Finally, let us denote by ε a vector $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m)$ with $\varepsilon_j \in \{\pm 1\}$ for $j = 1, \dots, m$, which will be called a *signature*.

Definition 2.1. *Given a signature $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ and a nonsingular $n \times n$ ($n \geq 2$) matrix A, we say that A has an ε bidiagonal decomposition (for brevity, A is ε-BD) if we can write A as*

$$A = L^{(1)} \cdots L^{(n-1)} D U^{(n-1)} \cdots U^{(1)}, \tag{2.1}$$

where $D = \operatorname{diag}(d_1, \dots, d_n)$, and, for $k = 1, \dots, n-1$, $L^{(k)}$ and $U^{(k)}$ are lower and upper bidiagonal matrices respectively with unit diagonal and off-diagonal entries $l_i^{(k)} := (L^{(k)})_{i+1,i}$ and $u_i^{(k)} := (U^{(k)})_{i,i+1}$, $(i = 1, \dots, n-1)$ satisfying

1. *$\varepsilon_n d_i > 0$ for all $i = 1, \dots, n$,*
2. *$l_i^{(k)} \varepsilon_i \geq 0$, $u_i^{(k)} \varepsilon_i \geq 0$ for all $i, k \leq n-1$.*

Clearly, ε-BD matrices are nonsingular matrices. Nonsingular TP matrices and those matrices opposite in sign to them are ε-BD, as the next result shows.

Proposition 2.2. *A matrix A is ε-BD with $\varepsilon = (1, \dots, 1)$ (respectively, $\varepsilon = (1, \dots, 1, -1)$) if and only if A (respectively, $-A$) is nonsingular TP.*

Proof. If A is ε-BD with signature $\varepsilon = (1, \dots, 1)$, then the matrix is TP because the bidiagonal nonnegative matrices of (2.1) are TP and the product of TP matrices is also TP (cf. [1, Theorem 3.1]). Conversely, if A is nonsingular TP then A satisfies Definition 2.1 with $\varepsilon = (1, \dots, 1)$ by [10, Theorem 4.2].

Let us observe that, if A is ε-BD, $-A$ can be decomposed as in (2.1) with the same bidiagonal factors as A but changing the sign of the diagonal elements of D. So, A is ε-BD with $\varepsilon = (1, \dots, 1)$ if and only if $-A$ is ε-BD with $\varepsilon = (1, \dots, 1, -1)$, and the result follows. ∎

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Recall that an elementary bidiagonal matrix is a matrix

$$E_i(x) := \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & x & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{bmatrix},$$

which has the entry x in position $(i, i-1)$. We denote by $E_i^T(x)$ the matrix $(E_i(x))^T$. Each bidiagonal matrix of the factorization (2.1) can be in turn decomposed as a product of elementary bidiagonal matrices. For instance,

$$E_2(l_1^{(k)}) \cdots E_n(l_{n-1}^{(k)}) = L^{(k)} = \begin{bmatrix} 1 & & & \\ l_1^{(k)} & 1 & & \\ & \ddots & \ddots & \\ & & l_{n-1}^{(k)} & 1 \end{bmatrix},$$

and an analogous decomposition can be applied for the upper triangular factors of (2.1). Therefore (2.1) can be written as

$$\begin{aligned} A = {} & (E_2(l_1^{(1)}) \cdots E_n(l_{n-1}^{(1)})) \cdots (E_2(l_1^{(n-1)}) \cdots E_n(l_{n-1}^{(n-1)}))D \cdot \\ & (E_n^T(u_{n-1}^{(n-1)}) \cdots E_2^T(u_1^{(n-1)})) \cdots (E_n^T(u_{n-1}^{(1)}) \cdots E_2^T(u_1^{(1)})), \end{aligned} \tag{2.2}$$

where all bidiagonal factors are elementary.

The equivalence of (2.1) and (2.2) leads in turn to the following characterization of ε-BD matrices.

PROPOSITION 2.3. *Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ be a signature and let A be a nonsingular $n \times n$ ($n \geq 2$) matrix. Then A is an ε-BD matrix if and only if we can write A as (2.2), where $D = \operatorname{diag}(d_1, \ldots, d_n)$, $l_i^{(k)}$ and $u_i^{(k)}$ satisfy conditions* 1 *and* 2 *of Definition 2.1.*

The following result shows that inverses of TP matrices and matrices opposite in sign to inverses of TP matrices are also ε-BD. Taking into account Proposition 2.2 and that when applying the inverse of A to its expression in (2.2) we have that $(E_i(x))^{-1} = E_i(-x)$ for all $i \leq n$ and for every real number x, the following result follows.

PROPOSITION 2.4. *A matrix A is ε-BD with $\varepsilon = (-1, \ldots, -1, 1)$ (respectively, $\varepsilon = (-1, \ldots, -1)$) if and only if A^{-1} (respectively, $-A^{-1}$) is TP.*

Given a matrix A that can be factorized as in (2.2), we say that *A has r nontrivial factors* if there are only r nonzero elements $l_i^{(k)}$ and $u_i^{(k)}$ in (2.2).

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The class of ε-BD matrices has some sign properties that allow us to know the sign of their elements and the sign of their minors from the corresponding signature, as the following two results show. We say that a minor has sign $+1$ (respectively, -1) if it is nonnegative (respectively, nonpositive). Observe that, with this definition, a zero minor can be considered with both signs: +1 or -1.

THEOREM 2.5. *Let A be an $n \times n$ ε-BD matrix. Then*

$$\mathit{sign}(\det A[\alpha|\beta]) = (\varepsilon_n)^k \prod_{i=1}^{k} \prod_{j=\min(\alpha_i,\beta_i)}^{\max(\alpha_i-1,\beta_i-1)} \varepsilon_j \tag{2.3}$$

for all $\alpha, \beta \in Q_{k,n}$ and all $k \leq n$.

Proof. By Proposition 2.3, A can be decomposed as in (2.2). We prove the result by induction on the number of nontrivial factors of A in (2.2). Let us assume that A has not nontrivial factors, i.e., A is a diagonal matrix with nonzero diagonal entries. Note that in this case, $\det A[\alpha|\beta]$ is a nonzero minor only if $\alpha = \beta$. Then it can be checked that (2.3) holds.

Now suppose that an ε-BD matrix with $r-1$ nontrivial factors satisfies (2.3) and let us prove that (2.3) holds for an ε-BD matrix, A, with r nontrivial factors. Without loss of generality, we can assume that $A = LB$, where L (the first factor of the decomposition (2.2) of A) is a lower elementary bidiagonal matrix with the entry $l \neq 0$ in a position (i_0, i_0-1) and $\text{sign}(l) = \varepsilon_{i_0-1}$ and $B = (b_{ij})_{1\leq i,j\leq n}$ satisfies (2.2) with $r-1$ nontrivial factors. The proof for the case $A = BU$, with U an upper elementary bidiagonal matrix, is analogous. Observe that $(A)_{ij} = b_{ij}$ if $i \neq i_0$ and $(A)_{i_0 j} = l b_{i_0-1,j} + b_{i_0 j}$ for all $1 \leq j \leq n$. Let $\alpha = (\alpha_1, \ldots, \alpha_k) \in Q_{k,n}$ be such that $\alpha_h = i_0$ for an $h \leq k$. We have that

$$\det A[\alpha|\beta] = l \det B[\alpha_1, \ldots, \alpha_{h-1}, i_0-1, \alpha_{h+1}, \ldots, \alpha_k|\beta] + \det B[\alpha|\beta]. \tag{2.4}$$

If we denote with $m := \det B[\alpha_1, \ldots, \alpha_{h-1}, i_0-1, \alpha_{h+1}, \ldots, \alpha_k|\beta]$, we have, by the induction hypotesis, that

$$\begin{aligned}\text{sign}(lm) = \varepsilon_{i_0-1} &\left((\varepsilon_n)^{h-1} \prod_{i=1}^{h-1} \prod_{j=\min(\alpha_i,\beta_i)}^{\max(\alpha_i-1,\beta_i-1)} \varepsilon_j \right) \cdot \\ &\left(\varepsilon_n \prod_{j=\min(i_0-1,\beta_h)}^{\max(i_0-2,\beta_h-1)} \varepsilon_j \right) \left((\varepsilon_n)^{k-h} \prod_{i=h+1}^{k} \prod_{j=\min(\alpha_i,\beta_i)}^{\max(\alpha_i-1,\beta_i-1)} \varepsilon_j \right).\end{aligned}$$

Observe that

$$\varepsilon_{i_0-1} \left(\prod_{j=\min(i_0-1,\beta_h)}^{\max(i_0-2,\beta_h-1)} \varepsilon_j \right) = \prod_{j=\min(i_0,\beta_h)}^{\max(i_0-1,\beta_h-1)} \varepsilon_j.$$

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A publication of the International Linear Algebra Society
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Then we have

$$\mathrm{sign}(lm) = (\varepsilon_n)^k \prod_{i=1}^{k} \prod_{j=\min(\alpha_i,\beta_i)}^{\max(\alpha_i-1,\beta_i-1)} \varepsilon_j = \mathrm{sign}(\det B[\alpha|\beta]).$$

Taking into account the previous formula and (2.4) we conclude that A satisfies (2.3) and so, the result holds. □

Applying the previous theorem with $\alpha, \beta \in Q_{1,n}$, we obtain the following result.

Corollary 2.6. *Let $A = (a_{ij})_{1\le i,j\le n}$ be an ε-BD matrix. Then*

$$sign(a_{ij}) = \varepsilon_n \prod_{k=\min(i,j)}^{\max(i-1,j-1)} \varepsilon_k \tag{2.5}$$

for all $1 \le i, j \le n$.

The following lemma extends to ε-BD matrices the well-known shadow's lemma for totally positive matices (see [5, Lemma A]): Given an n by n ε-BD matrix $A = (a_{ij})$ with a zero entry $a_{ij} = 0$, one of the following four regions of zero entries appears:

$$a_{kj} = 0 \quad \forall k, \tag{2.6}$$

$$a_{ik} = 0 \quad \forall k, \tag{2.7}$$

$$a_{kl} = 0 \quad \forall k \ge i, l \le j, \tag{2.8}$$

$$a_{kl} = 0 \quad \forall k \le i, l \ge j, \tag{2.9}$$

Lemma 2.7. *Let $A = (a_{ij})_{1\le i,j\le n}$ be an ε-BD matrix with $a_{ij} = 0$ for some i, j. Then one of the conditions (2.6), (2.7), (2.8), (2.9) holds.*

Proof. Suppose that $i < j$ (the case $i > j$ can be checked analogously) and let us assume that neither (2.6) nor (2.7) hold. Since (2.6) does not hold, $a_{tj} \ne 0$ for some $t \ne i$. Assume now that $t > i$. Then $\det A[i, t|j, l] = -a_{tj}a_{il}$ for all $l > j$. Consider the case $i < t < j < l$ (cases $i < j < t < l$ and $i < j < l < t$ are similar). By Corollary 2.6,

$$\begin{aligned}\mathrm{sign}(-a_{tj}a_{il}) &= -\left(\varepsilon_n \prod_{k=\min(t,j)}^{\max(t-1,j-1)} \varepsilon_k\right)\left(\varepsilon_n \prod_{k=\min(i,l)}^{\max(i-1,l-1)} \varepsilon_k\right)\\ &= -\prod_{k=t}^{j-1}\varepsilon_k \prod_{k=i}^{l-1}\varepsilon_k,\end{aligned}$$

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A publication of the International Linear Algebra Society
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However, by Theorem 2.5,

$$\begin{aligned}\operatorname{sign}(\det A[i,t|j,l]) &= (\varepsilon_n)^2 \prod_{k=\min(i,j)}^{\max(i-1,j-1)} \varepsilon_k \prod_{k=\min(t,l)}^{\max(t-1,l-1)} \varepsilon_k = \prod_{k=i}^{j-1} \varepsilon_k \prod_{k=t}^{l-1} \varepsilon_k \\ &= \prod_{k=i}^{l-1} \varepsilon_k \prod_{k=t}^{j-1} \varepsilon_k.\end{aligned}$$

Thus, $a_{il} = 0$ for all $l > j$.

Analogously, we deduce that $a_{rs} = 0$ for all $r < i$, $s \geq j$ and we have proved that (2.9) holds.

It can be proved by a similar reasoning that if $t < i$ then (2.8) holds. ∎

3. Backward stability of Gaussian elimination. In this section we are going to show that we can guarantee the backward stability without pivoting of a system $Ax = b$, where A is an ε-BD matrix and b is any vector.

It is well known that the backward error of Gaussian elimination depends on its growth factor. Several measures have been used for computing this factor. For instance, let us mention the classical growth factor introduced by Wilkinson, or the growth factor that appears in the following result and involves the triangular matrices L and U. See [11, chapter 9] and [3] for more information and comparisons of these and other growth factors.

THEOREM 3.1. *Let $A = (a_{ij})_{1\leq i,j\leq n}$ be an ε-BD matrix. Then $A = LU$, where L is lower triangular with unit diagonal and U is the upper triangular matrix obtained after the Gaussian elimination of A, and*

$$\frac{\| |L||U| \|_\infty}{\|A\|_\infty} = 1 \tag{3.1}$$

Proof. If we denote by $L := L^{(1)} \cdots L^{(n-1)}$ and $U := DU^{(n-1)} \cdots U^{(1)}$ the products of matrices of (2.1), we have the LU factorization of the statement. It is a consequence of the signs of the entries of L and U that

$$|A| = |LU| = |L||U|, \tag{3.2}$$

which in turn implies (3.1). ∎

Observe that the previous result shows that the growth factor of Gaussian elimination for ε-BD matrices is optimal. By formula (3.2) and [12, Theorem 2.2], no row exchanges is the optimal scaled partial pivoting strategy for any strictly monotone

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A publication of the International Linear Algebra Society
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norm for ε-BD matrices. Moreover, as we shall see at the end of this section, the fact that the last equality of formula (3.2) can be also guaranteed for the computed triangular factors $\hat{L}$, $\hat{U}$ implies backward stability of Gaussian elimination without row exchanges for ε-BD matrices. For this purpose, we need some additional results on the zero pattern of ε-BD matrices and the triangular factors.

The following auxiliary result shows that the leading principal minors of ε-BD matrices are nonzero.

LEMMA 3.2. *Let A be an ε-BD matrix. Then* $\det A[1,\dots,k] \neq 0$ *for all* $k \leq n$.

Proof. If we denote by $L := L^{(1)} \cdots L^{(n-1)}$ and $U := U^{(n-1)} \cdots L^{(1)}$ the products of matrices of (2.1), observe that we can factorize $A = LDU$ with L, U lower and upper triangular matrices, respectively, with unit diagonal and D a diagonal matrix with nonzero diagonal entries. Then is well known that

$$A[1,\dots,k] = L[1,\dots,k]D[1,\dots,k]U[1,\dots,k]$$

for all $k \leq n$. By the Cauchy-Binet identity (see [1, formula (1.23)]), we conclude that $\det A[1,\dots,k] = \det D[1,\dots,k] \neq 0$ for all $k \leq n$. $\square$

The following lemma is an extension of Lemma 1 of [9, p.94], valid for TP matrices, to the class of ε-BD matrices.

LEMMA 3.3. *Let $A = (a_{ij})_{1\leq i,j\leq n}$ be an ε-BD matrix such that*

$$\det A[1,\dots,p-1,q|1,\dots,p] = 0. \tag{3.3}$$

Then $a_{qk} = 0$ for all $1 \leq k \leq p$ and $q > p$.

Proof. By Lemma 3.2, we have that $\det A[1,\dots,k] \neq 0$ for all $k \leq n$.

Since $\det A[1,\dots,p-1] \neq 0$ and taking into account (3.3), we notice that the first $p-1$ rows of the submatrix $A[1,\dots,p-1,q|1,\dots,p]$ are linearly independent, and the q-th row is a linear combination of these $p-1$ rows, i.e.,

$$a_{qk} = \sum_{h=1}^{p-1} \lambda_h a_{hk}, \quad 1 \leq k \leq p.$$

Let us see now that $\lambda_h = 0$ for all $1 \leq h \leq p-1$. We have, for each $1 \leq h \leq p-1$,

$$\det A[1,\dots,h-1,h+1,\dots,p-1,q|1,\dots,p-1] = (-1)^{p-h-1}\lambda_h \det A[1,\dots,p-1]$$

and

$$\det A[1,\dots,h-1,h+1,\dots,p,q|1,\dots,p] = (-1)^{p-h}\lambda_h \det A[1,\dots,p].$$

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A publication of the International Linear Algebra Society
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By Theorem 2.5, the minors in the left hand sides of the previous formulas have the same sign $S := (\varepsilon_n)^p(\varepsilon_h \cdots \varepsilon_{q-1})$. So, since $S(-1)^{p-h-1}\lambda_h \geq 0$ and $S(-1)^{p-h}\lambda_h \geq 0$, we conclude that $\lambda_h = 0$ for all $1 \leq h \leq p-1$. □

The following result extends [4, Proposition 2] to ε-BD matrices and it will be useful to prove the announced backward stability result.

Proposition 3.4. *Let $A = (a_{ij})_{1\leq i,j\leq n}$ be an ε-BD matrix and consider the triangular factorization $A = LU$, where $L = (l_{ij})_{1\leq i,j\leq n}$ is a lower triangular matrix with unit diagonal and $U = (u_{ij})_{1\leq i,j\leq n}$ is a nonsingular upper triangular matrix. Then, for $i > j$, $\varepsilon_j \cdots \varepsilon_{i-1} l_{ij} \geq 0$ with equality if and only if $a_{qp} = 0$ for $p \geq i$, $q \leq j$. Also, for $i < j$, $\varepsilon_i \cdots \varepsilon_{j-1} u_{ij} \geq 0$ with equality if and only if $a_{qp} = 0$ for $p \leq i$, $q \geq j$.*

Proof. The LU factorization of the statement was proved in the proof of the Theorem 3.1. If $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$, then we have that L is ε_L-BD, where $\varepsilon_L = (\varepsilon_1, \ldots, \varepsilon_{n-1}, 1)$. Observe that, for $i > j$,

$$a_{ik} = l_{i1}u_{1k} + \cdots + l_{ik}u_{kk}, \tag{3.4}$$

for $1 \leq k \leq j$. Let us now see that $a_{i1} = \cdots = a_{ij} = 0$ if and only if $l_{ij} = 0$ for $i > j$. If $a_{i1} = \cdots = a_{ij} = 0$ then, by (3.4) and since $u_{hh} \neq 0$ for all $h \leq n$, we can prove recursively that $l_{i1} = \cdots = l_{ij} = 0$. If $l_{ij} = 0$, let us consider $k < j$. Note that $\det L[j, i|k, j] = l_{jk}l_{ij} - l_{jj}l_{ik} = -l_{jj}l_{ik} = -l_{ik}$. By Theorem 2.5, the minor has sign

$$\prod_{l=k}^{j-1} \varepsilon_l \prod_{l=j}^{i-1} \varepsilon_l = \prod_{l=k}^{i-1} \varepsilon_l. \tag{3.5}$$

But by Corollary 2.6, $-l_{ik}$ has sign

$$-\prod_{l=k}^{i-1} \varepsilon_l. \tag{3.6}$$

So, (3.5) and (3.6) imply that $l_{ik} = 0$ for all $k \leq j$. Then by (3.4), $a_{i1} = \cdots = a_{ij} = 0$. Since L is ε_L-BD, we have by Corollary 2.6 that $(\varepsilon_j \cdots \varepsilon_{i-1})l_{ij} \geq 0$ with equality if and only if $a_{i1} = \cdots = a_{ij} = 0$. Let us assume that the equality holds. Then since A is nonsingular, there exists an index $r > j$ such that $a_{ir} \neq 0$. Let us consider the minor $\det A[i, p|q, r] = -a_{ir}a_{pq}$ for $p > i$ and $q \leq j$. Assume that $q < r < i < p$ (cases $q < i < r < p$ and $q < i < p < r$ are similar). Then, by Theorem 2.5, the minor has sign

$$(\varepsilon_n)^2 \prod_{k=\min(i,q)}^{\max(i-1,q-1)} \varepsilon_k \prod_{k=\min(p,r)}^{\max(p-1,r-1)} \varepsilon_k = \prod_{k=q}^{i-1} \varepsilon_k \prod_{k=r}^{p-1} \varepsilon_k = \prod_{k=q}^{p-1} \varepsilon_k \prod_{k=r}^{i-1} \varepsilon_k. \tag{3.7}$$

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However, by Corollary 2.6, $-a_{ir}a_{pq}$ has sign

$$-\left(\varepsilon_n \prod_{k=\min(i,r)}^{\max(i-1,r-1)} \varepsilon_k\right)\left(\varepsilon_n \prod_{k=\min(p,q)}^{\max(p-1,q-1)} \varepsilon_k\right) = -\prod_{k=r}^{i-1}\varepsilon_k \prod_{k=q}^{p-1}\varepsilon_k,$$

which coincides with the opposite of (3.7).

So, we have that $a_{pq} = 0$ for $p \geq i$ and $q \leq j$. Analogously, we can prove the result for U. ☐

Let us now see that, when working in finite precision arithmetic with sufficiently small unit roundoff u, the computed triangular matrices $\hat{L}$, $\hat{U}$ also satisfy the second equality of (3.2), and so we can derive a backward stability result with a very small backward bound.

Theorem 3.5. *Let A be an ε-BD matrix. Let us assume that LU is the triangular factorization of A of Theorem 3.1 and that we perform Gaussian elimination without row exchanges in finite precision arithmetic producing the computed factors $\hat{L}, \hat{U}$ and the computed solution $\hat{x}$ to $Ax = b$. Let E and ΔA be matrices satisfying $\hat{L}\hat{U} = A+E$ and $(A + \Delta A)\hat{x} = b$. Then the following properties hold for a sufficiently small unit roundoff u:*

(i) $|\hat{L}||\hat{U}| = |\hat{L}\hat{U}|$
(ii) $|E| \leq \frac{\gamma_n}{1-\gamma_n}|A|, \quad |\Delta A| \leq \frac{\gamma_{3n}}{1-\gamma_n}|A|$

where $\gamma_n := \frac{nu}{1-nu}$, *assuming that* $nu < 1$.

Proof. (i) If $l_{ij} = 0$ for some $i > j$, then we have seen in the previous result that $a_{pq} = 0$, for $p \geq i$, $q \leq j$ and we must have $\hat{l}_{pq} = 0$ for $p \geq i$, $q \leq j$. Thus, L and $\hat{L}$ can only differ in the nonzero entries of L. Analogously, U and $\hat{U}$ only differ in the nonzero entries of U. It is well known that $\hat{L} \to L$ and $\hat{U} \to U$ as $u \to 0$. So, for a sufficiently small unit roundoff u, all nonzero entries of $\hat{L}$ and $\hat{U}$ have the same sign that the entries of L and U respectively. Finally, taking into account formula (3.2), we can conclude that

$$|\hat{L}\hat{U}| = |\hat{L}||\hat{U}|,$$

for a sufficiently small u.

(ii) By [11, formula (9.6)] we know that

$$|E| \leq \gamma_n |\hat{L}||\hat{U}|. \tag{3.8}$$

Besides $|\hat{L}||\hat{U}| = |\hat{L}\hat{U}|$ as we have seen in (i). So, we have that

$$|\hat{L}||\hat{U}| = |\hat{L}\hat{U}| = |A+E| \leq |A| + \gamma_n |\hat{L}||\hat{U}|$$

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and this implies that

$$|\hat{L}||\hat{U}| \leq \frac{1}{1-\gamma_n}|A|. \tag{3.9}$$

Taking into account this and (3.8), we conclude that $|E| \leq \frac{\gamma_n}{1-\gamma_n}|A|$. The inequality $|\Delta A| \leq \frac{\gamma_{3n}}{1-\gamma_n}|A|$ is a consequence of (3.9) and [11, Theorem 9.4], where it is shown that $|\Delta A| \leq \gamma_{3n}|\hat{L}||\hat{U}|$. ☐

4. Numerical experiments. In this section we present numerical experiments that illustrate the accuracy of Gaussian elimination without row exanges (GE) and that it is higher than that of Gaussian elimination with partial pivoting for solving linear systems with ε-BD matrices. We compute the exact solution x of the linear system $Ax = b$ by using the command `LinearSolve` of Mathematica and use it for comparing the accuracy of the results obtained in MATLAB by means of an algorithm of Gaussian elimination without row exchanges and the command $A\backslash b$ of MATLAB (which uses partial pivoting).

We compute the relative error of a solution x of the linear system $Ax = b$ by means of the formula:

$$err = \frac{\|x - \hat{x}\|_2}{\|x\|_2},$$

where $\hat{x}$ is the computed solution.

Example 4.1. *Let A be the following ε-BD matrix*

$$A = \begin{bmatrix} 8 & 16 & -32 & -32 & 64 & 192 & -192 & -384 \\ 16 & 38 & -88 & -124 & 284 & 888 & -960 & -1992 \\ -32 & -82 & 201 & 312 & -735 & -2316 & 2538 & 5298 \\ -96 & -258 & 655 & 1079 & -2610 & -8380 & 9620 & 20844 \\ 96 & 282 & -761 & -1401 & 3642 & 12472 & -16584 & -39972 \\ 0 & 0 & -4 & -72 & 444 & 2458 & -5920 & -18556 \\ 0 & 0 & 20 & 360 & -2220 & -12300 & 29684 & 93332 \\ 0 & 0 & 20 & 360 & -2220 & -12342 & 30052 & 95855 \end{bmatrix}.$$

And let b_1, b_2 and b_3 be the vectors

$$b_1^T = [6, 5, -4, 3, -1, -1, -3, 5]^T,$$
$$b_2^T = [5, 5, 7, 5, 4, 3, 0, -1]^T,$$
$$b_3^T = [9, -2, 3, -5, -4, 2, -3, -1]^T.$$

The relative errors obtained when using no row exchanges strategy (column GE of Table 4.1) and partial pivoting (column $A\backslash b_i$ of Table 4.1) for solving the systems $Ax = b_i$ for $i = 1, 2, 3$ are reported in Table 4.1.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 25, pp. 60-71, August 2012

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b_i	GE	$A\backslash b_i$
b_1	6.3184e-015	7.6074e-010
b_2	1.1155e-014	7.4450e-010
b_3	6.1846e-016	7.6540e-010

Table 4.1
Relative errors

n	GE	$A_i\backslash b_i$
8	3.3616e-015	9.1664e-010
10	2.5172e-014	4.3218e-007
16	2.9305e-012	2.570883242

Table 4.2
Average of relative errors

Let us observe that, while the order of the relative errors using partial pivoting is about 10^{-10}*, the order of the relative errors using no row exchanges oscillates between* 10^{-14} *and* 10^{-16}.

Example 4.2. *We have created 100 random systems ($A_i x = b_i$) where the $n \times n$ matrices A_i are ε-BD:*

- *40 matrices and vectors for $n = 8$.*
- *40 matrices and vectors for $n = 10$.*
- *20 matrices and vectors for $n = 16$.*

Matrices have been created in MATLAB by multiplying elementary bidiagonal factors as in (2.2). The entry $l_i^{(k)}$ (or $u_i^{(k)}$) of each elementary bidiagonal matrix $E_{i+1}(l_i^{(k)})$ (or $E_{i+1}^T(u_i^{(k)})$) is a random number that satisties condition 2 of Definition 2.1.

The average of the relative errors obtained when using no row exchanges strategy (column GE of Table 4.2) and partial pivoting (column $A_i\backslash b_i$ of Table 4.2) for solving the systems $A_i x = b_i$ are reported in Table 4.2.

Note that the order of the relative errors using partial pivoting increases very fast with the size of the systems. However, the increase of the order of the relative errors using no row exchanges is smoother. Besides, no rows exchanges strategy gives much better relative errors than partial pivoting strategy (for instance, we have errors of order 10^{-14} compared to errors of order 10^{-7} for systems of size $n = 10$).

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Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 25, pp. 60-71, August 2012

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