

THE SZEGÖ MATRIX RECURRENCE AND ITS ASSOCIATED LINEAR NON-AUTONOMOUS AREA-PRESERVING MAP*

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Abstract. A change to the Szegő matrix recurrence relation, satisfied by orthonormal polynomials on the unit circle, gives rise to a linear map by the action of matrices belonging to the group $SU(1; 1)$. The companion factorization of such matrices, via 2^{n^d} -order linear homogeneous difference equations, provides a compact representation of the orthogonal polynomial on the circle. Moreover, an isomorphism $SU(1; 1) \simeq SL(2; \mathbb{R})$ enables the introduction of a linear non-autonomous area-preserving map. This dynamical system has counterparts in those from the complex Szegő recurrence relation, and some basic results are outlined.

Key words. Companion factorization, Coquaternion, Difference Equation, Nested Sum, Szegő Matrix Recurrence.

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1. Introduction. The classical (forward) Szegő recursive relation on $\mathbb{C}^2$ is an essential tool for theoretical as well as applied aspects of orthogonal polynomials on the unit circle; see e.g. [7, 8, 15, 17, 19]. The theoretical results, e.g. asymptotic and spectral, have mainly been handled with methods of functional analysis. In addition, some connections between Lie algebras and quantum algebras with orthogonal polynomials have been given, see e.g. [11, 20]. Our aim here is to introduce a simple approach for the obtainment of some basic results related to the solutions of the Szegő matrix recurrence. For this task, we handle some properties of the group $SU(1; 1)$; the unit sphere of the coquaternion algebra [6], the companion factorization for the general linear group $GL(n; \mathbb{C})$ [1], and a linear non-autonomous area-preserving map (LNAAM for short) on $\mathbb{R}^2$ associated to the Szegő matrix recurrence relation.

The procedure of our study is the following. We found a transformation on the Szegő recurrence; that is, we normalize the Szegő matrices, so that the recurrence is decomposed in a product of a (variable) complex unit factor and another related recurrence. This new recurrence characterizes a map on $\mathbb{C}^2$ by the action

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of shift matrices belonging to $SU(1;1)$. These complex matrices are decomposed by using the companion factorization [1, 2]. Therefore, when applying the methods given in [2, 3], we provide a compact and explicit representation for the orthogonal polynomials on the circle, based on nested sums of the ratios of coefficients related to the transformed matrix recurrence. Moreover, by using a group isomorphism $SU(1;1) \simeq SL(2; \mathbb{R})$, we can introduce a LNAAM on $\mathbb{R}^2$ defined by $x(n) = L_n x(n-1)$, where $x(n) = (x_1(\theta; n); x_2(\theta; n)) \in \mathbb{R}^2$, with $n \in \mathbb{N}$ and $\theta \in [0, 2\pi[$, associated to the Szegő recurrence. This dynamical system gives us some qualitative results linked with the intricate behavior of the Szegő recurrence relation. As an illustration, we handle some basic results related with the features of asymptotics and spectral measures of the orthonormal polynomials on the unit circle.

The structure of the paper is as follows. In Section 2 we found an explicit representation for the solutions of the transformed Szegő recursive relation. As an extension of the representation given in [3] for orthogonal polynomials on the real line, which satisfy a general three-term recurrence relation, a compact representation for orthogonal polynomials on the unit circle is provided. In Section 3 the associated LNAAM in $\mathbb{R}^2$ of the transformed Szegő matrix recurrence is made explicit, which simplifies further analysis. More precisely, we follow a dynamical approach based on the shift and transfer matrices for the orthonormal Szegő recurrence. Finally, some consequences and illustrations of our study are given.

2. The transformed Szegő recurrence and representations of the orthonormal polynomials on the circle. Consider the sequence $\{\tilde{P}_n(z)\}_{n \geq 0}$ of monic orthogonal polynomials on the unit circle $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$ and its associated reversed polynomials $\{\tilde{P}_n^*(z)\}_{n \geq 0}$, where $\tilde{P}_n^*(z) = z^n \tilde{P}_n(\frac{1}{z})$; such that $\tilde{P}_0(z) = \tilde{P}_0^*(z) = 1$. It is well-known that these sequences can be used to build a map on $\mathbb{C}^2$, satisfying the (forward) Szegő matrix recurrence relation,

$$(2.1) \quad \begin{pmatrix} \tilde{P}_n(z) \\ \tilde{P}_n^*(z) \end{pmatrix} = \begin{bmatrix} z & \delta_n \\ \bar{\delta}_n z & 1 \end{bmatrix} \begin{pmatrix} \tilde{P}_{n-1}(z) \\ \tilde{P}_{n-1}^*(z) \end{pmatrix}, \quad n \in \mathbb{N},$$

where the $\delta_n = \tilde{P}_n(0)$ are the Verblunsky coefficients satisfying $\delta_0 = 1$ and $|\delta_n| < 1$ for $n \geq 1$; see e.g. [8, 19]. Let $\{\varphi_n(z)\}_{n \geq 0}$ be the sequence of orthonormal polynomials defined by $\varphi_n(z) = \frac{\tilde{P}_n(z)}{\|\tilde{P}_n(z)\|}$ and its sequence of reversed polynomials $\{\varphi_n^*(z)\}_{n \geq 0}$. When using the forward Szegő recurrence (2.1), we derive that

$$(2.2) \quad \begin{pmatrix} \varphi_n(z) \\ \varphi_n^*(z) \end{pmatrix} = \frac{1}{\rho_n} \begin{bmatrix} z & \delta_n \\ \bar{\delta}_n z & 1 \end{bmatrix} \begin{pmatrix} \varphi_{n-1}(z) \\ \varphi_{n-1}^*(z) \end{pmatrix}, \quad n \in \mathbb{N},$$

where $\varphi_0(z) = \varphi_0^*(z) = 1$ and $\rho_n = \sqrt{1 - |\delta_n|^2}$. The shift matrix from (2.2) is the so-called Szegő matrix [8]. Further reading on the Szegő recurrence and orthogonal polynomial on the unit circle can be found in [8, 17, 18, 19].

The following transformation can be introduced in the forward Szegő recurrence relation (2.1),

$$(2.3) \quad \begin{pmatrix} \tilde{P}_n(z) \\ \tilde{P}_n^*(z) \end{pmatrix} = e^{i\frac{n\theta}{2}} \|\tilde{P}_n(z)\| \begin{pmatrix} \hat{P}_n(z) \\ \overline{\hat{P}_n(z)} \end{pmatrix}.$$

The equivalent change in the orthonormal Szegő recurrence (2.2) is,

$$(2.4) \quad \begin{pmatrix} \varphi_n(z) \\ \varphi_n^*(z) \end{pmatrix} = e^{i\frac{n\theta}{2}} \begin{pmatrix} \hat{P}_n(z) \\ \overline{\hat{P}_n(z)} \end{pmatrix}.$$

Note that if $\hat{P}_n(z)$ is on the unit circle, then $\varphi_n(z)$ is a unit complex too. These new component functions $\hat{P}_n(z)$ and their conjugates $\overline{\hat{P}_n(z)}$ are not polynomials. They are truncated series (of Laurent or with semi-integer powers) on the unit circle. Moreover, the variable change (2.4) gives rise to the transformed matrix recurrence relation,

$$(2.5) \quad \begin{pmatrix} \hat{P}_n(z) \\ \overline{\hat{P}_n(z)} \end{pmatrix} = \begin{bmatrix} \beta_n & \gamma_n \\ \overline{\gamma_n} & \overline{\beta_n} \end{bmatrix} \begin{pmatrix} \hat{P}_{n-1}(z) \\ \overline{\hat{P}_{n-1}(z)} \end{pmatrix},$$

where $\beta_n = \frac{e^{i\frac{\theta}{2}}}{\rho_n}$ and $\gamma_n = \delta_n \overline{\beta_n}$.

Following the same procedure as in [2], taking into consideration the value of $\det M_n = |\beta_n|^2 - |\gamma_n|^2 = 1$, the determinant of the transformed Szegő matrix $M_n = \begin{bmatrix} \beta_n & \gamma_n \\ \overline{\gamma_n} & \overline{\beta_n} \end{bmatrix}$ from (2.5), and the fact that $\beta_j \neq 0$ for $j \geq 0$, the companion matrix factorization of these particular complex matrices is straightforward,

$$(2.6) \quad M_j = \begin{bmatrix} \beta_j & \gamma_j \\ \overline{\gamma_j} & \overline{\beta_j} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \beta_j^{-1} & \beta_j^{-1} \overline{\gamma_j} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \beta_j & \gamma_j \end{bmatrix}.$$

Therefore, the second order difference equation associated to (2.6), is given as follows

$$(2.7) \quad y_{n+2} = p_1(n)y_{n+1} + p_2(n)y_n,$$

where $p_1(2i) = \gamma_{i+1}$, $p_1(2i+1) = \frac{\overline{\gamma_{i+1}}}{\beta_{i+1}}$, $p_2(2i) = \beta_{i+1}$ and $p_2(2i+1) = \frac{1}{\beta_{i+1}}$. The equivalent companion matrix representation of the recurrence relation (2.7) is

$$(2.8) \quad \begin{pmatrix} y_{n+1} \\ y_{n+2} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ p_2(n) & p_1(n) \end{bmatrix} \begin{pmatrix} y_n \\ y_{n+1} \end{pmatrix}.$$

The general solution of the recurrence relations (2.7)-(2.8), for arbitrary initial conditions, is

$$y_n = y_0 \psi_n^{(1)} + y_1 \psi_n^{(0)},$$

where $\psi_n^{(0)}$ and $\psi_n^{(1)}$ are the canonical solutions of (2.7), satisfying the initial conditions $\psi_0^{(0)} = 0$, $\psi_1^{(0)} = 1$ and $\psi_0^{(1)} = 1$, $\psi_1^{(1)} = 0$, respectively. Compact representations of $\psi_n^{(0)}$ and $\psi_n^{(1)}$ were given in [10], under the determinants of special tridiagonal matrices. Their equivalent expressions based on nested sums are given with more details in [2]. For reason of clarity, we recall that the expansion of $\psi_n^{(0)}$, using the nested sums of the ratios of matrix coefficients $\alpha_2(k_j) = \frac{p_2(k_j)}{p_1(k_j-1)p_1(k_j)}$, is given by

$$(2.9) \quad \psi_n^{(0)} = \left(\prod_{j=0}^{n-2} p_1(j) \right) \sum_{q=0}^{\lfloor \frac{n-1}{2} \rfloor} \Delta_q^{(0)}(n),$$

where the nested sums $\Delta_q^{(0)}(n)$, with q and n determined, are

$$\Delta_q^{(0)}(n) = \sum_{k_1=2(q-1)+1}^{n-2} \alpha_2(k_1) \left(\sum_{k_2=2(q-2)+1}^{k_1-2} \alpha_2(k_2) \left(\dots \left(\sum_{k_q=1}^{k_{q-1}-2} \alpha_2(k_q) \right) \dots \right) \right).$$

When $q = 0$, we assume $\Delta_0^{(0)}(n) = 1$, and $k_0 = n$. If $q = 1$, the $\Delta_1^{(0)}(n)$ is a sum of ratios coefficients. When $q > 1$, the $\Delta_q^{(0)}(n)$ are nested sums of ratios coefficients. Similarly, an expansion of $\psi_n^{(1)}$ with the aid of nested sums can be provided.

REMARK 2.1. *The fundamental solutions $\psi_n^{(0)}$ and $\psi_n^{(1)}$ can be used for deriving the compact representation of the transition (transfer) matrix $\mathcal{T}_n = \prod_{i=0}^{n-1} M_i$ as follows,*

$$(2.10) \quad \mathcal{T}_n = \begin{bmatrix} \psi_{2n}^{(1)} & \psi_{2n}^{(0)} \\ \psi_{2n+1}^{(1)} & \psi_{2n+1}^{(0)} \end{bmatrix}.$$

More details on the properties and applications of the companion decomposition and LNAAM, can be founded in [1, 2].

The main result of this section is formulated as follows.

THEOREM 2.2. *Under the preceding data, the component solutions $\hat{P}_n(e^{i\theta})$ satisfying the transformed Szegő recurrence relation (2.5), have the following representation based on nested sums*

$$\hat{P}_n(e^{i\theta}) = \beta^{(n)}(e^{i\theta}) + \gamma^{(n)}(e^{i\theta}),$$

with the functions,

$$\beta^{(n)}(e^{i\theta}) = \left(\beta_1 \prod_{k=1}^{n-1} \frac{\gamma_{k+1} \bar{\gamma}_k}{\beta_k} \right) \sum_{q=0}^{n-1} \Delta_q^{(1)}(2n, e^{i\theta}).$$

For fixed θ , the nested sums $\Delta_q^{(1)}(2n, e^{i\theta})$ are, with determined pair $(2n, q)$,

$$(2.11) \quad \Delta_q^{(1)}(2n, e^{i\theta}) = \sum_{k_1=2q}^{2n-2} \sum_{k_2=2(q-1)}^{k_1-2} \cdots \sum_{k_q=2}^{k_{q-1}-2} \left(\prod_{j=k_1}^{k_q} w_j(e^{i\theta}) \right),$$

and the ratio coefficients are $w_j(e^{i\theta}) = \frac{\left(\beta_{\lfloor \frac{j+2}{2} \rfloor} \bar{\gamma}_{\lfloor \frac{j}{2} \rfloor} \right)^{e_j}}{\gamma_{\lfloor \frac{j+2}{2} \rfloor} \bar{\gamma}_{\lfloor \frac{j+1}{2} \rfloor}}$, with $e_j = \frac{1+(-1)^j}{2}$. The symbol $\lfloor x \rfloor$ represents the integer part of x . Similarly for the functions $\gamma^{(n)}(e^{i\theta})$, we have

$$\gamma^{(n)}(e^{i\theta}) = \left(\gamma_1 \prod_{k=1}^{n-1} \frac{\gamma_{k+1} \bar{\gamma}_k}{\beta_k} \right) \sum_{q=0}^{n-1} \Delta_q^{(0)}(2n, e^{i\theta}),$$

with the nested sums

$$(2.12) \quad \Delta_q^{(0)}(2n, e^{i\theta}) = \sum_{k_1=2q-1}^{2n-2} \sum_{k_2=2(q-1)-1}^{k_1-2} \cdots \sum_{k_q=1}^{k_{q-1}-2} \left(\prod_{j=k_1}^{k_q} w_j(e^{i\theta}) \right).$$

Note that, in our previous formulae the dependence on $z = e^{i\theta}$ of the entries of the shift matrices has been suppressed by simplicity.

Proof. We use equations (2.5)–(2.6), and some results given in [2]. The component solutions $\hat{P}_n(e^{i\theta})$ of the transformed Szegő recurrence relation (2.5) can be seen as the (unique) solution at step $2n$ of the difference equation (2.7), with initial conditions $y_1 = y_0 = 1$. Indeed, $\hat{P}_n(e^{i\theta}) = \psi_{2n}^{(1)} + \psi_{2n}^{(0)}$, where the functions $\psi_{2n}^{(1)} = \beta^{(n)}(e^{i\theta})$ and $\psi_{2n}^{(0)} = \gamma^{(n)}(e^{i\theta})$ are the canonical solutions of the difference equation (2.7). From (2.9) we have,

$$\psi_{2n}^{(0)} = \left(\prod_{j=0}^{2n-2} p_1(j) \right) \sum_{q=0}^{n-1} \Delta_q^{(0)}(2n, e^{i\theta}),$$

where $\Delta_q^{(0)}(2n, e^{i\theta})$ is given by (2.12). In an analogous way, from [2, Equation (13)], the other canonical solution $\psi_{2n}^{(1)}$ can be represented as,

$$\psi_{2n}^{(1)} = p_2(0) \left(\prod_{j=1}^{2n-2} p_1(j) \right) \sum_{q=0}^{n-1} \Delta_q^{(1)}(2n, e^{i\theta}),$$

where $\Delta_q^{(1)}(2n, e^{i\theta})$ is given by (2.11). We now obtain the compact expression for $w_j(e^{i\theta}) = \alpha_2(j) = \frac{p_2(j)}{p_1(j-1)p_1(j)}$ ($j \geq 1$). The variable coefficients are $p_1(j) = \gamma_{\lfloor \frac{j+2}{2} \rfloor}$ and $p_2(j) = \beta_{\lfloor \frac{j+2}{2} \rfloor}$ for j even, $p_1(j) = \bar{\gamma}_{\lfloor \frac{j+1}{2} \rfloor}$, $p_2(j) = \frac{1}{\beta_{\lfloor \frac{j+1}{2} \rfloor}}$, for j odd. The ratios

of coefficients yield $w_j(e^{i\theta}) = \frac{\beta_{\lfloor \frac{j+2}{2} \rfloor} \beta_{\lfloor \frac{j}{2} \rfloor}}{\gamma_{\lfloor \frac{j+2}{2} \rfloor} \overline{\gamma}_{\lfloor \frac{j}{2} \rfloor}}$ for j even, and $w_j(e^{i\theta}) = \frac{1}{\gamma_{\lfloor \frac{j+1}{2} \rfloor} \overline{\gamma}_{\lfloor \frac{j+1}{2} \rfloor}}$ for j odd. It can be expressed in a more compact form by introducing the exponent $e_j = \frac{1+(-1)^j}{2}$ and using the properties of $\lfloor \frac{x}{2} \rfloor$, when x is an even (or odd) number, $w_j(e^{i\theta}) = \frac{\left(\beta_{\lfloor \frac{j+2}{2} \rfloor} \beta_{\lfloor \frac{j}{2} \rfloor} \right)^{e_j}}{\gamma_{\lfloor \frac{j+2}{2} \rfloor} \overline{\gamma}_{\lfloor \frac{j+1}{2} \rfloor}} \quad (j \in \mathbb{N})$.

In order to complete the representation of $\gamma^{(n)}(e^{i\theta})$, we consider the multiplicative factor $\left(\prod_{j=0}^{2n-2} p_1(j) \right)$. We use the usual conventions on products. For $k = 1$, we have, $p_1(0) = \gamma_1$. For $k = n - 1$, we assume that $\left(\prod_{j=0}^{2(n-1)-2} p_1(j) \right) = \gamma_1 \prod_{k=1}^{n-2} \frac{\gamma_{k+1} \overline{\gamma}_k}{\beta_k}$. Thus, for $k = n$, we obtain $\left(\prod_{j=0}^{2n-2} p_1(j) \right) = p_1(2n-2) p_1(2n-3) \gamma_1 \prod_{k=1}^{n-2} \frac{\gamma_{k+1} \overline{\gamma}_k}{\beta_k}$. That is, we have $\left(\prod_{j=0}^{2n-2} p_1(j) \right) = \gamma_1 \prod_{k=1}^{n-1} \frac{\gamma_{k+1} \overline{\gamma}_k}{\beta_k}$. From this result, the representation of the multiplicative factor of $\beta^{(n)}(e^{i\theta})$, i.e. $\beta_1 \prod_{k=1}^{n-1} \frac{\gamma_{k+1} \overline{\gamma}_k}{\beta_k}$, is immediate. $\square$

The entries of the transfer matrix $\mathcal{T}_n$ (Equation (2.10)) of the map (2.5) at step n can be expressed with the results given in Theorem 2.2,

$$\mathcal{T}_n(e^{i\theta}) = \begin{bmatrix} \frac{\beta^{(n)}(e^{i\theta})}{\gamma^{(n)}(e^{i\theta})} & \frac{\gamma^{(n)}(e^{i\theta})}{\beta^{(n)}(e^{i\theta})} \\ \gamma^{(n)}(e^{i\theta}) & \beta^{(n)}(e^{i\theta}) \end{bmatrix}.$$

The explicit representation of monic and orthonormal polynomials, solutions of Szegő recurrences (2.1) and (2.2), based on nested sums of the ratios of coefficients from the transformed recurrence relation (2.5) is now simple. That is, from Theorem 2.2 and the transformations (2.3) and (2.4), we derive compact expressions for the polynomials $\tilde{P}_n(z)$ and $\varphi_n(z)$.

COROLLARY 2.3. *For every $n \geq 0$, we have the representations*

$$\begin{aligned} \tilde{P}_n(e^{i\theta}) &= e^{i\frac{n\theta}{2}} \left(\prod_{j=1}^n \rho_j \right) \left(\beta^{(n)}(e^{i\theta}) + \gamma^{(n)}(e^{i\theta}) \right), \\ \varphi_n(e^{i\theta}) &= e^{i\frac{n\theta}{2}} \left(\beta^{(n)}(e^{i\theta}) + \gamma^{(n)}(e^{i\theta}) \right), \end{aligned} \quad (2.13)$$

where $\beta^{(n)}(e^{i\theta})$, $\gamma^{(n)}(e^{i\theta})$, are as given in Theorem 2.2.

Thus, the model of representation introduced in [3], based on nested sums for the orthogonal polynomials on the real line, is extended to the unit circle. In the best of our knowledge, these compact expressions (2.13) for orthogonal polynomials on the unit circle are not current in the literature.

REMARK 2.4. *Note that besides the dependence on $z = e^{i\theta}$, equation (2.13) provides us a compact representation for the orthogonal polynomials on the unit circle*

in terms of the given Verblunsky coefficients. For example in the orthonormal case,

$$(2.14) \quad \varphi_n(e^{i\theta}) = \frac{1}{\delta_n} \left(\prod_{k=1}^n \frac{|\delta_k|^2}{\sqrt{1-|\delta_k|^2}} \right) \left(\frac{e^{i\theta}}{\delta_1} \sum_{q=0}^{n-1} \Delta_q^{(1)}(2n, e^{i\theta}) + \sum_{q=0}^{n-1} \Delta_q^{(0)}(2n, e^{i\theta}) \right),$$

where the ratios coefficients of the nested sums, $w_j(e^{i\theta})$, are also given in terms of the Verblunsky coefficients.

3. LNAAM associated to the Szegő recurrence relation.

3.1. Construction of the LNAAM. The transformed Szegő matrix of the recurrence (2.5) can be constructed from the coquaternion $\mathbb{R}$ -algebra $\mathcal{CQ}_{\mathbb{R}}$, see e.g. [6, 9, 12], generated by the basis $\{1; \mathbf{i}; \mathbf{j}; \mathbf{k}\}$, satisfying the following constants of structure, $1^2 = -\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = 1$; $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$; $\mathbf{ik} = -\mathbf{ki} = -\mathbf{j}$; $\mathbf{jk} = -\mathbf{kj} = \mathbf{i}$. That is, each coquaternion $q = \lambda + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ can be written under the form $q = \beta + \gamma\mathbf{j}$, with the complex entries $\beta = \lambda + x\mathbf{i}$ and $\gamma = y + z\mathbf{i}$. Therefore, each coquaternion q can be expressed through a particular 2×2 complex matrix, see e.g. [4],

$$(3.1) \quad M(q) = \begin{bmatrix} \beta & \gamma \\ \overline{\gamma} & \overline{\beta} \end{bmatrix},$$

where $\overline{\beta}$ and $\overline{\gamma}$ are the conjugate of β and γ , respectively. We can verify that $\mathcal{M}_q$, the set of matrices of the type (3.1), is a real subalgebra of dimension 4 of the real algebra of square matrices $\mathcal{M}(2; \mathbb{C})$. The $\mathbb{R}$ -algebras $\mathcal{CQ}_{\mathbb{R}}$ and $\mathcal{M}_q$ are isomorphic, via the isomorphism of real algebras $\Phi: \mathcal{CQ}_{\mathbb{R}} \rightarrow \mathcal{M}_q$ defined by $\Phi(q) = M(q)$, where $q = \lambda + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $M(q)$ is given by (3.1). Moreover, for $q \in \mathcal{CQ}_{\mathbb{R}}$ we have $q\overline{q} = \det(M(q)) = \lambda^2 + x^2 - y^2 - z^2$, where $\overline{q} = \lambda - x\mathbf{i} - y\mathbf{j} - z\mathbf{k}$ is the conjugate of q .

REMARK 3.1. *Our main goal is not to use here an algebraic sophisticated construction of the coquaternion algebra. Meanwhile, following the procedure of [13], one can elaborate an interesting constructive method for the real coquaternion algebra $\mathcal{CQ}_{\mathbb{R}}$.*

Furthermore, the transformed Szegő matrix M_n of (2.5) satisfies $\det M_n = |\beta_n|^2 - |\gamma_n|^2 = 1$. Hence, M_n is in $\mathcal{S}(\mathcal{CQ}_{\mathbb{R}})$ the set of elements of $\mathcal{CQ}_{\mathbb{R}}$ with modulus 1, representing the unit sphere of coquaternions. By the isomorphism Φ defined by (3.1), the set $\mathcal{S}(\mathcal{CQ}_{\mathbb{R}})$ is a group for the matrix product, identified with the group $SU(1; 1)$. Therefore, the transformed recurrence (2.5) can be seen as a map on $\mathbb{C}^2$ by the action of the transformed Szegő matrices belong to the group $SU(1; 1)$. When taking the initial conditions $\varphi_0(z) = \varphi_0^*(z) = 1$, we have

$$(3.2) \quad \begin{pmatrix} \widehat{P}_n(z) \\ \widehat{\overline{P}}_n(z) \end{pmatrix} = \prod_{j=1}^n \begin{bmatrix} \beta_j & \gamma_j \\ \overline{\gamma}_j & \overline{\beta}_j \end{bmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{bmatrix} \beta^{(n)} & \gamma^{(n)} \\ \overline{\gamma}^{(n)} & \overline{\beta}^{(n)} \end{bmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \beta^{(n)} + \gamma^{(n)} \\ \overline{\beta}^{(n)} + \overline{\gamma}^{(n)} \end{pmatrix}.$$

The following well-known result is of main interest in the sequel,

LEMMA 3.2. *The unit sphere of coquaternions, $SU(1;1)$, is a group (with the usual product of matrices) isomorphic to the special linear group $SL(2; \mathbb{R})$.*

Indeed, consider the application $\Psi : SL(2; \mathbb{R}) \rightarrow \mathcal{S}(\mathcal{CQ}_{\mathbb{R}})$ defined by $\Psi(L) = q$, where $L = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2; \mathbb{R})$ and $q = \lambda + x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \in \mathcal{S}(\mathcal{CQ}_{\mathbb{R}})$ are such that,

$$(3.3) \quad \lambda = \frac{a+d}{2}, \quad x = \frac{b-c}{2}, \quad y = \frac{b+c}{2}, \quad z = \frac{a-d}{2}.$$

Direct computation shows that Ψ is a group isomorphism. From the applications Ψ and Φ defined by (3.1), we can compose another group isomorphism between $SU(1;1)$ and $SL(2; \mathbb{R})$; see e.g. [4],

$$(3.4) \quad \beta = \frac{1}{2}[(a+d) + i(b-c)] ; \quad \gamma = \frac{1}{2}[(b+c) + i(a-d)].$$

Therefore, since $SU(1;1)$ is a group isomorphic to $SL(2; \mathbb{R})$, we can build a LNAAM associated to the Szegő recurrence relation (2.2), $x(n) = L_n x(n-1)$; where $x(n) = (x_1(\theta; n); x_2(\theta; n))$. The real matrix $L_n = \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix}$ is the shift matrix derived from the matrix M_n via the isomorphism of Lemma 3.2. From Equation (3.3), we have $L_n = \begin{bmatrix} \lambda_n + z_n & x_n + y_n \\ -x_n + y_n & \lambda_n - z_n \end{bmatrix}$.

Furthermore, using (3.3)–(3.4), and recalling that $\beta_n = \frac{e^{i\frac{\theta}{2}}}{\rho_n}$, $\gamma_n = \delta_n \overline{\beta}_n$, where $\delta_n = \tilde{P}_n(0) = |\delta_n|e^{i\phi_n}$, we obtain the following relations

$$\begin{aligned} \beta_n &= \lambda_n + ix_n = \frac{1}{\rho_n} \left(\cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \right), \\ \gamma_n &= y_n + iz_n = \frac{|\delta_n|}{\rho_n} \left(\cos \left(\phi_n - \frac{\theta}{2} \right) + i \sin \left(\phi_n - \frac{\theta}{2} \right) \right). \end{aligned}$$

Therefore, the explicit representation for the shift matrices L_n ($n \in \mathbb{N}$) is

$$(3.5) \quad L_n = \frac{1}{\rho_n} \begin{bmatrix} \cos(\frac{\theta}{2}) + |\delta_n| \sin(\phi_n - \frac{\theta}{2}) & \sin(\frac{\theta}{2}) + |\delta_n| \cos(\phi_n - \frac{\theta}{2}) \\ -\sin(\frac{\theta}{2}) + |\delta_n| \cos(\phi_n - \frac{\theta}{2}) & \cos(\frac{\theta}{2}) - |\delta_n| \sin(\phi_n - \frac{\theta}{2}) \end{bmatrix}.$$

Thus, the representation of the shift matrices L_n can be decomposed as a sum of two matrices by a common factor with a dependence on n via the Verblunsky coefficients. The first matrix is a fixed orthogonal matrix; the second one is variable, but always symmetric. More precisely, we have the following proposition.

PROPOSITION 3.3. *The shift matrices $L_n \in SL(2; \mathbb{R})$, belonging to the associated LNAAM (3.5), have the following decomposition*

$$(3.6) \quad L_n = \frac{1}{\rho_n}(O + S_n),$$

where O is a fixed orthogonal matrix and S_n is a symmetric one, defined by,

$$O = \begin{bmatrix} \cos(\frac{\theta}{2}) & \sin(\frac{\theta}{2}) \\ -\sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{bmatrix} \text{ and } S_n = |\delta_n| \begin{bmatrix} \sin(\phi_n - \frac{\theta}{2}) & \cos(\phi_n - \frac{\theta}{2}) \\ \cos(\phi_n - \frac{\theta}{2}) & -\sin(\phi_n - \frac{\theta}{2}) \end{bmatrix}.$$

Moreover, we have for the trace of L_n , $\text{tr}(L_n) = \frac{2 \cos(\frac{\theta}{2})}{\rho_n}$.

Proof. It follows directly from (3.5). $\square$

The family of shift matrices $\{L_n\}_{n \geq 0}$, given by (3.5)–(3.6), contains the dynamics of the LNAAM associated to the Szegő matrix difference equation (2.2). In the next subsection we explore some results on the behavior of the LNAAM, with the aid of the decomposition (3.6). In particular, as mentioned above, we will recover some known results in a simple way.

3.2. Limit behavior of the LNAAM and its connections with basic results from the Szegő recurrence. Results on the behavior of the simpler dynamical system, generated by the action of (real) shift matrices from (3.6), are linked to other equivalent results on the complex map (2.2) for the orthonormal polynomials satisfying the Szegő matrix recursive relation. Thus, as an application of preceding results, an overview on the limit behavior of the solutions of the LNAAM (3.6) is provided. We remark its relation with the ratio asymptotics and the basic structure of the spectral measures for orthonormal polynomials on the unit circle.

First, we recall that a non trivial solution $x(\theta; n)$ is *minimal solution* of this LNAAM if, for any other linearly independent solution $y(\theta; n)$, it is satisfied that

$$\lim_{n \rightarrow +\infty} \frac{x(\theta; n)}{y(\theta; n)} = 0.$$

The concept of minimal solution has a role in qualitative aspects of the spectral measures. For the sequence $\{\delta_n\}_{n \geq 0}$ of the Verblunsky coefficients, we can associate the subset $\mathcal{S}(\{\delta_n\}_{n \geq 0})$ of θ belonging to $[0; 2\pi[$ such that the minimal solution $x(\theta; n)$ for the LNAAM (3.6) does not exists. That is, we have

$$\mathcal{S}(\{\delta_n\}_{n \geq 0}) = \{\theta \in [0, 2\pi[\text{ such that the minimal solution } x(\theta; n) \text{ does not exist}\}.$$

The following statement holds between the trace of the transition matrix of the LNAAM, $\text{tr}(\mathcal{T}_n)$, and the set $\mathcal{S}(\{\delta_n\}_{n \geq 0})$. Here $\text{tr}(\mathcal{T}_n) = 2\Re(\beta^{(n)}) = 2\lambda^{(n)}$, with $\lambda^{(n)}$ the scalar part of the particular unit coquaternion at step n .

PROPOSITION 3.4. *Let $\mathcal{T}_n = \prod_{j=1}^n L_j$ be the transition (transfer) matrix associated to the family $\{L_n\}_{n \geq 1}$ of shift matrices introduced in Proposition 3.3. If there exists a natural number $n_0 \in \mathbb{N}$, such that $|\text{tr}(\mathcal{T}_n)| < 2$ for every $n \geq n_0$, then $\theta \in \mathcal{S}(\{\delta_n\}_{n \geq 0})$.*

Proof. Indeed, the proof is not difficult if we observe that the shift matrices (also the transfer matrices) belong to $SL(2, \mathbb{R})$. If a particular θ does not belong to $\mathcal{S}(\{\delta_n\}_{n \geq 0})$, then a minimal solution exists. Furthermore, if for some n_0 we have $|\text{tr}(\mathcal{T}_n)| < 2$ for every $n \geq n_0$, then the pair of eigenvalues should be bounded and self-conjugates, which is a contradiction with the existence of such minimal solution. $\square$

Let $d\mu$ be a non trivial probability measure (i.e. with infinite support) on the unit circle $\partial\mathbb{D}$ of $\mathbb{C}$, identified with the real interval $[0, 2\pi[$ via the usual map $\theta \rightarrow e^{i\theta}$. Thus, the probability measure $d\mu$ is related to the Lebesgue measure $d\theta$ on the interval $[0, 2\pi[$ as follows

$$d\mu(\theta) = \frac{\omega(\theta)}{2\pi} d\theta + d\mu_S(\theta),$$

where $d\mu_S$ is the singular part of $d\mu$. The norm associated to $d\mu(\theta)$ on the Hilbert space $L^2(d\mu)$ is defined as

$$\|\tilde{P}_n(z)\|_{L^2(d\mu)} = \|\tilde{P}_n(z)\| = \rho_n \|\tilde{P}_{n-1}(z)\|,$$

where $\rho_n = \sqrt{1 - |\delta_n|^2}$. Thus, we have $\|\tilde{P}_n(z)\| = \|\tilde{P}_n^*(z)\| = \prod_{j=1}^n \rho_j$; see e.g. [17] for more details. The Szegő Theorem, [19], provides us a relation between the norm at infinity with the geometrical mean of the density (weight) $\omega(\theta)$ of the absolutely continuous (a.c.) part of $d\mu$; namely,

$$(3.7) \quad \lim_{n \rightarrow +\infty} \|\tilde{P}_n(z)\|^2 = \prod_{j=1}^{+\infty} \rho_j^2 = \exp \left(\int_0^{2\pi} \log(\omega(\theta)) \frac{d\theta}{2\pi} \right).$$

From (3.7), the sequence of Verblunsky coefficients belong to $l^2(\mathbb{Z}_+)$ if and only if $\int_0^{2\pi} \log(\omega(\theta)) \frac{d\theta}{2\pi} > -\infty$. In other words, we have the Szegő condition,

$$(3.8) \quad \sum_{j=1}^{+\infty} |\delta_n|^2 < +\infty \Leftrightarrow \int_0^{2\pi} \log(\omega(\theta)) \frac{d\theta}{2\pi} > -\infty.$$

In the sequel, we assume $\sup_{n \geq 1} |\delta_n| < 1$. As a consequence of Proposition 3.4, if for a fixed θ there exists a minimal solution for the LNAAM (3.6), then this particular θ does not belong to the support of the a.c. spectrum. This result has its counterpart in the Szegő recurrence relation (2.2), concerning the a.c. part of the spectral measure, which was obtained with other techniques; see [8, Theorem 10].

THEOREM 3.5. (Golinskii-Nevai [8]). *The set $\mathcal{S}(\{\delta_n\}_{n \geq 0})$ is an essential support of the a.c. part of μ and for each Borel set $E \subset S$, $\mu_S(E) = 0$.*

Now we add the following assumption on the limit behavior of the Verblunsky coefficients, $\lim_{n \rightarrow \infty} \delta_n = 0$. Just consider as it is necessary for the Szegő condition (3.8). Then, there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{\sqrt{1-|\delta_n|^2}} = 1 + \frac{|\delta_n|^2}{2} + o(|\delta_n|^2)$, for $n \geq n_0$. Hence, the LNAAM (3.6) can be seen as

$$(3.9) \quad L_n = O + B_n \text{ with } \lim_{n \rightarrow \infty} \|B_n\| \rightarrow 0.$$

That is, we have a difference system of Poincaré type, see e.g. [5, 14]. The following result is achieved for the nontrivial solutions of the LNAAM with shift matrices given by (3.9); see [14, Theorem I],

$$(3.10) \quad \lim_{n \rightarrow \infty} \sqrt[n]{\|x(n)\|} = 1,$$

which is linked to the weak (logarithmic) asymptotic behavior of the orthonormal polynomials on the circle; see e.g. [16].

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\log \varphi_n(e^{i\theta})| = |e^{i\theta}|.$$

A more strengthened form of Equation (3.10) is the following; see [14, Remark 2],

$$(3.11) \quad \lim_{n \rightarrow \infty} \frac{\|x(n+1)\|}{\|x(n)\|} = 1,$$

linked to the ratio asymptotic of the orthonormal polynomials, [16],

$$(3.12) \quad \lim_{n \rightarrow \infty} \frac{\varphi_{n+1}(e^{i\theta})}{\varphi_n(e^{i\theta})} = e^{i\theta}.$$

As a final point, we can gather some additional results.

THEOREM 3.6. *The following statements on the limit behavior of the Verblunsky coefficients, the LNAAM (3.6), and the transformed Szegő matrix M_n from (2.5) are equivalent:*

- a) $\lim_{n \rightarrow \infty} \delta_n = 0$.
- b) $\lim_{n \rightarrow \infty} L_n = O$, the orthogonal matrix given in Proposition 3.3.
- c) $\lim_{n \rightarrow \infty} M_n = \begin{bmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{bmatrix}$.

Proof.

- a) $\Rightarrow$ b) As $\lim_{n \rightarrow \infty} |\delta_n| = 0$, then by Equation (3.6), we have $\lim_{n \rightarrow \infty} L_n = O$.
- b) $\Rightarrow$ c) Suppose that $\lim_{n \rightarrow \infty} L_n = O$. Then, the result is straightforward by

equations (2.5) and (3.4).

c) $\Rightarrow$ a) Suppose that $\lim_{n \rightarrow \infty} M_n = \begin{bmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{bmatrix}$. Thus, Equation (2.5) shows that $\lim_{n \rightarrow \infty} \beta_n = e^{i\frac{\theta}{2}}$ and $\lim_{n \rightarrow \infty} \gamma_n = 0$. Since $\beta_n = \frac{e^{i\frac{\theta}{2}}}{\rho_n}$ and $\gamma_n = \delta_n \overline{\beta_n}$, we have $\lim_{n \rightarrow \infty} \delta_n = 0$. $\square$

A result on orthonormal polynomials on the unit circle related to the equivalence between the (null) limit of the Verblunsky coefficients and the limit behavior of the LNAAM (3.6) given in Theorem 3.6, is nothing else but the celebrated Rakhmanov Theorem; see e.g. [15].

THEOREM 3.7. (*Rakhmanov [15]*). *If $\frac{d\mu}{d\theta} = \mu' > 0$, a. e. on $\partial\mathbb{D}$, the corresponding sequence of Verblunsky coefficients satisfies*

$$\lim_{n \rightarrow \infty} \delta_n = 0.$$

We show other known result linked to the equivalence, given in Theorem 3.6, between the (null) limit of the Verblunsky coefficients and the limit behavior of the transformed Szegő matrix M_n . This result provides the equivalence between the nullity of the limit of the Verblunsky coefficients and the ratio asymptotic (3.12) for the orthonormal polynomials,

$$\lim_{n \rightarrow \infty} \delta_n = 0 \iff \lim_{n \rightarrow \infty} \frac{\varphi_{n+1}(e^{i\theta})}{\varphi_n(e^{i\theta})} = e^{i\theta}.$$

It can easily be derived, under the assumption c) of Theorem 3.6, by using the equations (2.4) and (2.5).

In general, without any statement on the limit value of the Verblunsky coefficients, when the shift matrices $L_n(\theta)$ of the LNAAM converges to a particular matrix $L(\theta) \in SL(2; \mathbb{R})$, the ratio asymptotic of the nontrivial solutions from the LNAAM (3.6) is

$$\lim_{n \rightarrow \infty} \frac{x(n+1)}{x(n)} = \lambda(\theta) \pm \sqrt{(\lambda(\theta))^2 - 1},$$

depending on $\lambda(\theta) = \frac{\text{tr}(L(\theta))}{2}$, the scalar part of the coquaternion related to $L(\theta)$.

If $|\lambda(\theta)| < 1$, we have the ratio (3.11). When $|\lambda(\theta)| > 1$, the minimal solution exists, and the parametric Poincaré Theorem, [5], is applicable, with the spectral radius given by $\rho(\theta) = \max\{|\lambda(\theta) \pm \sqrt{(\lambda(\theta))^2 - 1}|\} = e^{\arg \cosh |\lambda(\theta)|}$. Non-minimal solutions $y(n)$ from the LNAAM (3.6) accomplish

$$\lim_{n \rightarrow \infty} \left| \frac{y(n+1)}{y(n)} \right| = e^{\arg \cosh |\lambda(\theta)|}.$$

A further analysis will depend on the particular sequence of Verblunsky coefficients.

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