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OPTIMAL GERŠGORIN-STYLE ESTIMATION OF THE LARGEST SINGULAR VALUE*

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Abstract. In estimating the largest singular value of an n-by-n complex matrix, a prior result [2] shows that it is attained at one of $n(n-1)$ sparse matrices in the equiradial class. Here, circumstances are identified under which the set of possible optimizers can be further narrowed. The results used to show this may be of independent interest.

Key words. Equiradial class; Geršgorin data; Singular values; Spectral norm.

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1. Introduction. Let $M_n(\mathbb{C})$ be the set of all n-by-n complex matrices. For a given matrix $A = (a_{ij}) \in M_n(\mathbb{C})$, we set $P_k(A) = \sum_{j \neq k} |a_{k,j}|$, $k = 1, \ldots, n$ and, for $X = (x_{i,j}) \in M_n(\mathcal{C})$, $D(X) = \operatorname{diag}(x_{1,1}, \ldots, x_{n,n})$, define the class $\Lambda(A)$ of matrices *equiradial* with A by

$$\Lambda(A) = \{B \in M_n(\mathbb{C}) : |D(B)| = |D(A)| \text{ and } P_k(B) = P_k(A), k = 1, \ldots, n\}.$$

In particular, we will focus on a subset of $\Lambda(A)$ consisting of $n(n-1)$ nonnegative matrices $A^{(s,k)} = (a_{i,j}^{(s,k)})$, with $s, k \in \{1, \ldots, n\}$ and $s \neq k$, such that

$$a_{i,j}^{(s,k)} = \begin{cases} |a_{i,i}| & \text{for } i = j, \\ P_i(A) & \text{for } i \neq j \text{ and } j = s, \\ P_s(A) & \text{for } (i,j) = (s,k), \\ 0 & \text{otherwise.} \end{cases}$$

EXAMPLE 1.1. *For a 3-by-3 complex matrix $A = (a_{i,j})$, we have*

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$$A^{(1,2)} = \begin{pmatrix} |a_{1,1}| & P_1(A) & 0 \\ P_2(A) & |a_{2,2}| & 0 \\ P_3(A) & 0 & |a_{3,3}| \end{pmatrix}, \quad A^{(1,3)} = \begin{pmatrix} |a_{1,1}| & 0 & P_1(A) \\ P_2(A) & |a_{2,2}| & 0 \\ P_3(A) & 0 & |a_{3,3}| \end{pmatrix},$$

$$A^{(2,1)} = \begin{pmatrix} |a_{1,1}| & P_1(A) & 0 \\ P_2(A) & |a_{2,2}| & 0 \\ 0 & P_3(A) & |a_{3,3}| \end{pmatrix}, \quad A^{(2,3)} = \begin{pmatrix} |a_{1,1}| & P_1(A) & 0 \\ 0 & |a_{2,2}| & P_2(A) \\ 0 & P_3(A) & |a_{3,3}| \end{pmatrix},$$

$$A^{(3,1)} = \begin{pmatrix} |a_{1,1}| & 0 & P_1(A) \\ 0 & |a_{2,2}| & P_2(A) \\ P_3(A) & 0 & |a_{3,3}| \end{pmatrix}, \quad A^{(3,2)} = \begin{pmatrix} |a_{1,1}| & 0 & P_1(A) \\ 0 & |a_{2,2}| & P_2(A) \\ 0 & P_3(A) & |a_{3,3}| \end{pmatrix}.$$

REMARK 1.2. *For our purposes, throughout this paper, it will be assumed that A has at least one nonzero off-diagonal entry in each row and that all its diagonal entries are nonzero. Then it is easy to observe that any matrix $A^{(s,k)}$ ($s, k \in \{1, \dots, n\}$ and $s \neq k$) has exactly $2n$ nonzero entries. Moreover, for a given (s, k) we have $n(n-1)$ matrices of the type "(s, k)" equiradial with $A^{(s,k)}$.*

The symbol $\|\cdot\|_2$ will be used to denote either the spectral norm of a matrix or the Euclidean norm of a vector. By a unit vector we will mean a vector x such that $\|x\|_2 = 1$. Columns of $A^{(s,k)}$ will be denoted by $A_i^{(s,k)}, i = 1, \dots, n$.

Matrices $A^{(s,k)}$ play an important role in estimation of the largest singular value among matrices equiradial with A (or, as the largest singular value of a matrix B is equal to the spectral norm of B, in estimation of the spectral norm of matrices equiradial with A). The spectral norm plays a crucial role in numerical linear algebra and, as it is difficult to compute, upper bounds for the largest singular value in terms of possible simple functions of the entries of a matrix are of interest. By simple functions we mean those which use Geršgorin data related to a matrix, i.e., diagonal entries and sums of the moduli of off-diagonal ones. Observe that for a given $A \in M_n(\mathbb{C})$ all matrices in $\Lambda(A)$ share this type of information and therefore, from this point of view, they are identified. In prior work [2] the question of the largest singular value among matrices in $\Lambda(A)$ was considered. Using Johnson's "concentration principle", it was shown (Theorem 3) that the maximum is attained at one of the matrices $A^{(s,k)}$. Thus, the question of estimating the largest singular value of A, based upon the information defining $\Lambda(A)$, is reduced to a finite number $n(n-1)$ of candidates. In general, it can be very difficult to distinguish among these candidates. Our purpose here is to further reduce the number of candidates, under certain circumstances that depend upon properties of the matrices $A^{(s,k)}$. Obviously, for all results of the paper, which are row oriented, there is also a column version.

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2. Results. Our first result characterizes a positive unit vector that realizes the spectral norm of a matrix $A^{(s,k)}$.

THEOREM 2.1. *Assume that there is a column $A_t^{(s,k)}$, $1 \le t \le n$, of $A^{(s,k)}$ such that*

$$\left\|A_t^{(s,k)}\right\|_2^2 > \sum_{i\neq t}\left\|A_i^{(s,k)}\right\|_2^2.$$

Then there is a positive vector $\tilde{x} = (\tilde{x}_1,\ldots,\tilde{x}_n)^T$ such that $\|\tilde{x}\|_2 = 1$, $\left\|A^{(s,k)}\right\|_2 = \left\|A^{(s,k)}\tilde{x}\right\|_2$ and

$$\tilde{x}_t > \max_{k\neq t}\{\tilde{x}_k\}.$$

Proof. We start by an easy observation that $(A^{(s,k)})^T A^{(s,k)}$ is irreducible. So, a unit vector $\tilde{x}$ that realizes the spectral norm of $A^{(s,k)}$ may be chosen positive by Perron-Frobenius theory [1]. Now, without loss of generality, we assume that $n = 3$ and $A^{(s,k)} = A^{(1,3)}$ – the argument used for $A^{(1,3)}$ can be easy applied to the general case. So, we have

$$\begin{aligned}
A_1^{(1,3)} &= (|a_{1,1}|, P_2(A), P_3(A))^T)\\
A_2^{(1,3)} &= (0, |a_{2,2}|, 0)^T, \text{ and}\\
A_3^{(1,3)} &= (P_1(A), 0, |a_{3,3}|)^T,
\end{aligned}$$

where $|a_{1,1}|$, $P_2(A)$, $P_3(A)$, $|a_{2,2}|$, $P_1(A)$, and $|a_{3,3}|$ are positive.

For a real unit vector $x = (x_1, x_2, x_3)^T$ we set

$$\begin{aligned}
F_1((x_1,x_2,x_3)) = {} & \left\|A_1^{(1,3)}\right\|_2^2 x_1^2 + \left\|A_2^{(1,3)}\right\|_2^2 x_2^2 + \left\|A_3^{(1,3)}\right\|_2^2 x_3^2\\
& +2P_2(A)|a_{2,2}|x_1x_2 + 2(|a_{1,1}|P_1(A) + P_3(A)|a_{3,3}|)x_1x_3.
\end{aligned}$$

Then it is easy to see that $F_1((\tilde{x}_1,\tilde{x}_2,\tilde{x}_3)) = \left\|A^{(1,3)}\right\|_2^2$.

So, $\tilde{x}$ maximizes $F_1((x_1,x_2,x_3))$ over all real unit vectors $x = (x_1,x_2,x_3)^T$ and therefore, as F_1 is differientiable, we have

$$\frac{\partial F_1((\tilde{x}_1, \sqrt{1-\tilde{x}_1^2-\tilde{x}_3^2}, \sqrt{1-\tilde{x}_1^2-\tilde{x}_2^2}))}{\partial x_1} = \frac{\partial F_1((\sqrt{1-\tilde{x}_2^2-\tilde{x}_3^2}, \tilde{x}_2, \sqrt{1-\tilde{x}_1^2-\tilde{x}_2^2}))}{\partial x_2}$$

$$= \frac{\partial F_1((\sqrt{1-\tilde{x}_2^2-\tilde{x}_3^2}, \sqrt{1-\tilde{x}_1^2-\tilde{x}_3^2}, \tilde{x}_3))}{\partial x_3} = 0.$$

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We will consider three cases.

Case 1: $A_t^{(1,3)} = A_1^{(1,3)}$. Then from the condition

$$\frac{\partial F_1((\tilde{x}_1, \sqrt{1-\tilde{x}_1^2-\tilde{x}_3^2}, \sqrt{1-\tilde{x}_1^2-\tilde{x}_2^2}))}{\partial x_1} = 0, \tag{2.1}$$

since

$$\left\|A_1^{(1,3)}\right\|_2^2 > \left\|A_2^{(1,3)}\right\|_2^2 + \left\|A_3^{(1,3)}\right\|_2^2,$$

we get that either

$$\tilde{x}_1 > \tilde{x}_2 \tag{2.2}$$

or

$$\tilde{x}_1 > \tilde{x}_3. \tag{2.3}$$

From the condition

$$\frac{\partial F_1((\sqrt{1-\tilde{x}_2^2-\tilde{x}_3^2}, \tilde{x}_2, \sqrt{1-\tilde{x}_1^2-\tilde{x}_2^2}))}{\partial x_2} = 0, \tag{2.4}$$

since

$$\left\|A_2^{(1,3)}\right\|_2^2 < \left\|A_1^{(1,3)}\right\|_2^2 + \left\|A_3^{(1,3)}\right\|_2^2,$$

we get

$$\tilde{x}_1 > \tilde{x}_2. \tag{2.5}$$

Finally, from the condition

$$\frac{\partial F_1((\sqrt{1-\tilde{x}_2^2-\tilde{x}_3^2}, \sqrt{1-\tilde{x}_1^2-\tilde{x}_3^2}, \tilde{x}_3))}{\partial x_3} = 0. \tag{2.6}$$

since

$$\left\|A_3^{(1,3)}\right\|_2^2 < \left\|A_1^{(1,3)}\right\|_2^2 + \left\|A_2^{(1,3)}\right\|_2^2,$$

we get

$$\tilde{x}_1 > \tilde{x}_3. \tag{2.7}$$

So, (2.5) and (2.7) yields

$$\tilde{x}_1 > \max\{\tilde{x}_2, \tilde{x}_3\}$$

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completing Case 1.

Case 2: $A_t^{(1,3)} = A_2^{(1,3)}$.

Then, from the condition (2.1), since

$$\left\|A_1^{(1,3)}\right\|_2^2 < \left\|A_2^{(1,3)}\right\|_2^2 + \left\|A_3^{(1,3)}\right\|_2^2,$$

we get that either

$$\tilde{x}_2 > \tilde{x}_1 \tag{2.8}$$

or

$$\tilde{x}_3 > \tilde{x}_1. \tag{2.9}$$

From the condition (2.4), as

$$\left\|A_2^{(1,3)}\right\|_2^2 > \left\|A_1^{(1,3)}\right\|_2^2 + \left\|A_3^{(1,3)}\right\|_2^2,$$

we are not able to derive any conclusion on properties of $\tilde{x}$. Finally, from the condition (2.6), since

$$\left\|A_3^{(1,3)}\right\|_2^2 < \left\|A_1^{(1,3)}\right\|_2^2 + \left\|A_2^{(1,3)}\right\|_2^2,$$

we get that

$$\tilde{x}_1 > \tilde{x}_3. \tag{2.10}$$

So, from (2.8), (2.9) and (2.10) we get that $\tilde{x}_2 > \tilde{x}_1 > \tilde{x}_3$ which concludes Case 2.

Case 3: $A_t^{(1,3)} = A_3^{(1,3)}$. Then, from the condition (2.1), as

$$\left\|A_1^{(1,3)}\right\|_2^2 < \left\|A_2^{(1,3)}\right\|_2^2 + \left\|A_3^{(1,3)}\right\|_2^2,$$

we get that either

$$\tilde{x}_2 > \tilde{x}_1 \tag{2.11}$$

or

$$\tilde{x}_3 > \tilde{x}_1. \tag{2.12}$$

From the condition (2.4), as

$$\left\|A_2^{(1,3)}\right\|_2^2 < \left\|A_1^{(1,3)}\right\|_2^2 + \left\|A_3^{(1,3)}\right\|_2^2,$$

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we get

$$\tilde{x}_1 > \tilde{x}_2. \tag{2.13}$$

So, from (2.11), (2.12) and (2.13) we get that $\tilde{x}_3 > \tilde{x}_1 > \tilde{x}_2$ which completes Case 3 and the proof of the lemma. ◻

Remark 2.2. *Observe that in Case 3, since*

$$\left\| A_3^{(1,3)} \right\|_2^2 > \left\| A_1^{(1,3)} \right\|_2^2 + \left\| A_2^{(1,3)} \right\|_2^2,$$

we are not able to derive any conclusion on properties of $\tilde{x}$ from the condition (2.6).

Corollary 2.3. *Let the rth diagonal entry, $1 \le r \le n$, of $A^{(s,k)}$ be such that its square is greater than the sum of the squares of all remaning entries of the matrix. Then for any matrix B of the type (s,k) that is equiradial with $A^{(s,k)}$ there is a positive vector $\tilde{z}(B) = (\tilde{z}(B)_1, \ldots, \tilde{z}(B)_n)^T$ such that $\|\tilde{z}(B)\|_2 = 1$, $\|B\|_2 = \|B\tilde{z}(B)\|_2$ and*

$$\tilde{z}(B)_r > \max_{k \neq r} \{\tilde{z}(B)_k\}.$$

Proof. The assertion follows by observing that rth column of any matrix B of the type (s,k) that is equiradial with $A^{(s,k)}$ sartisfies the hypothesis of Theorem 2.1. ◻

Theorem 2.4. *Let the rth, $r \neq k$, column of $A^{(k,s)}$ be such that*

$$\left\| A_r^{(k,s)} \right\|_2^2 > \sum_{i \neq r} \left\| A_i^{(k,s)} \right\|_2^2. \tag{2.14}$$

If $r = s$,then

$$\left\| A^{(s,k)} \right\|_2 > \left\| A^{(k,s)} \right\|_2, \tag{2.15}$$

and if $r \neq s$ then

$$\max_{X \in \Lambda(A)} \sigma_1(X) = \max_{l \neq r} \sigma_1(A^{(r,l)}). \tag{2.16}$$

Proof. The proof will be split into two parts.

Part 1: $r = s$.
Without loss of generality, we my assume that $(k,s) = (n,1)$. Then from (2.14) and the definition of $A^{(1,n)}$ and $A^{(n,1)}$ we get

$$\left\| A_1^{(n,1)} \right\|_2^2 > \sum_{i \neq 1} \left\| A_i^{(n,1)} \right\|_2^2.$$

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and

$$\left\|A_1^{(1,n)}\right\|_2^2 > \sum_{i\neq 1}\left\|A_i^{(1,n)}\right\|_2^2.$$

So, by Theorem 2.1, there are positive unit vectors $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n)^T$ and $\tilde{y} = (\tilde{y}_1, \ldots, \tilde{y}_n)^T$ such that $\|A^{(1,n)}\|_2 = \|A^{(1,n)}\tilde{x}\|_2$, $\tilde{x}_1 > \max_{k\neq 1}\{\tilde{x}_k\}$ and

$$\left\|A^{(n,1)}\right\|_2 = \left\|A^{(n,1)}\tilde{y}\right\|_2 \tag{2.17}$$

and

$$\tilde{y}_1 > \max_{k\neq 1}\{\tilde{y}_k\}. \tag{2.18}$$

Then we get

$$\left\|A^{(1,n)}\right\|_2^2 = \left\|A^{(1,n)}\tilde{x}\right\|_2^2 \geq \left\|A^{(1,n)}\tilde{y}\right\|_2^2 = \tag{2.19}$$

$$\sum_{i=1}^{n}\left\|A_i^{(1,n)}\right\|_2^2\tilde{y}_i^2 + 2\sum_{i=2}^{n-1} a_{i,1}^{(1,n)}a_{i,i}^{(1,n)}\tilde{y}_1\tilde{y}_i + 2(a_{1,1}^{(1,n)}a_{1,n}^{(1,n)} + a_{n,1}^{(1,n)}a_{n,n}^{(1,n)})\tilde{y}_1\tilde{y}_n.$$

So, by (2.18) and the definition of $A^{(n,1)}$, (2.19) becomes

$$\left\|A^{(1,n)}\right\|_2^2 \geq \left\|A^{(1,n)}\tilde{y}\right\|_2^2 > \sum_{i=1}^{n}\left\|A_i^{(n,1)}\right\|_2^2\tilde{y}_i^2 +$$

$$+2\sum_{i=2}^{n-1} a_{i,n}^{(n,1)}a_{i,i}^{(n,1)}\tilde{y}_n\tilde{y}_i + 2(a_{1,1}^{(n,1)}a_{1,n}^{(n,1)} + a_{n,1}^{(n,1)}a_{n,n}^{(n,1)})\tilde{y}_1\tilde{y}_n = \left\|A^{(n,1)}\tilde{y}\right\|_2^2,$$

from which, by (2.17), we get

$$\left\|A^{(s,k)}\right\|_2^2 > \left\|A^{(k,s)}\right\|_2^2$$

and (2.15) follows.
Part 2: $r \neq s$.

From the definition of $A^{(k,s)}$, as $r \neq k$, we have

$$\left\|A_r^{(k,s)}\right\|_2^2 = (a_{r,r}^{(k,s)})^2 > \sum_{i\neq r}\left\|A_i^{(k,s)}\right\|_2^2 = \sum_{i=1}^{n}\sum_{j\neq r}\left(a_{i,j}^{(k,s)}\right)^2. \tag{2.20}$$

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Without loss of generality, we may assume that $r = 1$, $s = n$ and $l = 2$. Then, by (2.20) and Corollary 2.3, there is a positive unit vector $\tilde{z}(A^{(1,n)})$, with components $(\tilde{z}(A^{(1,n)})_1, \ldots, \tilde{z}(A^{(1,n)})_n)^T$ and such that $\left\|A^{(1,n)}\right\|_2 = \left\|A^{(1,n)}\tilde{z}(A^{(1,n)})\right\|_2$ and

$$\tilde{z}(A^{(1,n)})_1 > \max_{k\neq 1}\{\tilde{z}(A^{(1,n)})_k\}.$$

Using again (2.20), by Corollary 2.3, we get

$$\left\|A^{(1,n)}\right\|_2^2 = \left\|A^{(1,n)}\tilde{z}(A^{(1,n)})\right\|_2^2 \geq \left\|A^{(1,n)}\tilde{z}(A^{(n,2)})\right\|_2^2, \tag{2.21}$$

where $\tilde{z}(A^{(n,2)})$ is a positive unit vector such that

$$\left\|A^{(n,2)}\right\|_2^2 = \left\|A^{(n,2)}\tilde{z}(A^{(n,2)})\right\|_2^2.$$

Applying the argument from the respective portion of the proof of Part 1, the inequality (2.21) becomes

$$\left\|A^{(1,n)}\right\|_2^2 > \left\|A^{(2,n)}\right\|_2^2.$$

So, (2.16) follows from Theorem 3 from [2] and by applying the argument from Part 2 to any $l \neq r$. $\square$

The application of Theorem 2.4 to $A^{(k,s)}$ when $r \neq s$ (and $r \neq k$) requires that the rth diagonal entry of the matrix satisfies (2.20). However, the application of Theorem 2.4 when $r = s$ requires a weaker condition: the square of a diagonal entry a_{ss}^2 plus the square of $P_k(A)$ (for $k \neq s$) is greater than the sum of the squares of the remaining diagonal entries a_{pp}^2 $(p \neq s)$ plus the sum of the squares of the remaining $P_q(A)$ (for $q \neq k$). Below we present two examples that illustrate these comments (the first one for $(k, s) = (3, 1)$, $(k, s) = (2, 1)$ and the second one for $(k, s) = (1, 2)$ and $(k, s) = (1, 3)$). In both examples there is no any diagonal entry of the matrix satisfyng (2.20).

Example 2.5. *Let*

$$A = \begin{pmatrix} 5 & -1 & 2 \\ -2 & 1 & 2 \\ 2 & -1 & 2 \end{pmatrix}.$$

So,

$$A^{(3,1)} = \begin{pmatrix} 5 & 0 & 3 \\ 0 & 1 & 4 \\ 3 & 0 & 2 \end{pmatrix} \text{ and } A^{(1,3)} = \begin{pmatrix} 5 & 0 & 3 \\ 4 & 1 & 0 \\ 3 & 0 & 2 \end{pmatrix}.$$

Then it is easy to see that

$$\left\|A_1^{(3,1)}\right\|_2^2 > \left\|A_2^{(3,1)}\right\|_2^2 + \left\|A_3^{(3,1)}\right\|_2^2,$$

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and therefore, by Theorem 2.4,

$$\left\|A^{(1,3)}\right\|_2 > \left\|A^{(3,1)}\right\|_2 .$$

Indeed, a calculation yields $\left\|A^{(1,3)}\right\|_2 = \sigma_1(A^{(1,3)}) = 7.7274$ *and finally* $\left\|A^{(3,1)}\right\|_2 = \sigma_1(A^{(3,1)}) = 7.2653$.
Analogously,

$$A^{(2,1)} = \begin{pmatrix} 5 & 3 & 0 \\ 4 & 1 & 0 \\ 0 & 3 & 2 \end{pmatrix} \text{ and } A^{(1,2)} = \begin{pmatrix} 5 & 3 & 0 \\ 4 & 1 & 0 \\ 3 & 0 & 2 \end{pmatrix}.$$

Then it is easy to see that

$$\left\|A_1^{(2,1)}\right\|_2^2 > \left\|A_2^{(2,1)}\right\|_2^2 + \left\|A_3^{(2,1)}\right\|_2^2,$$

and therefore, by Theorem 2.4,

$$\left\|A^{(1,2)}\right\|_2 > \left\|A^{(2,1)}\right\|_2 .$$

Indeed, a calculation yields $\left\|A^{(1,2)}\right\|_2 = \sigma_1(A^{(1,2)}) = 7.6263$ *and finally* $\left\|A^{(2,1)}\right\|_2 = \sigma_1(A^{(2,1)}) = 7.2210$.

Example 2.6. *Let*

$$A = \begin{pmatrix} -2 & 6 & -3 \\ 1 & 4 & 2 \\ -2 & -2 & 5 \end{pmatrix}.$$

So,

$$A^{(1,2)} = \begin{pmatrix} 2 & 9 & 0 \\ 3 & 4 & 0 \\ 4 & 0 & 5 \end{pmatrix} \text{ and } A^{(2,1)} = \begin{pmatrix} 2 & 9 & 0 \\ 3 & 4 & 0 \\ 0 & 4 & 5 \end{pmatrix}.$$

Then it is easy to see that

$$\left\|A_2^{(1,2)}\right\|_2^2 > \left\|A_1^{(1,2)}\right\|_2^2 + \left\|A_3^{(1,2)}\right\|_2^2,$$

and therefore, by Theorem 2.4,

$$\left\|A^{(2,1)}\right\|_2 > \left\|A^{(1,2)}\right\|_2 .$$

Indeed, a calculation yields $\left\|A^{(2,1)}\right\|_2 = \sigma_1(A^{(2,1)}) = 11.1818$ *and finally* $\left\|A^{(1,2)}\right\|_2 = \sigma_1(A^{(1,2)}) = 10.4387$.
Analogously,

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$$A^{(1,3)} = \begin{pmatrix} 2 & 0 & 9 \\ 3 & 4 & 0 \\ 4 & 0 & 5 \end{pmatrix} \text{ and } A^{(3,1)} = \begin{pmatrix} 2 & 0 & 9 \\ 0 & 4 & 3 \\ 4 & 0 & 5 \end{pmatrix}.$$

Then it is easy to see that

$$\left\| A_3^{(1,3)} \right\|_2^2 > \left\| A_1^{(1,3)} \right\|_2^2 + \left\| A_2^{(1,3)} \right\|_2^2,$$

and therefore, by Theorem 2.4,

$$\left\| A^{(3,1)} \right\|_2 > \left\| A^{(1,3)} \right\|_2.$$

A calculation yields $\left\| A^{(3,1)} \right\|_2 = \sigma_1(A^{(3,1)}) = 11.3781$ *and* $\left\| A^{(1,3)} \right\|_2 = \sigma_1(A^{(1,3)}) = 11.036$. *In fact,* $\max_{X \in \Lambda(A)} \sigma_1(X) = \sigma_1(A^{(3,1)}) = 11.3781$ *and so* $A^{(3,1)}$ *is a* σ_1*-maximizer in this case. Indeed, following Theorem 3 from [2] for remaining candidates for a* σ_1*-maximizer in* $\Lambda(A)$*, i.e., for matrices*

$$A^{(2,3)} = \begin{pmatrix} 2 & 9 & 0 \\ 0 & 4 & 3 \\ 0 & 4 & 5 \end{pmatrix}, \quad A^{(3,2)} = \begin{pmatrix} 2 & 0 & 9 \\ 0 & 4 & 3 \\ 0 & 4 & 5 \end{pmatrix},$$

we have $\sigma_1(A^{(2,3)}) = 11.036, \sigma_1(A^{(3,2)}) = 11.3246$.

From Theorem 2.4 we can conclude the following corollary.

Corollary 2.7. *Let* $A = (a_{ij}) \in M_n(\mathbb{C})$ *be such that its* r*th,* $1 \le r \le n$*, diagonal entry satisfies*

$$|a_{r,r}|^2 > \sum_{i \neq r} |a_{i,i}|^2 + \sum_{i=1}^{n} P_i(A)^2. \tag{2.22}$$

Then

$$\max_{X \in \Lambda(A)} \sigma_1(X) = \max_{l \neq r} \sigma_1(A^{(r,l)}).$$

Proof. The assertion follows directly from Theorem 3 in [2] and Theorem 2.4. ∎

Example 2.8. *Let*

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 5 & -2 \\ 1 & -1 & 1 \end{pmatrix}.$$

Observe that (2.22) holds for $r = 2$*. Then, by Corollary 2.7,*

$$\max_{X \in \Lambda(A)} \sigma_1(X) = \max \{\sigma_1(A^{(2,1)}), \sigma_1(A^{(2,3)})\},$$

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where

$$A^{(2,1)} = \begin{pmatrix} 1 & 3 & 0 \\ 3 & 5 & 0 \\ 0 & 2 & 1 \end{pmatrix} \text{ and } A^{(2,3)} = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 5 & 3 \\ 0 & 2 & 1 \end{pmatrix}.$$

A calculation yields $\sigma_1(A^{(2,1)}) = 6.8465$ *and* $\sigma_1(A^{(2,3)}) = 6.7960$ *and so we have that* $\max_{X\in\Lambda(A)} \ \sigma_1(X) = 6.8465$ *and* $A^{(2,1)}$ *is a* σ_1*-maximazing matrix in* $\Lambda(A)$*. Indeed, following Theorem 3 from [2] for remaining candidates for a* σ_1*-maximizer in* $\Lambda(A)$*, i.e., for matrices*

$$A^{(1,2)} = \begin{pmatrix} 1 & 3 & 0 \\ 3 & 5 & 0 \\ 2 & 0 & 1 \end{pmatrix}, \quad A^{(1,3)} = \begin{pmatrix} 1 & 0 & 3 \\ 3 & 5 & 0 \\ 2 & 0 & 1 \end{pmatrix},$$
$$A^{(3,1)} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 5 & 3 \\ 2 & 0 & 1 \end{pmatrix}, \quad A^{(3,2)} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 5 & 3 \\ 0 & 2 & 1 \end{pmatrix},$$

we have $\sigma_1(A^{(1,2)}) = 6.6794, \sigma_1(A^{(1,3)}) = 5.9834, \sigma_1(A^{(3,1)}) = 6.1327, \sigma_1(A^{(3,2)}) = 6.4653$.

In some cases the following, more general, version of Corollary 2.7 can be useful.

COROLLARY 2.9. *For any matrix* $A \in M_n(\mathbb{C})$ *there is a matrix* $\tilde{A} \in M_n(\mathbb{C})$ *such that, for a positive integer* r*,* $1 \le r \le n$*, it satisfies (2.22) and*

$$\max_{X\in\Lambda(A)} \ \sigma_1(X) \le \max_{l\neq r} \ \sigma_1(\tilde{A}^{(r,l)}).$$

Proof. Suppose that $A \in M_n(\mathbb{C})$ does not satisfy (2.22) and let r, $1 \le r \le n$, be the smallest positive integer such that $|a_{r,r}| = \max_{1\le i\le n} \ |a_{i,i}|$. Now define $\tilde{A} = (\tilde{a}_{ij})_{1\le i,j\le n} \in M_n(\mathbb{C})$ as the matrix which differs from A only in the r-th diagonal entry $\tilde{a}_{r,r}$ for which we have such that

$$|\tilde{a}_{r,r}|^2 > \sum_{i\neq r} |\tilde{a}_{i,i}|^2 + \sum_{i=1}^{n} P_i(\tilde{A})^2.$$

Taking into account the well known Wielandt's result for the spectral radius of dominating nonnegative matrices (cf. Corollary 2.1 of Chapter 2 of [3]), we can deduce that, for any (s,k) with $s,k \in \{1,\dots,n\}$ and $s \neq k$, $\sigma_1(A^{(s,k)}) \le \sigma_1(\tilde{A}^{(s,k)})$ and the assertion follows from Corollary 2.7. ∎

EXAMPLE 2.10. *Let* A *be the matrix from Example 2.6. Then we can take the following matrix* $\tilde{A}$ *satisfying the conditions of Corollary 2.9:*

$$\tilde{A} = \begin{pmatrix} -2 & 6 & -3 \\ 1 & 4 & 2 \\ -2 & -2 & 12 \end{pmatrix}.$$

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Observe that (2.22) holds for $r = 3$. *Then, by Corollary 2.7,*

$$\max_{X \in \Lambda(A)} \sigma_1(X) \leq \max \{\sigma_1(\tilde{A}^{(3,1)}), \sigma_1(\tilde{A}^{(3,2)})\},$$

where

$$\tilde{A}^{(3,1)} = \begin{pmatrix} 2 & 0 & 9 \\ 0 & 4 & 3 \\ 4 & 0 & 12 \end{pmatrix} \text{ and } \tilde{A}^{(3,2)} = \begin{pmatrix} 2 & 0 & 9 \\ 0 & 4 & 3 \\ 0 & 4 & 12 \end{pmatrix}.$$

A calculation yields $\sigma_1(\tilde{A}^{(3,1)}) = 15.9148$ *and* $\sigma_1(\tilde{A}^{(3,.2)}) = 15.8649$. *So, we get the bound for* $\max_{X \in \Lambda(A)} \sigma_1(X)$ *greater than that obtained in Example 2.6.*

We close by mentioning that, with no additional information about an n-by-n matrix A, any one of the $n(n-1)$ candidates $A^{(s,k)}$ for a σ_1-maximizer in $\Lambda(A)$ can give the largest singular value. In fact, if we multiply on both sides the matrix A of Example 2.8 by any of the five 3-by-3 permutation matrices P different from the identity (P^TAP), then we obtain that the matrix $PA^{(2,1)}P^T$ is the corresponding σ_1-maximizer, and this procedure leads to the remaining 5 possibilities for the σ_1-maximizer. Let us also recall that the singular values do not change because they are invariant under permutation equivalence.

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