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WEAK MONOTONICITY OF MATRICES AND SUBCLASSES OF PROPER SPLITTINGS*

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Abstract. This article concerns weak monotonicity of matrices, with specific emphasis on its relationship with a certain class of proper splittings. The matrix $A \in \mathbb{R}^{m\times n}$ is weak monotone provided $Ax \geq 0 \Longrightarrow x \in \mathbb{R}^n_+ + N(A)$, where $N(A)$ is the nullspace of A. In particular, the following extension of well known characterizations for M-matrices is obtained. Suppose that $\mathrm{int}(\mathbb{R}^m_+) \cap R(A) \neq \phi$. Then the statements

(a) A is weak-monotone.
(b) $\mathbb{R}^m_+ \cap R(A) \subseteq A\mathbb{R}^n_+$.
(c) There exists $x^0 \geq 0$ such that $Ax^0 > 0$.

satisfy (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c). Suppose further that A can be written as $A = U - V$, where A and U have the same range space and null space, U and V are nonnegative, $VU^\dagger \geq 0$ (where $U^\dagger$ denotes the Moore-Penrose inverse of U), and $Ax \geq 0$, $Ux \geq 0 \Longrightarrow x \in \mathbb{R}^n_+ + N(A)$. Then each of the above statements is equivalent to the statement

(d) $\rho(VU^\dagger) < 1$.

Key words. Proper splittings; Weak monotonicity; Nonnegativity; Moore-Penrose inverse.

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1. Introduction.

A real square matrix A is a *Z-matrix* if the off-diagonal entries of A are nonpositive. A Z-matrix A is an *M-matrix* if A can be written as $A = sI - B$, where $B \geq 0$ (meaning that all the entries are nonnegative) and $s \geq \rho(B)$, where $\rho(\cdot)$ denotes the spectral radius. There are several characterizations for nonsingular M-matrices (see [3]). The following result is a sample of such a characterization. In the rest of the article, the notation $x \geq 0$ ($x > 0$) means that all coordinates of x are nonnegative (strictly positive). $\mathbb{R}^k_+$ denotes the nonnegative orthant of the real Euclidean space $\mathbb{R}^k$, $u \geq 0$ denotes that $u \in \mathbb{R}^k_+$, whereas $u > 0$ denotes that $u \in \mathrm{int}(\mathbb{R}^k_+)$.

THEOREM 1.1. *Let A be an $n \times n$ Z-matrix. Then the following are equivalent:*

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(a) A is invertible and $A^{-1} \geq 0$;
(b) There exists $x > 0$ such that $Ax > 0$;
(c) A can be written as $A = sI - B$, where $B \geq 0$ and $s > \rho(B)$.

Let us briefly discuss real square matrices satisfying the first condition of Theorem 1.1. A square real matrix A is *monotone* if $Ax \geq 0$ implies $x \geq 0$. The concept of monotonicity was first proposed by Collatz [7], who showed that a real square matrix A is monotone if and only if it is invertible and $A^{-1} \geq 0$. Mangasarian [9] extended the notion to rectangular real matrices using the same implication $Ax \geq 0$ implies $x \geq 0$. He showed that a real rectangular matrix A is monotone if and only if A has a nonnegative left inverse.

Berman and Plemmons extended the notion of monotonicity to more general classes of matrices, using generalized inverses. We shall be particularly concerned with the following two types.

Definition 1.2. Let $A \in \mathbb{R}^{m \times n}$. Then

(a) A is row monotone if $Ax \geq 0$ and $x \in R(A^T)$ implies that $x \geq 0$;
(b) A is weak monotone if $Ax \geq 0 \Longrightarrow x \in \mathbb{R}^n_+ + N(A)$.

It is clear that if A is monotone then A is row monotone, which in turn implies that A is weak monotone. The implications in the reverse direction are not true. It can be seen that if A is weak monotone and $Ax \geq 0$, then there exists $y \geq 0$ such that $Ax = Ay$. Stated informally, weak monotonicity is equivalent to the statement that for any consistent linear system $Ax = b$, where the requirement vector b is nonnegative, there is always a nonnegative solution.

In the literature, several results deal with relationships between the notion of inverse positivity and splittings of the matrix under consideration. In what follows, we mention those most relevant to the present work.

Definition 1.3. Let $A \in \mathbb{R}^{m \times n}$. A decomposition $A = U - V$ is a *positive splitting* if $U \geq 0$ and $V \geq 0$; a *proper splitting* if $R(A) = R(U)$ and $N(A) = N(U)$; a *regular splitting* if U is invertible, $U^{-1} \geq 0$ and $V \geq 0$; a *positive pseudo sub-proper splitting* if $U \geq 0$, $V \geq 0$ and $R(U) \subseteq R(A)$; and a *semi-positive pseudo sub-proper splitting* if $U \geq 0$ and $R(U) \subseteq R(A)$.

It was proved that for any regular splitting $A = U - V$, A is inverse positive if and only if $U^{-1}V$ has spectral radius strictly less than 1. It is well known that this latter condition ensures convergence of iterative schemes defined in terms of U and V. For more details see [15]. In this context, we would also like to point out the notion of a B-splitting, introduced and studied by Peris [12]. More pertinent to the present work is the notion of a generalized B-splitting, again considered by Peris [13], given

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as follows:

DEFINITION 1.4. Let $A \in \mathbb{R}^{m\times n}$. A positive splitting $A = U - V$ of A is a generalized B-splitting of A if it satisfies $Ux \geq 0 \Longrightarrow Vx \geq 0$ and $Ax \geq 0$, $Ux \geq 0 \Longrightarrow x \in \mathbb{R}^n_+ + N(A)$.

We remark that first condition in Definition 1.4 is equivalent to the existence of a matrix $T \in \mathbb{R}^{m\times m}$ such that $V = TU$ with $T \geq 0$. If $m = n$ and U is invertible, it follows that $VU^{-1} \geq 0$. The second condition can be interpreted as a joint weak monotonicity condition involving A and U. With the aid of such a generalized B-splitting, Peris characterized weak monotonicity as follows:

THEOREM 1.5. *(Theorem 1, [13]) Let $A \in \mathbb{R}^{m\times n}$ and $A = U - V$ be a generalized B-splitting. Suppose that* $\mathrm{int}(\mathbb{R}^m_+) \cap R(A) \neq \phi$. *Then the following conditions are equivalent:*

(a) *A is weak monotone.*
(b) There exists $x \geq 0$ such that $Ax > 0$.
(c) $\rho(T) < 1$, where T is as given above.

The existence of a generalized B-splitting *for a class of matrices* was also settled, as stated next.

THEOREM 1.6. *(Theorem 3, [13]) Let $A \in \mathbb{R}^{m\times n}$ such that $\mathbb{R}^n_+ \cap N(A) = \{0\}$. Then the following conditions are equivalent:*

(a) A is weak monotone.
(b) A allows a generalized B-splitting $A = U - V$ such that $V = TU$ with $\rho(T) < 1$.

Next, we turn our attention to three generalizations of B-splittings. The first was introduced and studied in [10], the second in [11], while the third is proposed in this article. In what follows and in the rest of the article, we use the notion of the Moore-Penrose inverse of a matrix. Briefly, given $A \in \mathbb{R}^{m\times n}$, the Moore-Penrose inverse of A denoted by $A^\dagger$ is the unique matrix $G \in \mathbb{R}^{n\times m}$ that satisfies the matrix equations $AGA = A$, $GAG = G$, $(AG)^T = AG$ and $(GA)^T = GA$.

DEFINITION 1.7. (Definition 3.6, [10]) A positive proper splitting $A = U - V$ of $A \in \mathbb{R}^{m\times n}$ is a $B_\dagger$-splitting if $VU^\dagger \geq 0$, and Ax, $Ux \in \mathbb{R}^m_+ + N(A^T)$ and $x \in R(A^T) \Rightarrow x \geq 0$.

DEFINITION 1.8. (Definition 2.6, [11]) A positive proper splitting $A = U - V$ of $A \in \mathbb{R}^{m\times n}$ is a B_{row}-splitting if $VU^\dagger \geq 0$, and Ax, $Ux \geq 0$ and $x \in R(A^T)$ implies $x \geq 0$

DEFINITION 1.9. A positive proper splitting $A = U - V$ of $A \in \mathbb{R}^{m\times n}$ is a

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B_{weak}-splitting of A if $VU^\dagger \geq 0$ and $Ax \geq 0$, $Ux \geq 0 \Longrightarrow x \in \mathbb{R}^n_+ + N(A)$.

It is clear that if A has a $B_\dagger$-splitting, then the same splitting is also a B_{row}-splitting, which in turn is a B_{weak}-splitting. The condition $VU^\dagger \geq 0$ is an obvious extension of the condition $VU^{-1} \geq 0$ for the rectangular case. It can be shown that $VU^\dagger \geq 0$ implies $Ux \geq 0 \Longrightarrow Vx \geq 0$. It then follows that any B_{weak}-splitting of a matrix is a generalized B-splitting, while the converse is not true.

Let us now summarize the contents of the article. The next section gives additional preliminaries. In the third section, we prove the main results of the paper. The first important result viz., Theorem 3.3, is an extension of Theorem 1.1 for weak monotone matrices. The existence of a B_{weak}-splitting for a *class of matrices* is established in Theorem 3.4. In Theorem 3.5, we prove that any pair of matrices U and V giving rise to a splitting of a particular class of weak monotone matrices, satisfies a certain generalized eigenvalue property. Theorem 3.6 studies the converse of Theorem 3.5. Theorem 3.8 provides a decomposition for the Moore-Penrose inverse of weak monotone matrices and answers a partial converse in the affirmative.

2. Preliminary Notions. Let $A \in \mathbb{R}^{n\times n}$. If A is nonsingular, then $A^{-1} = A^\dagger$. Let K and L be complementary subspaces of $\mathbb{R}^n$, i.e., $K \oplus L = \mathbb{R}^n$. Then $P_{K,L}$ denotes the projection of $\mathbb{R}^n$ onto K along L. So, we have $P^2_{K,L} = P_{K,L}$, $R(P_{K,L}) = K$ and $N(P_{K,L}) = L$. If, in addition, $K \perp L$, then $P_{K,L}$ will be denoted by P_K. In such a case, we also have $P_K^T = P_K$. Some of the well known properties of $A^\dagger$ which will be frequently used in this paper are: $R(A^T) = R(A^\dagger)$; $N(A^T) = N(A^\dagger)$; and $AA^\dagger = P_{R(A)}$; $A^\dagger A = P_{R(A^T)}$. In particular, if $x \in R(A^T)$ then $x = A^\dagger Ax$. We refer to [2] for more details.

Let $A \in \mathbb{R}^{m\times n}$ with $r = \text{rank}(A) > 0$. Then a full rank factorization of A is a factorization $A = FG$ where $\text{rank}(F) = \text{rank}(G) = r$. Full rank factorizations have proven to be a useful tool in the study of generalized inverses. A full rank factorization $A = FG$ of a nonnegative matrix A is said to be nonnegative if in addition F and G are nonnegative.

We will also make use of the following results.

THEOREM 2.1. *(Theorem 3, [4]) Let $A \in \mathbb{R}^{m\times n}$. Then the following are equivalent.*

(a) *A is row monotone;*
(b) *$AX \geq 0$ implies $A^\dagger AX \geq 0$ for every X;*
(c) *There exists $Y \geq 0$, such that $A^\dagger A = YA$.*

The existence of a B_{row}-splitting was proved in [11]. We state this next, and give

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a short proof for the sake of completeness and ready reference.

THEOREM 2.2. *(Theorem 2.12, [11]) Suppose that A is row monotone and*

$$R(A) \cap \operatorname{int}(\mathbb{R}^m_+) \neq \phi \text{ for } A \in \mathbb{R}^{m\times n}.$$

Furthermore, if $A^\dagger A \geq 0$ then A possesses a B_{row}-splitting $A = U - V$ with $\rho(VU^\dagger) < 1$.

Proof. Let $p \in R(A) \cap \operatorname{int}(\mathbb{R}^m_+)$ and $q \in \operatorname{int}(\mathbb{R}^n_+)$. Set $E = pq^T \in \mathbb{R}^{m\times n}$. Let $y = A^\dagger p$. It can be shown that $y \neq 0$. For $\alpha > 0$, define $W = \frac{1}{\alpha + q^T A^\dagger p} EA^\dagger \in \mathbb{R}^{m\times m}$. Then it can be shown that the non-zero eigenvalue λ of W satisfies $0 < \lambda = \frac{q^T A^\dagger p}{\alpha + q^T A^\dagger p} < 1$. Hence $\rho(W) < 1$. So $(I-W)^{-1}$ exists and $(I-W)^{-1} = \sum_{k=0}^{\infty} W^k$. By choosing α and η such that $\frac{1}{\alpha} \geq \eta > \max|a_{ij}|$, where $A = (a_{ij})$, it can be shown that $(I-W)^{-1}A \geq 0$. Set $U = (I-W)^{-1}A$ and $V = WU$. Then $U \geq 0$. It follows that $R(A) = R(U)$ and $N(A) = N(U)$. It can be shown that $A^\dagger U \geq 0$ and so $V = WU = \frac{1}{\alpha + q^T A^\dagger p} EA^\dagger U \geq 0$. Thus, $A = (I-W)U = U - WU = U - V$ is a positive proper splitting. Also, $R(W^T) \subseteq R(A) = R(U)$ gives $UU^\dagger W^T = W^T$. So, $W = WUU^\dagger = VU^\dagger \geq 0$. Since A is row monotone, $Ax \geq 0$ and $x \in R(A^T)$ implies $x \geq 0$. So the second condition of B_{row}-splitting trivially holds. Finally, since $W = VU^\dagger$ it follows that $\rho(VU^\dagger) < 1$. ∎

LEMMA 2.3. *Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. The system $Ax = b$ has a solution if and only if $AA^\dagger b = b$. In that case, the general solution is given by $x = A^\dagger b + z$ for some $z \in N(A)$.*

The next result is part of the well known Perron-Frobenius theorem.

THEOREM 2.4. *Let $A \in \mathbb{R}^{n\times n}$. If $A \geq 0$, then*

(a) *A has a nonnegative real eigenvalue equal to its spectral radius.*
(b) *There exists a nonnegative eigenvector for its spectral radius.*

The next result is well known as Farkas' Lemma.

LEMMA 2.5. *Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Then, exactly one of the following two statements is true:*

(a) *$Ax = b$ and $x \geq 0$ has a solution for $x \in \mathbb{R}^n$.*
(b) *$A^T y \geq 0$ and $b^T y < 0$ has a solution for $y \in \mathbb{R}^m$.*

The next two results are finite dimensional versions of corresponding results which hold in Banach spaces.

THEOREM 2.6. *(Theorem 3.15, [15])*
Let $X \in \mathbb{R}^{n\times n}$ and $X \geq 0$. Then $\rho(X) < 1$ if and only if $(I-X)^{-1}$ exists and

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$$(I-X)^{-1}=\sum_{k=0}^{\infty}X^k \geq 0.$$

THEOREM 2.7. *(Theorem 25.4, [8])*
Suppose that $C,B \in \mathbb{R}^{n\times n}$ *with* $C \leq B$, B^{-1} *exists and* $B^{-1}\geq 0$. *Then* C^{-1} *exists and* $C^{-1}\geq 0$ *if and only if* $C\mathbb{R}^n_+ \cap \mathrm{int}(\mathbb{R}^n_+) \neq \phi$.

3. Main Results. In this section, we discuss various aspects of weak monotonicity and prove new results. Let us observe that if $A \in \mathbb{R}^{m\times n}$ is such that $A^\dagger \geq 0$, then A is weak monotone. Let us recall the following characterization for $A^\dagger \geq 0$, proved in Theorem 3.8, [10].

THEOREM 3.1. *Let* $A \in \mathbb{R}^{m\times n}$. *Consider the following statements.*

(a) $A^\dagger \geq 0$.
(b) $Ax \in \mathbb{R}^m_+ + N(A^T)$ *and* $x \in R(A^T) \Rightarrow x \geq 0$.
(c) $\mathbb{R}^m_+ \subseteq A\mathbb{R}^n_+ + N(A^T)$.
(d) *There exists* $x^0 \in \mathbb{R}^n_+$ *and* $z^0 \in N(A^T)$ *such that* $Ax^0+z^0 \in \mathrm{int}(\mathbb{R}^m_+)$.

Then (a) $\Leftrightarrow$ *(b)* $\Rightarrow$ *(c)* $\Rightarrow$ *(d). Assume further that* $A=U-V$ *is a* $B_\dagger$*-splitting. Then each of the above is equivalent to the following:*

(e) $\rho(VU^\dagger)<1$.

Our first main result is an extension of Thereom 3.1 to a *class* of weak monotone matrices. In proving this result, the following properties of a proper splitting will be used.

THEOREM 3.2. *(Theorem 1, [5]) Let* $A=U-V$ *be a proper splitting. Then*

(a) $AA^\dagger = UU^\dagger$, $A^\dagger A = U^\dagger U$, $VU^\dagger U = V$;
(b) $A=(I-VU^\dagger)U$;
(c) $I-VU^\dagger$ *is invertible; and*
(d) $A^\dagger = U^\dagger(I-VU^\dagger)^{-1}$.

We are now in a position to prove our first main result.

THEOREM 3.3. *Let* $A \in \mathbb{R}^{m\times n}$. *Suppose that* $\mathrm{int}(\mathbb{R}^m_+)\cap R(A) \neq \phi$. *Consider the following statements:*

(a) A *is weak-monotone;*
(b) $\mathbb{R}^m_+ \cap R(A) \subseteq A\mathbb{R}^n_+$;
(c) *There exists* $x^0 \geq 0$ *such that* $Ax^0 > 0$.

Then (a) $\Leftrightarrow$ *(b)* $\Rightarrow$ *(c).*
Assume further that $A=U-V$ *is* B_{weak}*-splitting. Then, each of the above statements*

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is equivalent to the following:

(d) $\rho(VU^{\dagger}) < 1$.

Proof. (a) $\Rightarrow$ (b): Let $p \in \mathbb{R}^m_+ \cap R(A)$. Then $p = Ax \geq 0$ for some $x \in \mathbb{R}^n$. Then $x \in \mathbb{R}^n_+ + N(A)$ so that $Ax \in A(\mathbb{R}^n_+ + N(A)) = A\mathbb{R}^n_+$. Thus $\mathbb{R}^m_+ \cap R(A) \subseteq A\mathbb{R}^n_+$.

(b) $\Rightarrow$ (a): Let $z = Ax \geq 0$. Then $z \in \mathbb{R}^m_+ \cap R(A)$. So there exists $y \in \mathbb{R}^n,\ y \geq 0$ such that $Ay = z = Ax$. This is weak monotonicity of A.

(b) $\Rightarrow$ (c): Let $u^0 \in int(\mathbb{R}^m_+) \cap R(A)$. Then there exists $x^0 \geq 0$ such that $u^0 = Ax^0$. Thus $Ax^0 > 0$.

(c) $\Rightarrow$ (d): Next, we suppose that A has a B_{weak} splitting $A = U - V$. By definition, $U \geq 0$, $V \geq 0$, $VU^{\dagger} \geq 0$, $R(A) = R(U)$ and $N(A) = N(U)$. Set $C = I - VU^{\dagger}$ and $B = I$. Then $C \leq B$, B^{-1} exists and $B^{-1} \geq 0$. Next we show that there exists a vector $w^0 \geq 0$ such that $Cw^0 > 0$.

By (c), there exists $x^0 \geq 0$ such that $Ax^0 > 0$. Set $w^0 = Ux^0$. Then $w^0 = Ax^0 + Vx^0$. Since $V \geq 0$ and $x^0 \geq 0$, $Vx^0 \geq 0$ which in turn implies that $w^0 \geq 0$. Also $Cw^0 = (I - VU^{\dagger})w^0 = (I - VU^{\dagger})Ux^0 = (U - V)x^0 = Ax^0 > 0$.
It now follows from Theorem 2.7 that C^{-1} exists and $(I - VU^{\dagger})^{-1} = C^{-1} \geq 0$. By Theorem 2.6, we then have $\rho(VU^{\dagger}) < 1$.

(d) $\Rightarrow$ (a): Once again, suppose that $A = U - V$ is a B_{weak}-splitting. Let $Ax \geq 0$. We show that $Ux \geq 0$. It would then follow from the definition of a B_{weak}-splitting that $x \in \mathbb{R}^n_+ + N(A)$. Since $\rho(VU^{\dagger}) < 1$ and $VU^{\dagger} \geq 0$, it follows from Theorem 2.6 that $I - VU^{\dagger}$ is invertible and that $(I - VU^{\dagger})^{-1} \geq 0$. Then $Ux = (I - VU^{\dagger})^{-1}Ax \geq 0$. Hence A is weak monotone. ∎

Theorem 3.3 is a generalization of Theorem 1.1 as explained below. Let $A \in \mathbb{R}^{n\times n}$ be a Z-matrix. Let us first observe that conditions (a), (b) (trivially) and (c) (guaranteeing that A is nonsingular) imply that $\text{int}(\mathbb{R}^m_+) \cap R(A) \neq \phi$. This ensures that the hypothesis of Theorem 3.3 holds. First suppose that A is nonsingular and $A^{-1} \geq 0$ (so that condition (a) of Theorem 1.1 holds). Then A is weak monotone. So, (c) of Theorem 3.3 holds, and there exists $x^0 \geq 0$ such that $Ax^0 > 0$. Then $x^0 = A^{-1}Ax^0 > 0$. Thus condition (b) of Theorem 1.1 holds. Next, if (b) of Theorem 1.1 holds, then (c) of Theorem 3.3 is applicable. Also, since A is a Z-matrix, A could be decomposed as $A = sI - B$ where $B \geq 0$ and $s \geq \rho(B)$. Clearly, this is a B_{weak}-splitting with $U = sI$ and $V = B$. So, condition (d) of Theorem 3.3 becomes $\rho(B(sI)^{-1}) < 1$, i.e., $\rho(B) < s$, which is condition (c) of Theorem 1.1. Finally, if (c) of Theorem 1.1 holds, then A is invertible and since A is weak monotone (due to the fact that A now has a B_{weak}-splitting with condition (c) of Theorem 3.3 being satisfied), it follows that $A^{-1} \geq 0$. This is condition (a) of Theorem 1.1.

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In view of the remark given earlier that any B_{weak}-splitting is a generalized B-splitting, it also follows that Theorem 1.5 is a particular case of Theorem 3.3.

In the next result, it is shown that a B_{weak}-splitting exists for those weak monotone matrices that have a nonnegative full rank factorization.

THEOREM 3.4. *Let $A \in \mathbb{R}^{m\times n}$ such that*

(a) *A is weak monotone,*
(b) *A has a nonnegative full rank factorization, and*
(c) *$R(A) \cap \text{int}(\mathbb{R}^m_+) \neq \phi$.*

Then A possesses a B_{weak}-splitting $A = U - V$ with $\rho(VU^\dagger) < 1$.

Proof. Let $A = FG$ be a nonnegative full rank factorization of A. Then F is monotone and hence row monotone. Also, $R(A) = R(FG) = R(F)$ so that,

$$R(F) \cap \text{int}(\mathbb{R}^m_+) \neq \phi$$

by (c). Since F is of full column rank, we have $F^\dagger F = I \geq 0$. By Theorem 2.2, F has a B_{row} splitting, which we denote by $F = X - Y$. Then $X \geq 0$, $Y \geq 0$, $R(X) = R(F)$, $N(X) = N(F)$ and $YX^\dagger \geq 0$. We also have $Fx \geq 0$, $Xx \geq 0$ and $x \in R(F^T) \Rightarrow x \geq 0$. Further, $\rho(YX^\dagger) < 1$. Set $U = XG$ and $V = YG$. Then $A = FG = (X - Y)G = XG - YG = U - V$.

We prove that $A = U - V$ is a B_{weak}-splitting. Clearly, $U \geq 0$ and $V \geq 0$. Also, since the rows of G are linearly independent, it follows that $R(U) = R(XG) = R(X) = R(F) = R(FG) = R(A)$. Further, $N(A) = N(G) \subseteq N(U)$. If $Ux = 0$ then $XGx = 0$, so that $Gx \in N(X) = N(F)$. Thus $Ax = FGx = 0$, so that $x \in N(A)$. Thus, $N(A) = N(U)$.

Next, we prove that $VU^\dagger \geq 0$. Recall that $F = X - Y$ is a B_{row}-splitting. So, rank F = rank X and X is of the same order as F. Thus, X is a full column rank matrix and hence $(XG)^\dagger = G^\dagger X^\dagger$. Now $VU^\dagger = YG(XG)^\dagger = YGG^\dagger X^\dagger = YX^\dagger \geq 0$. It follows that $\rho(VU^\dagger) < 1$. This completes the proof that $A = U - V$ is a B_{weak}-splitting. $\square$

The next result shows that for a subclass of weak monotone matrices, every positive proper splitting satisfies a specific generalized eigenvalue property.

THEOREM 3.5. *Let $0 \neq A \in \mathbb{R}^{m\times n}$ be weak monotone. Assume that $A^\dagger A \geq 0$. If $A = U - V$ is a positive proper splitting with $U \neq 0$, then there exist $0 \neq x \in \mathbb{R}^n_+ \cap R(A^T)$ and $\mu \in [0, 1)$ with $Vx = \mu Ux$.*

Proof. First we prove that $A^\dagger U \geq 0$. Let $x \geq 0$ and $y = A^\dagger Ux$. Then $AA^\dagger Ux = Ay$, i.e., $Ux = Ay$ (since $R(U) \subseteq R(A)$). This implies that $Ay \geq 0$, as $U \geq 0$.

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Since A is weak monotone, there exists $u \geq 0$ such that $Au = Ay = Ux$. Now, $y = A^{\dagger}Ux = A^{\dagger}Au \geq 0$, so that $A^{\dagger}U \geq 0$.

So, by the Perron-Frobenius theorem, there exist $\lambda = \rho(A^{\dagger}U) \geq 0$ and $0 \neq p \in \mathbb{R}^n_+$ such that $A^{\dagger}Up = \lambda p$. Suppose that $\lambda = 0$. Then $A^{\dagger}U$ is nilpotent. Let k be the least positive integer such that $(A^{\dagger}U)^k = 0$.

If $k = 1$, then $A^{\dagger}U = 0$ so that $U = AA^{\dagger}U = 0$, a contradiction. Hence $k \geq 2$. Set $T = (A^{\dagger}U)^{k-1}$. Then, $R(T) \subseteq R(A^{\dagger}) = R(A^T) = R(U^T)$. Also, for every $z \in \mathbb{R}^n$, we have $0 = A(A^{\dagger}U)^k z = AA^{\dagger}U(A^{\dagger}U)^{k-1}z = UTz$. This means that $R(T) \subseteq N(U)$. This can happen only if $0 = T = (A^{\dagger}U)^{k-1}$, contradicting the minimality of k. Thus $\lambda > 0$ and so $p \in R(A^T)$. Pre-multiplying $A^{\dagger}Up = \lambda p$ by A, we get $AA^{\dagger}Up = \lambda Ap$, i.e., $Up = \lambda Ap$. Thus, $Vp = \frac{\lambda-1}{\lambda}Up$.

If $\lambda < 1$, then $Vp \leq 0$. Since $V \geq 0$ and $p \geq 0$, this implies that $Vp = 0$. Consequently, $Up = 0$ so that $Ap = 0$. So, $p \in N(A) \cap R(A^T)$, a contradiction, since $p \neq 0$. Thus $\lambda \geq 1$. By setting $\mu = \frac{\lambda-1}{\lambda}$, the conclusion follows. ◻

The next result shows that a certain type of converse of Theorem 3.5 holds for nonnegative matrices. Specifically, if for every semi-positive pseudo subproper splitting (being more general than a positive splitting) of a nonnegative matrix A there exists a generalized eigenvalue, then A is weak monotone.

Theorem 3.6. *Let $0 \neq A \in \mathbb{R}^{m\times n}$ with $A \geq 0$. Suppose that for every semi-positive pseudo sub-proper splitting $A = U - V$, there exist $0 \neq y \in \mathbb{R}^n_+ \cap R(A^T)$ and $\mu \in [0, 1)$ such that $Vy = \mu Uy$. Then A is weak monotone.*

Proof. The proof is by contradiction. Suppose that A is not weak monotone. Then there exists x^0 such that $Ax^0 \geq 0$ and $x^0 \notin \mathbb{R}^n_+ + N(A)$. Set $p = Ax^0$. So $p \in R(A)$, $0 \neq p \in \mathbb{R}^m_+$ and $A^{\dagger}p = A^{\dagger}Ax^0$. If $A^{\dagger}Ax^0 \in \mathbb{R}^n_+ + N(A)$, then $AA^{\dagger}Ax^0 \in A\mathbb{R}^n_+$; that is $Ax^0 = Az$ for some $z \geq 0$. This implies that $x^0 = z + w$ where $w \in N(A)$, a contradiction. Hence $A^{\dagger}Ax^0 \notin \mathbb{R}^n_+ + N(A)$. Since $\mathbb{R}^n_+ + N(A)$ is closed, we can find $r > 0$ such that the closed ball $B(A^{\dagger}p; r)$ does not intersect $\mathbb{R}^n_+ + N(A)$. Then for any $\alpha > 0$, $\alpha B(A^{\dagger}p; r) \cap (\mathbb{R}^n_+ + N(A)) = \phi$.

Consider a decomposition of the form $A = U_1 - V_1$ with $U_1 \geq 0$ and $R(U_1) \subseteq R(A)$. (such a decomposition does exist, as A could be taken to be U_1.) Define the operator $W : \mathbb{R}^n \to \mathbb{R}^m$ by

$$W(x) = l(e^T x)p,$$

where $e = (1, 1, \ldots, 1)^T \in \mathbb{R}^n$ and l is a fixed number such that $l > \frac{\|A^{\dagger}\| \, \|U_1\|}{r}$, where the operator norm used here is induced by the 1-norm (for vectors) defined for $z \in \mathbb{R}^n$ by $\|z\| = \|z\|_1 = \sum_{i=1}^{n} |z_i|$.

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Clearly, $W \geq 0$ and $R(W) \subseteq R(A)$, since $p \in R(A)$. Setting $U = U_1 + W$ and $V = V_1 + W$, we have $U \geq 0$, $R(U) \subseteq R(A)$ and $A = U - V$. By the hypotheses, there exist $0 \neq y^0 \in \mathbb{R}^n_+ \cap R(A^T)$ and $\mu \in [0, 1)$ such that $Vy^0 = \mu U y^0$. Without loss of generality we may assume that $e^T y^0 = 1$ (by replacing y^0 by $\frac{1}{e^T y^0} y^0$, if necessary) so that $W(y^0) = lp$. Note that $\|y^0\| \leq 1$. Then

$$Ay^0 = Uy^0 - Vy^0 = (1 - \mu)Uy^0.$$

Setting $z = Uy^0$, we have $A^\dagger z = A^\dagger U y^0 = A^\dagger \left(\frac{1}{1-\mu} A y^0\right) = \frac{1}{1-\mu} y^0 \in \mathbb{R}^n_+ \subseteq \mathbb{R}^n_+ + N(A)$.

On the other hand, $A^\dagger z = A^\dagger U y^0 = A^\dagger U_1 y^0 + l A^\dagger p$. Thus, $\|A^\dagger z - l A^\dagger p\| = \|A^\dagger U_1 y^0\| \leq \|A^\dagger\| \, \|U_1\| \|y^0\| < rl$. Thus $A^\dagger z \in B(lA^\dagger p; rl) = lB(A^\dagger p; r)$, so that $A^\dagger z \notin \mathbb{R}^n_+ + N(A)$, a contradiction. □

Remark 3.7. Theorem 1.6 includes the result that under certain conditions on a weak monotone matrix A, a generalized B-splitting for A exists. In a similar spirit, Theorem 3.4 provides conditions under which a weak monotone matrix admits a B_{weak}-splitting.

In the next result the concept of the dual cone of a convex cone will be used. A subset K of $\mathbb{R}^n$ is a cone if $x + y \in K$ for all $x, y \in K$ and $\alpha x \in K$ for all $\alpha \geq 0$. For a cone $K \subseteq \mathbb{R}^n$, let $K^* = \{y \in \mathbb{R}^n : \langle x, y\rangle \geq 0 \text{ for all } x \in K\}$. K^* is the dual cone of K. If $K^{**} = (K^*)^*$, then $K^{**} = \mathrm{cl}K$, where $\mathrm{cl}K$ denotes the closure of K. If $K = \mathbb{R}^n_+$, then $K^{**} = K^* = \mathbb{R}^n_+$. If $K = \mathbb{R}^n_+ \cap M$ for some subspace M of $\mathbb{R}^n$, then $K^* = \mathbb{R}^n_+ + M^\perp$, where $M^\perp$ denotes the orthogonal complement of M in $\mathbb{R}^n$. It follows that in this case also, $K^{**} = K$. For details, we refer to [1].

Theorem 3.8. *Let $A \in \mathbb{R}^{m\times n}$. Suppose that $A^\dagger$ has a decomposition $A^\dagger = K - L$, where $K \geq 0$ and $R(A) \subseteq N(L)$. Then A is weak monotone. The converse holds if $A^\dagger A \geq 0$.*

Proof. Suppose that $A^\dagger = K - L$, where $K \geq 0$ and $R(A) \subseteq N(L)$. Let $y = Ax \geq 0$. Then, $x = A^\dagger y + z$, for some $z \in N(A)$. Now, $A^\dagger y = A^\dagger(Ax) = (K - L)Ax = KAx - LAx = KAx \geq 0$, since $K \geq 0$. Thus, $x \in \mathbb{R}^n_+ + N(A)$, proving that A is weak monotone.

Conversely, suppose that A is weak monotone. Let $x \in \mathbb{R}^n$ be such that $Ax \geq 0$. Then $x \in \mathbb{R}^n_+ + N(A)$. Since $\mathbb{R}^n_+ + N(A) = (\mathbb{R}^n_+ \cap R(A^T))^*$, it follows that $Ax \geq 0 \Rightarrow \langle x, w\rangle \geq 0$ for all $w \in \mathbb{R}^n_+ \cap R(A^T)$. By Farkas' Theorem, the system $A^T u = w$, $u \geq 0$ has a solution for all $w \in \mathbb{R}^n_+ \cap R(A^T)$. This means that the system $A^T u = A^\dagger z$, $u \geq 0$ has a solution for all $z \in \mathbb{R}^m$ whenever $A^\dagger z \in \mathbb{R}^n_+$.

Let a^i denote the i-th column of A for $i = 1, 2, \ldots, n$. Then, $A^\dagger a^i \geq 0$ holds for $i = 1, 2, \ldots, n$. Then the system $A^T u = A^\dagger a^i, u \geq 0$ has a solution for each

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$i = 1, 2, \ldots, n$. Let x^i be a solution of this system for each i. Let X be the matrix whose i-th column is x^i. Then $X \geq 0$ and $A^T X = A^\dagger A$, so that $X^T A = A^\dagger A$. Thus, $(X^T - A^\dagger)A = 0$. Now, set $K = X^T$ and $L = X^T - A^\dagger$. Then $K \geq 0$, $A^\dagger = K - L$ and $R(A) \subseteq N(L)$. $\square$

Remark 3.9. *We show by an example that the condition* $A^\dagger A \geq 0$ *is indispensable in the above result. Let* $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$. *It can be shown that* A *is weak monotone. Then* $A^\dagger = \frac{1}{9}\begin{pmatrix} 5 & -4 & 1 \\ -4 & 5 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ *so that* $A^\dagger A \not\geq 0$. *Suppose that* $A^\dagger = K - L$, *with* $K \geq 0$ *and* $R(A) \subseteq N(L)$. *Then* $A^\dagger A = KA - LA = KA \geq 0$, *a contradiction.*

4. Conclusions. In this article, a classical notion of splittings of matrices, namely a regular splitting employed in iterative methods for obtaining the unique solution of nonsingular linear systems, is extended to include singular matrices belonging to a very general class. This splitting is a B_{weak}-splitting. This concept is motivated by the idea of weak monotonicity of matrices and derives its name from there. The important result that has been shown here is a characterization of weak monotone matrices (similar to the one for M-matrices), which includes a statement on the existence of a B_{weak}-splitting of the matrix concerned. Results discussing the relationship between a B_{weak}-splitting and a certain generalized eigenvalue property being satisfied by the components of the splitting are also presented. Finally, a connection between weak monotonicity of a matrix and the existence of a splitting of a particular type for the Moore-Penrose inverse of the matrix, is established.

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