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A NOTE ON STRONGLY REGULAR GRAPHS AND (k, τ)-REGULAR SETS*

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Abstract. A subset of the vertex set of a graph G, $S \subseteq V(G)$, is a (k, τ)-regular set if it induces a k-regular subgraph of G and every vertex not in the subset has τ neighbors in it. This paper is a contribution to the given problem of existence of (k, τ)-regular sets associated with all distinct eigenvalues of integral strongly regular graphs. The minimal idempotents of the Bose-Mesner algebra of strongly regular graphs are used to obtain a necessary and sufficient condition on the existence of (k, τ)-regular sets for its two restricted eigenvalues.

Key words. Graph theory; Graph spectra; Integral graphs; Strongly regular graphs; Dominating sets.

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1. Introduction. The study of graphs with (k, τ)-regular sets emerged in [1, 6] related with strongly regular graphs and designs and later, in the study of graphs with domination constraints [9, 10]. Recent applications of (k, τ)-regular sets may be found in [3, 4, 5].

The problem of characterizing integral graphs with (k, τ)-regular sets corresponding to all distinct eigenvalues [7] have been studied and some results can be found in [11]. These integral graphs are necessarily regular and there are some well known classes of connected integral graphs with (k, τ)-regular sets for all distinct eigenvalues; complete graphs, complete multipartite graph, hypercube graphs, the n-crown graph with n even and triangle free strongly regular graphs are some examples. Although triangle free strongly regular graphs have this property not all strongly regular graphs have it. For example, the generalized quadrangle $Q(2, 4)$, the complement of the Schläfli graph, is a strongly regular graph with parameters $(27, 10, 1, 5)$ and eigenvalues -5, 1 and 10 (avoiding multiplicities) and has no (k, τ)-regular sets associated to the eigenvalue -5, as noted by Haemers (private communication).

In this paper, we find conditions on the existence of (k, τ)-regular sets corresponding to the restricted eigenvalues of integral strongly regular graphs. We use

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the minimal idempotents of the associated Bose-Mesner algebra of strongly regular graphs to obtain a necessary and sufficient condition for the existence of (k, τ)-regular sets for the two restricted eigenvalues. In particular, we show that the characteristic vectors of such sets are 0-1 eigenvectors of those minimal idempotents. It would be very helpful to find those vectors; in the general case this remains an unsolved problem.

2. Basic definitions and properties. A graph G of order n has vertex set $V(G)$ with $n = |V(G)|$ and edge set $E(G)$ consisting of unordered pairs of vertices. The distance between two vertices v and w of G will be denoted by $\partial(v, w)$. The neighborhood of a vertex $v \in V(G)$ is defined as $N(v) = \{w : \partial(v, w) = 1\}$. Given $S \subseteq V(G)$, the characteristic vector of S is the vector $x_S \in \mathbb{R}^n$ such that the $v-$th component is 1 if $v \in S$ and 0 otherwise. We refer to the *spectrum of G* as the spectrum of its adjacency matrix.

In this paper we consider integral connected strongly regular graphs. An integral graph is a graph whose spectrum consists entirely of integer values. A strongly regular graph with parameters (n, p, a, c) is a p-regular graph (graph with valency p) of order n $(0 < p < n-1)$ such that each pair of adjacent vertices has a common neighbors and each pair of nonadjacent vertices has c common neighbors. The adjacency matrices of G are the matrices $A_m \in \mathbb{R}^{n \times n}$, for $m = 0, 1, 2$, where $(A_m)_{ij} = 1$ if $\partial(i, j) = m$ and $(A_m)_{ij} = 0$ otherwise. In particular, A_0 is the identity matrix and $A_1 = A$ is the usual adjacency matrix of G. It is well known (see, for instance, [2]) that a connected strongly regular graph has diameter two and three distinct eigenvalues, where p is the largest one with multiplicity 1. The spectrum of G will be represented as $\sigma(G) = \{p, [\lambda_1]^{m_1}, [\lambda_2]^{m_2}\}$, where p, λ_1, λ_2 are the distinct eigenvalues of G and m_1, m_2 are the corresponding multiplicities. We refer to λ_1 and λ_2 as the restricted eigenvalues of G. Throughout the text **j** will denote the all-one vector and I, J and **0** denote the identity matrix, the square all-one matrix and the square null matrix, respectively. All other terminology and notations can be found in [2].

A non-empty subset $S \subseteq V(G)$ of a graph G is a (k, τ)-*regular set* of G if S induces a k-regular subgraph of G and every vertex not in the subset has τ neighbors in it. It is clear that, if G is a p-regular graph then $V(G)$ is a (p, τ)-regular set of G for every $\tau \in \mathbb{N}_0$; for convenience, we consider $V(G)$ a $(p, 0)$-regular set of G (associated with the eigenvalue p) thus, from now on, we only consider the restricted eigenvalues of G. Also note that if S is a (k, τ)-regular set of G, then $\overline{S} := V(G) \setminus S$ is a $(p - \tau, p - k)$-regular set of G and S is a $(|S| - k - 1, |S| - \tau)$-regular set of the complement of G.

We recall the following known result:

PROPOSITION 2.1. *[1, 6] Let G be a p-regular graph of order n. Then $\emptyset \neq S \subset$*

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$V(G)$, with characteristic vector x_S, is a (k, τ)-regular set of G if and only if $k - \tau$ is an eigenvalue of G with corresponding eigenvector $(p - k + \tau)x_S - \tau\mathbf{j}$. Moreover, $|S| = \frac{n\tau}{p-k+\tau}$ and if S_i is a (k_i, τ_i)-regular set of G, $i = 1, 2$, with $k_1 - \tau_1 \neq k_2 - \tau_2$ then $|S_1 \cap S_2| = \frac{n\tau_1\tau_2}{(p-k_1+\tau_1)(p-k_2+\tau_2)}$.

An eigenvalue of a graph G is said to be a main eigenvalue if its eigenspace is not orthogonal to the all-one vector $\mathbf{j}$; otherwise, it is called non-main.

A graph G is p-regular if and only if p is the only main eigenvalue of G with eigenvector $\mathbf{j}$ [2].

3. Results. Let G be a connected strongly regular graph of order n and valency p, and $p > \lambda_1 > \lambda_2$ the three distinct eigenvalues of G (λ_1 is nonnegative and λ_2 is negative). It is known that strongly regular graphs are distance regular graphs of diameter two and association schemes with two classes.

The Bose-Mesner algebra $\mathcal{A}$ of G is the $\mathbb{R}-$algebra spanned by the adjacency matrices A_0, A_1, A_2 [12]. These $n \times n$ matrices are symmetric and linearly independent. This algebra has a basis of minimal idempotents, E_0, E_1, E_2, mutually orthogonal, corresponding to the eigenvalues of G, p, λ_1 and λ_2, such that $\mathcal{A} = \langle E_0, E_1, E_2 \rangle$,

- $E_0 = \frac{1}{n}J$;
- $E_1 = \frac{1}{\lambda_1 - \lambda_2}(A - \lambda_2 I - \frac{p-\lambda_2}{n}J)$;
- $E_2 = \frac{1}{\lambda_2 - \lambda_1}(A - \lambda_1 I - \frac{p-\lambda_1}{n}J)$;
- $E_0 + E_1 + E_2 = I$;

satisfying $A = \lambda_0 E_0 + \lambda_1 E_1 + \lambda_2 E_2 = \frac{p}{n}J + \lambda_1 E_1 + \lambda_2 E_2$ and $E_i\mathbf{j} = \mathbf{0}$, for $i = 1, 2$. Let S be a nonempty proper subset of vertices of $V(G)$ with characteristic vector x_S; since $E_i E_j = 0$ for any two distinct $i, j \in \{0, 1, 2\}$, the vectors $E_0 x_S$, $E_1 x_S$ and $E_2 x_S$ are mutually orthogonal. Consider the subspace

$$\mathcal{A}x_S = \langle E_0 x_S, E_1 x_S, E_2 x_S \rangle.$$

PROPOSITION 3.1. *Let G be a strongly regular graph of order n and valency p and $\emptyset \neq S \subset V(G)$. Then $\langle x_S, x_{\overline{S}} \rangle = \mathcal{A}x_S$ if and only if there exist k, $\tau \in \mathbb{N}_0$ such that S is a (k, τ)-regular set of G.*

Proof. As I and $J = nE_0$ belong to $\mathcal{A}$ then, $x_S = Ix_S$ and $\mathbf{j} = \frac{1}{|S|}Jx_S$ belong to $\mathcal{A}x_S$. Therefore, the vector $x_{\overline{S}} = \mathbf{j} - x_S$ is also in $\mathcal{A}x_S$, that is, $\langle x_S, x_{\overline{S}} \rangle \subseteq \mathcal{A}x_S$. If S is a (k, τ)-regular set of G then, by definition, each vertex of S is adjacent to k vertices of S and a vertex of $\overline{S}$ is adjacent to τ vertices of S; so, $Ax_S = kx_S + \tau x_{\overline{S}}$ and this is sufficient to conclude that $\mathcal{A}x_S \subseteq \langle x_S, x_{\overline{S}} \rangle$.

Conversely, suppose that $\langle x_S, x_{\overline{S}} \rangle = \mathcal{A}x_S$. Since Ax_S is in $\mathcal{A}x_S$, there are real numbers, say k and τ, such that $Ax_S = kx_S + \tau x_{\overline{S}}$; this means that each vertex of S

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is adjacent to k vertices of S and each vertex of $\overline{S}$ is adjacent to τ vertices of S so, S is a (k,τ)-regular set of G. $\square$

The following proposition is a direct consequence of [13] (Theorem 3.5) and G being strongly regular.

PROPOSITION 3.2. *Let G be a connected strongly regular graph of order n and valency p and restricted eigenvalues λ_1, λ_2, and S a nonempty set of vertices of G that induces a subgraph of G with m edges. Then*

$$\lambda_2 + (p - \lambda_2)\frac{|S|}{n} \leq \frac{2m}{|S|} \leq \lambda_1 + (p - \lambda_1)\frac{|S|}{n}.$$

If equality holds in one of the above inequalities then S is a (k,τ)-regular set.

If the upper (respectively, lower) bound is attained then S is a $(k,\tau)-$regular set associated with λ_1 (respectively, λ_2), with $k = \frac{2m}{|S|}$ and $\tau = (p - \lambda_1)\frac{|S|}{n}$ (resp., $\tau = (p - \lambda_2)\frac{|S|}{n}$).

LEMMA 3.3. *Let G be a strongly regular graph, p, λ_1, λ_2 the three distinct eigenvalues of G with corresponding minimal idempotents E_0, E_1, E_2; let S be a nonempty proper subset of vertices of G with characteristic vector x_S. If S is a $(k,\tau)-$regular set of G then either $E_1 x_S = 0$ or $E_2 x_S = 0$.*

Proof. Let S be a $(k,\tau)-$regular set of G and $E_1 x_S = 0$, that is, $Ax_S = \lambda_2 x_S + (p - \lambda_2)\frac{|S|}{n}\mathbf{j}$. In this case,

$$\begin{aligned}
E_2 x_S &= \frac{-1}{\lambda_1 - \lambda_2}(Ax_S - \lambda_1 I x_S - \frac{p - \lambda_1}{n}|S|\mathbf{j}) \\
&= \frac{-1}{\lambda_1 - \lambda_2}\left((\lambda_2 - \lambda_1)x_S + (\lambda_1 - \lambda_2)\frac{|S|}{n}\mathbf{j}\right) \\
&= x_S - \frac{|S|}{n}\mathbf{j}
\end{aligned}$$

is a nonzero vector. Similarly, if $E_2 x_S = 0$ then $E_1 x_S \neq 0$.

Now, if S is a $(k,\tau)-$regular set of G with characteristic vector x_S such that $E_1 x_S \neq 0$ and $E_2 x_S \neq 0$, we have

(3.1) $$Ax_S = kx_S + \tau x_{\overline{S}} = (k - \tau)x_S + \tau\mathbf{j}.$$

From $E_1 x_S \neq 0$ comes $Ax_S \neq \lambda_2 x_S + \tau\mathbf{j}$ and from $E_2 x_S \neq 0$ comes $Ax_S \neq \lambda_1 x_S + \tau\mathbf{j}$; this contradicts (3.1) since $k - \tau = \lambda_1$ or $k - \tau = \lambda_2$. $\square$

The (k,τ)-regular sets associated with the restricted eigenvalues λ_1 and λ_2 are the 0-1 solutions of $E_2 x = 0$ and $E_1 x = 0$, respectively, as is stated in the next result.

PROPOSITION 3.4. *Let G be a strongly regular graph of order n and valency p*

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with distinct eigenvalues p, λ_1, λ_2 and correspondent idempotents E_0, E_1, E_2. Let x_S be the characteristic vector of a nonempty proper subset of vertices of G. For $i = 1, 2$, $E_{3-i}x_S = 0$ if and only if S is a $(k, \tau)-$regular set corresponding to the eigenvalue λ_i with $k = \lambda_i + \frac{|S|}{n}(p - \lambda_i)$ and $\tau = \frac{|S|}{n}(p - \lambda_i)$.

Proof. Assume, without loss of generality, that $i = 2$, $E_2x_S \neq 0$ and $E_1x_S = 0$, that is,

$$Ax_S = \lambda_2 x_S + \frac{p - \lambda_2}{n}|S|\mathbf{j}.$$

Therefore,

$$Ax_S = (\lambda_2 + \frac{p - \lambda_2}{n}|S|)x_S + \frac{p - \lambda_2}{n}|S|x_{\overline{S}},$$

which means that S is a $(k, \tau)-$ regular set with $k = \lambda_2 + \frac{|S|}{n}(p - \lambda_2)$ and $\tau = \frac{|S|}{n}(p - \lambda_2)$.

Suppose now that S is a $(k, \tau)-$ regular set corresponding to the eigenvalue λ_2, i.e., $k - \tau = \lambda_2$ and $Ax_S = kx_S + \tau x_{\overline{S}} = (k - \tau)x_S + \tau\mathbf{j}$, where k is the number of neighbors of a vertex $s \in S$ in S and τ is the number of neighbors of a vertex $s \notin S$ in S.

Simple counting leads to $|S|(p - k) = \tau(n - |S|)$. This together with $k - \tau = \lambda_2$ gives:

$$\tau = \frac{|S|}{n - |S|}(p - \lambda_2 - \tau) = \frac{|S|}{n - |S|}(p - \lambda_2) - \tau\frac{|S|}{n - |S|},$$

so,

$$\tau(1 + \frac{|S|}{n - |S|}) = \frac{|S|}{n - |S|}(p - \lambda_2) \quad \text{that is} \quad \tau = (p - \lambda_2)\frac{|S|}{n}.$$

Therefore, we have:

$$(\lambda_1 - \lambda_2)E_1x_S = Ax_S - \lambda_2 x_S - (p - \lambda_2)\frac{|S|}{n}\mathbf{j}$$

$$= (k - \tau)x_S + \tau\mathbf{j} - \lambda_2 x_S - (p - \lambda_2)\frac{|S|}{n}\mathbf{j}$$

$$= (k - \tau - \lambda_2)x_S + (\tau - (p - \lambda_2)\frac{|S|}{n})\mathbf{j} = 0. \ \square$$

The multiplicity of λ_i as an eigenvalue of A is the rank of E_i, $i = 1, 2$ [8]. Therefore, for a $(n, p, a, b)-$strongly regular graph G, there is a $(k, \tau)-$regular set S, associated with a restricted eigenvalue λ if its characteristic vector x_S is a solution of the

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linear system

$$Mx = 0, \quad x \in \{0,1\}^n,$$

where the matrix M is closely related with A, the the adjacency matrix of G,

$$(M)_{ij} = \begin{cases} -n\lambda - p + \lambda & \text{if} \quad i = j \\ -n - p + \lambda & \text{if} \quad (A)_{ij} = 1 \\ -p + \lambda & \text{if} \quad (A)_{ij} = 0 \text{ and } i \neq j \end{cases} .$$

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