

ON GRADED MATRIX HOM-ALGEBRAS*

MARÍA J. ARAGÓN PERIÑÁN[†] AND ANTONIO J. CALDERÓN MARTÍN[‡]

Abstract. Consider an (associative) matrix algebra $\mathcal{M}_I(\mathcal{R})$ graded by means of an abelian group G , and a graded automorphism ϕ on $\mathcal{M}_I(\mathcal{R})$. By defining a new product by $x \star y := \phi(x)\phi(y)$ on $\mathcal{M}_I(\mathcal{R})$, $(\mathcal{M}_I(\mathcal{R}), \star)$ becomes a hom-associative algebra graded by a twist of G . The structure of $(\mathcal{M}_I(\mathcal{R}), \star)$ is studied, by showing that $\mathcal{M}_I(\mathcal{R})$ is of the form

$$\mathcal{M}_I(\mathcal{R}) = U + \sum_j \mathfrak{I}_j$$

with U an $\mathcal{R}$ -submodule of the 0-homogeneous component and any $\mathfrak{I}_j$ a well described graded ideal of $\mathcal{M}_I(\mathcal{R})$, satisfying $\mathfrak{I}_j \star \mathfrak{I}_k = 0$ if $j \neq k$. Under certain conditions, the graded simplicity of an arbitrary graded hom-associative algebra $\mathcal{M}$ is characterized and it is shown that $\mathcal{M}$ is the direct sum of the family of its simple graded ideals.

Key words. Hom-associative algebra, Matrix algebra, Graded algebra, Structure theory.

AMS subject classifications. 16W50, 16S35, 16S80.

1. Introduction and first definitions. We begin by recalling the fundamental concepts of hom-associative algebras and graded algebras theory. We note that, under otherwise stated, our algebras will be consider over an arbitrary commutative ring of scalars $\mathcal{R}$.

DEFINITION 1.1. An $\mathcal{R}$ -module $\mathcal{M}$ endowed with a bilinear product

$$\begin{aligned} \star: \mathcal{M} \times \mathcal{M} &\rightarrow \mathcal{M} \\ (x, y) &\mapsto x \star y \end{aligned}$$

is called a *hom-associative algebra* if there exists a module homomorphism

$$\phi: \mathcal{M} \rightarrow \mathcal{M},$$

called the *twisting map* of $\mathcal{M}$, such that the following identity holds for all $x, y, z \in \mathcal{M}$:

$$(x \star y) \star \phi(z) = \phi(x) \star (y \star z).$$

*Received by the editors on November 11, 2011. Accepted for publication on May 25, 2012.
Handling Editor: Shmuel Friedland.

[†]Departamento de Matemáticas, Universidad de Cádiz, 11510 Puerto Real, Cádiz, Spain (mariajesus.aragonperin@alum.uca.es).

[‡]Universidad de Cádiz, 11510 Puerto Real, Cádiz, Spain (ajesus.calderon@uca.es). Supported by the PCI of the UCA ‘Teoría de Lie y Teoría de Espacios de Banach’, by the PAI with project numbers FQM298, FQM7156 and by the project of the Spanish Ministerio de Educación y Ciencia MTM2012-15223.

The concept of hom-algebra, (over a base field $\mathbb{K}$), was introduced in [22], where a notion of hom-Lie algebra appeared in the study of deformations of Witt and Virasoro algebras. Later, this notion was transferred to other categories like the ones of hom-associative algebras, hom-alternative algebras, hom-Leibniz algebras, hom-Lie superalgebras, etc. (see [1], [11], [26], [27], [28], [30], [24], [35]). Hom-associative algebras has been recently studied by different authors like Fregier, Gohr, Makhlouf, Silvestrov or Yau who have been interested, in particular, in establishing conditions under which a hom-associative algebra is a twist of an associative algebra or in developing a deformation and homology theory of hom-associative algebras, ([17], [18], [21], [29], [34]). A *subalgebra* of a hom-associative algebra $(\mathcal{M}, \star)$ is an $\mathcal{R}$ -submodule $\mathcal{N}$ of $\mathcal{M}$ closed under $\star$ and the twisting map ϕ . A subalgebra $\mathfrak{I}$ of $(\mathcal{M}, \star)$ is called an *ideal* if $\mathfrak{I} \star \mathcal{M} \subset \mathfrak{I}$ and $\mathcal{M} \star \mathfrak{I} \subset \mathfrak{I}$. Finally, it should be noted that in all of the above references the base ring of scalars $\mathcal{R}$ is always a field $\mathbb{K}$. So the concept of hom-associative algebra given in Definition 1.1 is, in this sense, more general.

Now consider a (non-necessarily associative) algebra $(A, \cdot)$. That is, A is just an $\mathcal{R}$ -module endowed with a bilinear product $\cdot : A \times A \rightarrow A$, $(x, y) \mapsto x \cdot y$.

DEFINITION 1.2. Let $(A, \cdot)$ be a (non-necessarily associative) algebra and $(G, +)$ a set together a binary operation $+$. It is said that A is a *graded algebra*, by means of G , if

$$A = \bigoplus_{g \in G} A_g$$

where any A_g is an $\mathcal{R}$ -submodule satisfying $A_g \cdot A_h \subset A_{g+h}$ for any $h \in G$.

The study of gradings on different classes of algebras has been remarkable in recent years, especially those gradings in which $(G, +)$ is an abelian group, (see [6, 14, 23, 31, 32]). In particular, graded matrix algebras have been considered in [3, 5, 9, 13, 15, 20, 25], not only for the interest by themselves but also because we can derive from them many examples of graded Lie algebras, which play an important role in the theory of strings, color supergravity, Walsh functions or electroweak interactions [10, 12, 33].

Consider the set $\mathcal{M}_I(\mathcal{R})$ of all of the $I \times I$ -matrices over an arbitrary commutative ring $\mathcal{R}$ with finitely many nonzero entries, endowed with its natural structure of associative algebra (whose product will be denoted by juxtaposition), and

$$\phi : \mathcal{M}_I(\mathcal{R}) \rightarrow \mathcal{M}_I(\mathcal{R})$$

an (associative) algebras automorphism. Then, if we endow the $\mathcal{R}$ -module $\mathcal{M}_I(\mathcal{R})$

with a new product

$$x \star y := \phi(x)\phi(y)$$

for $x, y \in \mathcal{M}_I(\mathcal{R})$, we have that

$$(\mathcal{M}_I(\mathcal{R}), \star)$$

is a hom-associative algebra, which is also called a *twist* of $\mathcal{M}_I(\mathcal{R})$ by ϕ , (see [18, 21]). Observe that, as consequence, we also have

$$(1.1) \quad \phi(x \star y) = \phi(x) \star \phi(y)$$

for $x, y \in \mathcal{M}_I(\mathcal{R})$.

Now suppose that the associative algebra $\mathcal{M}_I(\mathcal{R})$ is graded by means of an abelian group $(G, +)$, $\mathcal{M}_I(\mathcal{R}) = \bigoplus_{g \in G} \mathcal{M}_I(\mathcal{R})_g$, and that the algebra automorphism ϕ of $\mathcal{M}_I(\mathcal{R})$ is graded in the sense that there exists a group automorphism $\sigma : G \rightarrow G$ such

$$(1.2) \quad \phi(\mathcal{M}_I(\mathcal{R})_g) \subset \mathcal{M}_I(\mathcal{R})_{\sigma(g)}$$

for each $g \in G$, (see [25]). Taking into account equation (1.2), it is easy to check that $(\mathcal{M}_I(\mathcal{R}), \star)$ becomes a hom-associative algebra graded by the *twist* $(G, +_\sigma)$ of $(G, +)$, (the set G endowed with the binary operation $g_1 +_\sigma g_2 := \sigma(g_1) + \sigma(g_2)$). That is, we can express $(\mathcal{M}_I(\mathcal{R}), \star)$ as the direct sum of the $\mathcal{R}$ -submodules

$$\mathcal{M}_I(\mathcal{R}) = \bigoplus_{g \in G} \mathcal{M}_I(\mathcal{R})_g,$$

in such a way that

$$\mathcal{M}_I(\mathcal{R})_{g_1} \star \mathcal{M}_I(\mathcal{R})_{g_2} \subset \mathcal{M}_I(\mathcal{R})_{g_1 +_\sigma g_2}$$

for $g_1, g_2 \in G$.

The present paper is devoted to the study of the structure of the graded, by means of $(G, +_\sigma)$, hom-associative algebra $(\mathcal{M}_I(\mathcal{R}), \star)$. In §2 we develop connections in the support techniques which becomes the main tool in our study of graded matrix hom-algebras. In §3 we apply the machinery introduced in the previous section to show that $\mathcal{M}_I(\mathcal{R})$ is of the form $\mathcal{M}_I(\mathcal{R}) = U + \sum_j \mathfrak{I}_j$ with U an $\mathcal{R}$ -submodule of the 0-homogeneous component and each $\mathfrak{I}_j$ a well described graded ideal of $\mathcal{M}_I(\mathcal{R})$, satisfying $\mathfrak{I}_j \star \mathfrak{I}_k = 0$ if $j \neq k$. Finally, in §4, and under certain conditions, the graded simplicity of an arbitrary graded hom-associative algebra $\mathcal{M}$ is characterized and it

is shown that $\mathcal{M}$ is the direct sum of the family of its simple graded ideals. These results extend the ones for graded associative algebras in [5].

As it is usual in the theory of graded algebras, the regularity concepts will be understood in the graded sense. That is, a *graded ideal* of the hom-associative algebra $(\mathcal{M}_I(\mathcal{R}), \star)$ graded as above by $\mathcal{M}_I(\mathcal{R}) = \bigoplus_{g \in G} \mathcal{M}_I(\mathcal{R})_g$, is an $\mathcal{R}$ -submodule $\mathfrak{I}$ of $\mathcal{M}_I(\mathcal{R})$ satisfying $\mathfrak{I} \star \mathcal{M}_I(\mathcal{R}) + \mathcal{M}_I(\mathcal{R}) \star \mathfrak{I} \subset \mathfrak{I}$ and $\phi(\mathfrak{I}) = \mathfrak{I}$, and such that splits as $\mathfrak{I} = \bigoplus_{g \in G} \mathfrak{I}_g$ with each $\mathfrak{I}_g = \mathfrak{I} \cap \mathcal{M}_I(\mathcal{R})_g$. The graded hom-associative algebra $(\mathcal{M}_I(\mathcal{R}), \star)$ will be called *graded simple* if $\mathcal{M}_I(\mathcal{R}) \star \mathcal{M}_I(\mathcal{R}) \neq 0$ and its only graded ideals are $\{0\}$ and $\mathcal{M}_I(\mathcal{R})$.

Throughout the paper, $\mathbb{N}$ will denote the set of non-negative integers and $\mathbb{Z}$ the set of integers.

2. Connections in the support techniques. We recall that $(\mathcal{M}_I(\mathcal{R}), \star)$ denotes the twist by the automorphism ϕ of the associative matrix algebra $\mathcal{M}_I(\mathcal{R})$. That is, the hom-associative algebra with product

$$x \star y := \phi(x)\phi(y)$$

for $x, y \in \mathcal{M}_I(\mathcal{R})$, graded by means of $(G, +_\sigma)$, (the twist by an automorphism σ of $(G, +)$, that is, $g_1 +_\sigma g_2 := \sigma(g_1) + \sigma(g_2)$ for $g_1, g_2 \in G$). That means that we have an associative algebras automorphism $\phi : \mathcal{M}_I(\mathcal{R}) \rightarrow \mathcal{M}_I(\mathcal{R})$, an abelian group $(G, +)$ and a groups automorphism $\sigma : G \rightarrow G$ such that we can decompose

$$(2.1) \quad \mathcal{M}_I(\mathcal{R}) = \bigoplus_{g \in G} \mathcal{M}_I(\mathcal{R})_g,$$

as the direct sum of $\mathcal{R}$ -submodules satisfying $\phi(\mathcal{M}_I(\mathcal{R})_g) \subset \mathcal{M}_I(\mathcal{R})_{\sigma(g)}$, actually the direct sum decomposition (2.1) lets us assert

$$(2.2) \quad \phi(\mathcal{M}_I(\mathcal{R})_g) = \mathcal{M}_I(\mathcal{R})_{\sigma(g)},$$

and

$$(2.3) \quad \mathcal{M}_I(\mathcal{R})_g \star \mathcal{M}_I(\mathcal{R})_{g'} \subset \mathcal{M}_I(\mathcal{R})_{\sigma(g) + \sigma(g')}$$

for $g, g' \in G$.

DEFINITION 2.1. We call the *support* of the grading (2.1) the set

$$\Sigma := \{g \in G \setminus 0 : \mathcal{M}_I(\mathcal{R})_g \neq 0\}.$$

The support of the grading is called *symmetric* if $g \in \Sigma$ implies $-g \in \Sigma$.

REMARK 2.2. Observe that equation (2.2), lets us assert that if $g \in \Sigma$, then

$$\{\sigma^n(g) : n \in \mathbb{N}\} \subset \Sigma.$$

In this section we are going to develop connections in the support of the grading techniques as the main tool to the study of the structure of the graded hom-associative algebra $(\mathcal{M}_I(\mathcal{R}), \star)$. In the following, we will suppose that the grading of $(\mathcal{M}_I(\mathcal{R}), \star)$ has a symmetric support Σ and will denote by

$$\mathcal{M}_I(\mathcal{R}) = \bigoplus_{g \in G} \mathcal{M}_I(\mathcal{R})_g = \mathcal{M}_I(\mathcal{R})_0 \oplus \left(\bigoplus_{g \in \Sigma} \mathcal{M}_I(\mathcal{R})_g \right)$$

the corresponding decomposition.

DEFINITION 2.3. Let g and g' be two elements in Σ . We shall say that g is *connected* to g' if there exists $\{g_1, g_2, \dots, g_k\} \subset \Sigma$ such that:

If $k = 1$,

1. $g_1 \in \{\sigma^n(g) : n \in \mathbb{N}\} \cap \{\pm\sigma^m(g') : m \in \mathbb{N}\}$.

If $k \geq 2$,

1. $g_1 \in \{\sigma^n(g) : n \in \mathbb{N}\}$.
2. $\sigma(g_1) + \sigma(g_2) \in \Sigma$
 $\sigma^2(g_1) + \sigma^2(g_2) + \sigma(g_3) \in \Sigma$
 $\sigma^3(g_1) + \sigma^3(g_2) + \sigma^2(g_3) + \sigma(g_4) \in \Sigma$
 $\dots\dots\dots$
 $\sigma^i(g_1) + \sigma^i(g_2) + \sigma^{i-1}(g_3) + \dots + \sigma(g_{i+1}) \in \Sigma$
 $\dots\dots\dots$
 $\sigma^{k-2}(g_1) + \sigma^{k-2}(g_2) + \sigma^{k-3}(g_3) + \dots + \sigma^{k-i}(g_i) + \dots + \sigma(g_{k-1}) \in \Sigma$
3. $\sigma^{k-1}(g_1) + \sigma^{k-1}(g_2) + \sigma^{k-2}(g_3) + \dots + \sigma^{k-i+1}(g_i) + \dots + \sigma(g_k) \in \{\pm\sigma^m(g') : m \in \mathbb{N}\}$.

We shall also say that $\{g_1, \dots, g_k\}$ is a *connection* from g to g' .

We note that the concept of connection from g to g' given in Definition 2.3 for the case $k = 1$ is equivalent to the fact $g' = \epsilon\sigma^z(g)$ for some $z \in \mathbb{Z}$ and $\epsilon \in \{\pm\}$.

Our next goal is to show that the connection relation is an equivalence relation. First, we need to state a series of preliminary results.

LEMMA 2.4. For each $g \in \Sigma$, we have that $\sigma^p(g)$ is connected to $\epsilon\sigma^q(g)$ for all $p, q \in \mathbb{N}$ and $\epsilon \in \{\pm\}$.

Proof. Taking into account Remark 2.2, we have that $\sigma^p(g), \sigma^q(g) \in \Sigma$. Let $r = \max\{p, q\}$ be, then we have that $\{\sigma^r(g)\}$ is a connection from $\sigma^p(g)$ to $\epsilon\sigma^q(g)$. $\square$

LEMMA 2.5. Let $\{g_1, \dots, g_k\}$ be a connection from g to g' satisfying $g_1 = \sigma^n(g)$, $n \in \mathbb{N}$. Then for all $r \in \mathbb{N}$ such that $r \geq n$, there exists a connection $\{\bar{g}_1, \dots, \bar{g}_k\}$ from g to g' such that $\bar{g}_1 = \sigma^r(g)$.

Proof. By Remark 2.2 we have $\{\sigma^{r-n}(g_1), \dots, \sigma^{r-n}(g_k)\} \subset \Sigma$. By defining $\bar{g}_i :=$

$\sigma^{r-n}(g_i)$, $i = 1, \dots, k$, Remark 2.2 lets us easily verify that $\{\bar{g}_1, \dots, \bar{g}_k\}$ is a connection from g to g' which clearly satisfies $\bar{g}_1 = \sigma^{r-n}(\sigma^n(g)) = \sigma^r(g)$. $\square$

LEMMA 2.6. *Let $\{g_1, \dots, g_k\}$ be a connection from g to g' satisfying $g_1 = \epsilon\sigma^m(g')$ in case $k = 1$ or*

$$\sigma^{k-1}(g_1) + \sigma^{k-1}(g_2) + \sigma^{k-2}(g_3) + \dots + \sigma(g_k) = \epsilon\sigma^m(g')$$

in case $k \geq 2$, with $m \in \mathbb{N}$ and $\epsilon \in \{\pm\}$. Then for each $r \in \mathbb{N}$ such that $r \geq m$, there exists a connection $\{\bar{g}_1, \dots, \bar{g}_k\}$ from g to g' such that $\bar{g}_1 = \epsilon\sigma^r(g')$ in case $k = 1$ or

$$\sigma^{k-1}(\bar{g}_1) + \sigma^{k-1}(\bar{g}_2) + \sigma^{k-2}(\bar{g}_3) + \dots + \sigma(\bar{g}_k) = \epsilon\sigma^r(g')$$

in case $k \geq 2$.

Proof. Remark 2.2 lets us assert that

$$\{\sigma^{r-m}(g_1), \dots, \sigma^{r-m}(g_k)\} \subset \Sigma.$$

By defining $\bar{g}_i := \sigma^{r-m}(g_i)$, $i = 1, \dots, k$, Remark 2.2 gives us that $\{\bar{g}_1, \dots, \bar{g}_k\}$ is a connection from g to g' . It is clear that $\bar{g}_1 = \epsilon\sigma^r(g')$ in case $k = 1$, and also $\sigma^{k-1}(\bar{g}_1) + \sigma^{k-1}(\bar{g}_2) + \sigma^{k-2}(\bar{g}_3) + \dots + \sigma(\bar{g}_k) = \sigma^{r-m}(\sigma^{k-1}(g_1) + \sigma^{k-1}(g_2) + \sigma^{k-2}(g_3) + \dots + \sigma(g_k)) = \epsilon\sigma^r(g')$ in case $k \geq 2$. $\square$

PROPOSITION 2.7. *The relation $\sim$ in Σ defined by $g \sim g'$ if and only if g is connected to g' is an equivalence relation.*

Proof. For each $g \in \Sigma$, Lemma 2.4 gives us $g \sim g$ and so $\sim$ is reflexive.

Let us see the symmetric character of $\sim$. If $g \sim g'$, there exists a connection

$$(2.4) \quad \{g_1, g_2, g_3, \dots, g_{k-1}, g_k\} \subset \Sigma$$

from g to g' .

If $k = 1$, we have $g_1 = \sigma^n(g)$ and $g_1 = \epsilon\sigma^m(g')$ with $n, m \in \mathbb{N}$ and $\epsilon \in \{\pm\}$. From here, it is clear that $\{\epsilon g_1\}$ is a connection from g' to g and so $g' \sim g$.

If $k \geq 2$, we have that the connection (2.4) satisfies conditions 1, 2 and 3 of Definition 2.3. Observe that condition 3 let us distinguish two possibilities. In the first one

$$(2.5) \quad \sigma^{k-1}(g_1) + \sigma^{k-1}(g_2) + \sigma^{k-2}(g_3) + \dots + \sigma^i(g_{k-i+1}) + \dots + \sigma(g_k) = \sigma^m(g'),$$

and in the second one

$$(2.6) \quad \sigma^{k-1}(g_1) + \sigma^{k-1}(g_2) + \sigma^{k-2}(g_3) + \dots + \sigma^i(g_{k-i+1}) + \dots + \sigma(g_k) = -\sigma^m(g')$$

for some $m \in \mathbb{N}$.

Suppose we have the first possibility (2.5). By Remark 2.2 and the symmetry of Σ , we can consider the set

$$(2.7) \quad \{\sigma^m(g'), -\sigma(g_k), -\sigma^3(g_{k-1}), -\sigma^5(g_{k-2}), \dots, -\sigma^{2i+1}(g_{k-i}), \dots, -\sigma^{2k-3}(g_2)\}$$

$$\subset \Sigma.$$

Let us show that this set is a connection from g' to g . It is clear that (2.7) satisfies condition 1 of Definition 2.3, so let us verify that the set (2.7) satisfies condition 2. We have

$$\sigma(\sigma^m(g')) - \sigma(\sigma(g_k)) = \sigma(\sigma^m(g')) - \sigma(g_k) =$$

$$\sigma(\sigma^{k-1}(g_1) + \sigma^{k-1}(g_2) + \sigma^{k-2}(g_3) + \dots + \sigma^{k-i+1}(g_i) + \dots + \sigma^2(g_{k-1})),$$

where the last equality is a consequence of equation (2.5), and so

$$\sigma(\sigma^m(g')) - \sigma(\sigma(g_k)) =$$

$$\sigma^2(\sigma^{k-2}(g_1) + \sigma^{k-2}(g_2) + \sigma^{k-3}(g_3) + \dots + \sigma^{k-i}(g_i) + \dots + \sigma(g_{k-1})).$$

Since $\sigma^{k-2}(g_1) + \sigma^{k-2}(g_2) + \sigma^{k-3}(g_3) + \dots + \sigma^{k-i}(g_i) + \dots + \sigma(g_{k-1}) \in \Sigma$ by condition 2 of Definition 2.3 applied to the connection (2.4), Remark 2.2 lets us conclude

$$\sigma(\sigma^m(g')) - \sigma(\sigma(g_k)) \in \Sigma.$$

For each $1 \leq i \leq k-2$ we also have that,

$$\sigma^i(\sigma^m(g')) - \sigma^i(\sigma(g_k)) - \sigma^{i-1}(\sigma^3(g_{k-1})) - \dots - \sigma(\sigma^{2i-1}(g_{k-(i-1)})) =$$

$$\sigma^i(\sigma^m(g')) - \sigma(g_k) - \sigma^2(g_{k-1}) - \dots - \sigma^i(g_{k-(i-1)}) =$$

$$\sigma^i(\sigma^{k-1}(g_1) + \sigma^{k-1}(g_2) + \dots + \sigma^{i+1}(g_{k-i})),$$

with the last equality being a consequence of equation (2.5). From here,

$$\sigma^i(\sigma^m(g')) - \sigma^i(\sigma(g_k)) - \sigma^{i-1}(\sigma^3(g_{k-1})) - \dots - \sigma(\sigma^{2i-1}(g_{k-(i-1)})) =$$

$$\sigma^{2i}(\sigma^{k-1-i}(g_1) + \sigma^{k-1-i}(g_2) + \dots + \sigma(g_{k-i})).$$

Taking into account that, by condition 2 of Definition 2.3 applied to (2.4),

$$\sigma^{k-1-i}(g_1) + \sigma^{k-1-i}(g_2) + \cdots + \sigma(g_{k-i}) \in \Sigma,$$

we get as consequence of Remark 2.2 that

$$\sigma^i(\sigma^m(g')) - \sigma^i(\sigma(g_k)) - \sigma^{i-1}(\sigma^3(g_{k-1})) - \cdots - \sigma(\sigma^{2i-1}(g_{k-(i-1)})) \in \Sigma.$$

We have showed that the set (2.7) satisfies condition 2 of Definition 2.3. It just remains to prove that this set also satisfies condition 3 of this definition. We have as above that

$$\begin{aligned} \sigma^{k-1}(\sigma^m(g')) - \sigma^{k-1}(\sigma(g_k)) - \sigma^{k-2}(\sigma^3(g_{k-1})) - \cdots - \sigma(\sigma^{2k-3}(g_2)) = \\ \sigma^{k-1}(\sigma^m(g') - \sigma(g_k) - \sigma^2(g_{k-1}) - \cdots - \sigma^{k-1}(g_2)) = \\ \sigma^{k-1}(\sigma^{k-1}(g_1)). \end{aligned}$$

Condition 1 of Definition 2.3 applied to the connection (2.4) gives us that $g_1 = \sigma^n(g)$ for some $n \in \mathbb{N}$ and so

$$\begin{aligned} \sigma^{k-1}(\sigma^m(g')) - \sigma^{k-1}(\sigma(g_k)) - \sigma^{k-2}(\sigma^3(g_{k-1})) - \cdots - \sigma(\sigma^{2k-3}(g_2)) = \\ \sigma^{2k+n-2}(g) \in \{\sigma^h(g) : h \in \mathbb{N}\}. \end{aligned}$$

We have showed that the set (2.7) is actually a connection from g' to g .

Now suppose we are in the second possibility given by equation (2.6). Then we can prove as in the above first possibility, given by equation (2.5), that

$$\{\sigma^m(g'), \sigma(g_k), \sigma^3(g_{k-1}), \sigma^5(g_{k-2}), \dots, \sigma^{2i+1}(g_{k-i}), \dots, \sigma^{2k-3}(g_2)\}$$

is a connection from g' to g . We conclude $g' \sim g$ and so the relation $\sim$ is symmetric.

Finally, let us verify that $\sim$ is transitive. Suppose $g \sim g'$ and $g' \sim g''$, and write $\{g_1, \dots, g_k\}$ for a connection from g to g' and $\{h_1, \dots, h_p\}$ for a connection from g' to g'' . From here, we have

$$(2.8) \quad g_1 = \epsilon \sigma^m(g') \text{ if } k = 1$$

or

$$(2.9) \quad \sigma^{k-1}(g_1) + \sigma^{k-1}(g_2) + \sigma^{k-2}(g_3) + \cdots + \sigma(g_k) = \epsilon \sigma^m(g') \text{ if } k \geq 2,$$

for some $m \in \mathbb{N}$, $\epsilon \in \{\pm\}$; and

$$(2.10) \quad h_1 = \sigma^r(g')$$

for some $r \in \mathbb{N}$. Lemmas 2.5 and 2.6 let us suppose $m = r$.

In the case $p = 1$, we have $h_1 = \tau\sigma^t(g'')$ with $t \in \mathbb{N}$ and $\tau \in \{\pm\}$. Since $m = r$, then $g_1 = \epsilon\sigma^m(g') = \epsilon h_1 = \epsilon\tau\sigma^t(g'')$ if $k = 1$, and

$$\sigma^{k-1}(g_1) + \sigma^{k-1}(g_2) + \sigma^{k-2}(g_3) + \cdots + \sigma(g_k) = \epsilon\sigma^m(g') = \epsilon h_1 = \epsilon\tau\sigma^t(g'')$$

if $k \geq 2$. From here, we get that $\{g_1, \dots, g_k\}$ is also a connection from g to g'' .

In the case $p \geq 2$, it is straightforward to verify, taking into account equations (2.8), (2.9), and (2.10); and the fact $m = r$, that $\{g_1, \dots, g_k, h_2, \dots, h_p\}$ is a connection from g to g'' if $\epsilon = +$ in (2.8) or (2.9); and taking also into account the symmetry of Σ , that $\{g_1, \dots, g_k, -h_2, \dots, -h_p\}$ it is if $\epsilon = -$ in (2.8) or (2.9). We have showed the connection relation is also transitive and so it is an equivalence relation. $\square$

3. Decompositions. Proposition 2.7 let us consider the quotient set

$$\Sigma / \sim = \{[g] : g \in \Sigma\}.$$

Our next goal is to associate an (adequate) graded ideal $\mathfrak{J}_{[g]}$ of the graded hom-associative algebra $(\mathcal{M}_I(\mathcal{R}), \star)$ to each $[g]$. For each $[g]$, $g \in \Sigma$, we define (see equation (2.3)),

$$\mathfrak{J}_{0,[g]} := \text{span}_{\mathcal{R}}\{\mathcal{M}_I(\mathcal{R})_{g'} \star \mathcal{M}_I(\mathcal{R})_{-g'} : g' \in [g]\} \subset \mathcal{M}_I(\mathcal{R})_0$$

and

$$V_{[g]} := \bigoplus_{g' \in [g]} \mathcal{M}_I(\mathcal{R})_{g'}.$$

Finally, we denote by $\mathfrak{J}_{[g]}$ the following (graded) $\mathcal{R}$ -submodule of $\mathcal{M}_I(\mathcal{R})$,

$$\mathfrak{J}_{[g]} := \mathfrak{J}_{0,[g]} \oplus V_{[g]}.$$

PROPOSITION 3.1. *Let $g \in \Sigma$. Then the following assertions hold:*

1. $\mathfrak{J}_{[g]} \star \mathfrak{J}_{[g]} \subset \mathfrak{J}_{[g]}$ and $\phi(\mathfrak{J}_{[g]}) = \mathfrak{J}_{[g]}$.
2. If $g' \notin [g]$ then $\mathfrak{J}_{[g]} \star \mathfrak{J}_{[g']} = 0$.

Proof. 1. We have

$$(3.1) \quad (\mathfrak{J}_{0,[g]} \oplus V_{[g]}) \star (\mathfrak{J}_{0,[g]} \oplus V_{[g]}) \subset \mathfrak{J}_{0,[g]} \star \mathfrak{J}_{0,[g]} + \mathfrak{J}_{0,[g]} \star V_{[g]} + V_{[g]} \star \mathfrak{J}_{0,[g]} + V_{[g]} \star V_{[g]}.$$

Let us consider the last summand in equation (3.1), that is, $V_{[g]} \star V_{[g]}$. Given $g', g'' \in [g]$ such that $\mathcal{M}_I(\mathcal{R})_{g'} \star \mathcal{M}_I(\mathcal{R})_{g''} \neq 0$, if $g'' = -g'$ then clearly

$$\mathcal{M}_I(\mathcal{R})_{g'} \star \mathcal{M}_I(\mathcal{R})_{g''} = \mathcal{M}_I(\mathcal{R})_{g'} \star \mathcal{M}_I(\mathcal{R})_{-g'} \subset \mathfrak{I}_{0,[g]}.$$

Suppose then $g'' \neq -g'$. Taking into account that the fact $\mathcal{M}_I(\mathcal{R})_{g'} \star \mathcal{M}_I(\mathcal{R})_{g''} \neq 0$ joint with equation (2.3) ensure $\sigma(g') + \sigma(g'') \in \Sigma$, we have that $\{g', g''\}$ is a connection from g' to $\sigma(g') + \sigma(g'')$. The transitivity of $\sim$ gives that $\sigma(g') + \sigma(g'') \in [g]$ and so $\mathcal{M}_I(\mathcal{R})_{g'} \star \mathcal{M}_I(\mathcal{R})_{g''} \subset \mathcal{M}_I(\mathcal{R})_{\sigma(g') + \sigma(g'')} \subset V_{[g]}$. Consequently $(\bigoplus_{g' \in [g]} \mathcal{M}_I(\mathcal{R})_{g'}) \star$

$(\bigoplus_{g' \in [g]} \mathcal{M}_I(\mathcal{R})_{g'}) \subset \mathfrak{I}_{0,[g]} \oplus V_{[g]}$. That is,

$$(3.2) \quad V_{[g]} \star V_{[g]} \subset \mathfrak{I}_{0,[g]} \oplus V_{[g]}.$$

Consider the first summand $\mathfrak{I}_{0,[g]} \star \mathfrak{I}_{0,[g]}$ in equation (3.1). By hom-associativity, (and taking also into account equation (2.2)), given $g', g'' \in [g]$ we have

$$\begin{aligned} & (\mathcal{M}_I(\mathcal{R})_{g'} \star \mathcal{M}_I(\mathcal{R})_{-g'}) \star (\mathcal{M}_I(\mathcal{R})_{g''} \star \mathcal{M}_I(\mathcal{R})_{-g''}) = \\ & (\mathcal{M}_I(\mathcal{R})_{g'} \star \mathcal{M}_I(\mathcal{R})_{-g'}) \star \phi(\phi^{-1}(\mathcal{M}_I(\mathcal{R})_{g''} \star \mathcal{M}_I(\mathcal{R})_{-g''})) \subset \\ & \phi(\mathcal{M}_I(\mathcal{R})_{g'}) \star (\mathcal{M}_I(\mathcal{R})_{-g'} \star \phi^{-1}(\mathcal{M}_I(\mathcal{R})_{g''} \star \mathcal{M}_I(\mathcal{R})_{-g''})) \subset \\ & \mathcal{M}_I(\mathcal{R})_{\sigma(g')} \star \mathcal{M}_I(\mathcal{R})_{-\sigma(g')} \subset \mathfrak{I}_{0,[g]}, \end{aligned}$$

where last inclusion is a consequence of the fact that $g' \in [g]$ implies $\sigma(g') \in [g]$ by Lemma 2.4. Hence,

$$(3.3) \quad \mathfrak{I}_{0,[g]} \star \mathfrak{I}_{0,[g]} \subset \mathfrak{I}_{0,[g]}.$$

In a similar way we can show

$$(3.4) \quad \mathfrak{I}_{0,[g]} \star V_{[g]} + V_{[g]} \star \mathfrak{I}_{0,[g]} \subset V_{[g]}.$$

From equations (3.1), (3.2), (3.3) and (3.4) we get

$$\mathfrak{I}_{[g]} \star \mathfrak{I}_{[g]} = (\mathfrak{I}_{0,[g]} \oplus V_{[g]}) \star (\mathfrak{I}_{0,[g]} \oplus V_{[g]}) \subset \mathfrak{I}_{[g]}.$$

Finally, observe that equation (2.2) and the above mentioned fact that $g' \in [g]$ implies $\sigma(g') \in [g]$ give us $\phi(\mathfrak{I}_{0,[g]}) = \mathfrak{I}_{0,[g]}$ and $\phi(V_{[g]}) = V_{[g]}$. Hence, we conclude $\phi(\mathfrak{I}_{[g]}) = \mathfrak{I}_{[g]}$.

2. We have

$$(\mathfrak{I}_{0,[g]} \oplus V_{[g]}) \star (\mathfrak{I}_{0,[g']} \oplus V_{[g']}) \subset$$

$$(3.5) \quad \mathfrak{I}_{0,[g]} \star \mathfrak{I}_{0,[g']} + \mathfrak{I}_{0,[g]} \star V_{[g']} + V_{[g]} \star \mathfrak{I}_{0,[g']} + V_{[g]} \star V_{[g']}.$$

Consider the fourth summand $V_{[g]} \star V_{[g']}$ above and suppose there exist $g_1 \in [g]$ and $g_2 \in [g']$ such that $\mathcal{M}_I(\mathcal{R})_{g_1} \star \mathcal{M}_I(\mathcal{R})_{g_2} \neq 0$. As necessarily $g_1 \neq -g_2$, then $\sigma(g_1) + \sigma(g_2) \in \Sigma$. So $\{g_1, g_2, -\sigma(g_1)\}$ is a connection between g_1 and g_2 . By the transitivity of the connection relation we have $g \in [g']$, a contradiction. Hence $\mathcal{M}_I(\mathcal{R})_{g_1} \star \mathcal{M}_I(\mathcal{R})_{g_2} = 0$ and so

$$(3.6) \quad V_{[g]} \star V_{[g']} = 0.$$

Now consider the first summand $\mathfrak{I}_{0,[g]} \star \mathfrak{I}_{0,[g']}$ in equation (3.5) and suppose there exist $g_1 \in [g]$ and $g_2 \in [g']$ such that

$$(\mathcal{M}_I(\mathcal{R})_{g_1} \star \mathcal{M}_I(\mathcal{R})_{-g_1}) \star (\mathcal{M}_I(\mathcal{R})_{g_2} \star \mathcal{M}_I(\mathcal{R})_{-g_2}) \neq 0.$$

By hom-associativity, we have as in item 1 that

$$\phi(\mathcal{M}_I(\mathcal{R})_{g_1}) \star (\mathcal{M}_I(\mathcal{R})_{-g_1} \star \phi^{-1}(\mathcal{M}_I(\mathcal{R})_{g_2} \star \mathcal{M}_I(\mathcal{R})_{-g_2})) \neq 0$$

and so, taking into account equation (1.1), that

$$0 \neq \mathcal{M}_I(\mathcal{R})_{-g_1} \star \phi^{-1}(\mathcal{M}_I(\mathcal{R})_{g_2} \star \mathcal{M}_I(\mathcal{R})_{-g_2}) =$$

$$\mathcal{M}_I(\mathcal{R})_{-g_1} \star (\phi^{-1}(\mathcal{M}_I(\mathcal{R})_{g_2}) \star \phi^{-1}(\mathcal{M}_I(\mathcal{R})_{-g_2})).$$

Finally, hom-associativity gives

$$(\phi^{-1}(\mathcal{M}_I(\mathcal{R})_{-g_1}) \star \phi^{-1}(\mathcal{M}_I(\mathcal{R})_{g_2})) \star \mathcal{M}_I(\mathcal{R})_{-g_2} \neq 0.$$

From here, $\phi^{-1}(\mathcal{M}_I(\mathcal{R})_{-g_1}) \star \phi^{-1}(\mathcal{M}_I(\mathcal{R})_{g_2}) \neq 0$ and by equation (1.1),

$$\mathcal{M}_I(\mathcal{R})_{-g_1} \star \mathcal{M}_I(\mathcal{R})_{g_2} \neq 0,$$

what contradicts equation (3.6). Hence

$$\mathfrak{I}_{0,[g]} \star \mathfrak{I}_{0,[g']} = 0.$$

Finally, we note that the same above arguments also show

$$\mathfrak{I}_{0,[g]} \star V_{[g']} + V_{[g]} \star \mathfrak{I}_{0,[g']} = 0.$$

By equation (3.5) we get

$$(\mathfrak{I}_{0,[g]} \oplus V_{[g]}) \star (\mathfrak{I}_{0,[g']} \oplus V_{[g']}) = 0$$

and so $\mathfrak{I}_{[g]} \star \mathfrak{I}_{[g']} = 0$. $\square$

Proposition 3.1-1 lets us assert that for each $g \in \Sigma$, the graded $\mathcal{R}$ -submodule $\mathfrak{I}_{[g]}$ is closed under the product in $(\mathcal{M}_I(\mathcal{R}), \star)$ and under ϕ . That is, $\mathfrak{I}_{[g]}$ is a graded subalgebra of $(\mathcal{M}_I(\mathcal{R}), \star)$. Now we show that each $\mathfrak{I}_{[g]}$ is actually a graded ideal of $(\mathcal{M}_I(\mathcal{R}), \star)$.

THEOREM 3.2. *The following assertions hold.*

1. *For each $g \in \Sigma$, the graded $\mathcal{R}$ -submodule*

$$\mathfrak{I}_{[g]} = \mathfrak{I}_{0,[g]} \oplus V_{[g]}$$

of $(\mathcal{M}_I(\mathcal{R}), \star)$ associated to $[g]$ is a graded ideal of $(\mathcal{M}_I(\mathcal{R}), \star)$.

2. *If $(\mathcal{M}_I(\mathcal{R}), \star)$ is graded simple, then there exists a connection from g to g' for each $g, g' \in \Sigma$ and $\mathcal{M}_I(\mathcal{R})_0 = \sum_{g \in \Sigma} (\mathcal{M}_I(\mathcal{R})_g \star \mathcal{M}_I(\mathcal{R})_{-g})$.*

Proof.

1. For each $g' \in [g]$ we have $\mathcal{M}_I(\mathcal{R})_{g'} \star \mathcal{M}_I(\mathcal{R})_0 \subset \mathcal{M}_I(\mathcal{R})_{\sigma(g')} \subset V_{[g]}$ and so

$$(3.7) \quad V_{[g]} \star \mathcal{M}_I(\mathcal{R})_0 \subset V_{[g]}.$$

By hom-associativity, we also have

$$(\mathcal{M}_I(\mathcal{R})_{g'} \star \mathcal{M}_I(\mathcal{R})_{-g'}) \star \mathcal{M}_I(\mathcal{R})_0 =$$

$$(\mathcal{M}_I(\mathcal{R})_{g'} \star \mathcal{M}_I(\mathcal{R})_{-g'}) \star \phi(\phi^{-1}(\mathcal{M}_I(\mathcal{R})_0)) =$$

$$\phi(\mathcal{M}_I(\mathcal{R})_{g'}) \star (\mathcal{M}_I(\mathcal{R})_{-g'} \star \phi^{-1}(\mathcal{M}_I(\mathcal{R})_0)) \subset$$

$$\mathcal{M}_I(\mathcal{R})_{\sigma(g')} \star \mathcal{M}_I(\mathcal{R})_{-\sigma(g')} \subset \mathfrak{I}_{0,[g]}$$

and then

$$(3.8) \quad \mathfrak{I}_{0,[g]} \star \mathcal{M}_I(\mathcal{R})_0 \subset \mathfrak{I}_{0,[g]}.$$

From equations (3.7) and (3.8) we get

$$\mathfrak{I}_{[g]} \star \mathcal{M}_I(\mathcal{R})_0 \subset \mathfrak{I}_{[g]}.$$

Taking into account the above observation and Proposition 3.1, we have

$$\mathfrak{I}_{[g]} \star \mathcal{M}_I(\mathcal{R}) = \mathfrak{I}_{[g]} \star (\mathcal{M}_I(\mathcal{R})_0 \oplus (\bigoplus_{g' \in [g]} \mathcal{M}_I(\mathcal{R})_{g'}) \oplus (\bigoplus_{g'' \notin [g]} \mathcal{M}_I(\mathcal{R})_{g''})) \subset \mathfrak{I}_{[g]}.$$

In a similar way we get $\mathcal{M}_I(\mathcal{R}) \star \mathfrak{I}_{[g]} \subset \mathfrak{I}_{[g]}$ and, since $\phi(\mathfrak{I}_{[g]}) = \mathfrak{I}_{[g]}$ by Proposition 3.1-1, we conclude $\mathfrak{I}_{[g]}$ is a graded ideal of I .

2. The graded simplicity of $(\mathcal{M}_I(\mathcal{R}), \star)$ implies $\mathfrak{I}_{[g]} = \mathcal{M}_I(\mathcal{R})$. From here, it is clear that $[g] = \Sigma$ and $\mathcal{M}_I(\mathcal{R})_0 = \sum_{g \in \Sigma} (\mathcal{M}_I(\mathcal{R})_g \star \mathcal{M}_I(\mathcal{R})_{-g})$. $\square$

THEOREM 3.3. *We have*

$$\mathcal{M}_I(\mathcal{R}) = U + \sum_{[g] \in \Sigma / \sim} \mathfrak{I}_{[g]},$$

where U is an $\mathcal{R}$ -submodule of $\mathcal{M}_I(\mathcal{R})_0$ and each $\mathfrak{I}_{[g]}$ is one of the graded ideals of $(\mathcal{M}_I(\mathcal{R}), \star)$ described in Theorem 3.2-1, satisfying $\mathfrak{I}_{[g]} \star \mathfrak{I}_{[g']} = 0$ if $[g] \neq [g']$.

Proof. We have $\mathfrak{I}_{[g]}$ is well defined and, by Theorem 3.2-1, a graded ideal of $\mathcal{M}_I(\mathcal{R})$. By considering an $\mathcal{R}$ -submodule U of $\mathcal{M}_I(\mathcal{R})_0$ such that $\mathcal{M}_I(\mathcal{R})_0 = U + \text{span}_{\mathcal{R}}\{\mathcal{M}_I(\mathcal{R})_g \star \mathcal{M}_I(\mathcal{R})_{-g} : g \in \Sigma\}$, we have

$$\mathcal{M}_I(\mathcal{R}) = \mathcal{M}_I(\mathcal{R})_0 \oplus \left(\bigoplus_{g \in \Sigma} \mathcal{M}_I(\mathcal{R})_g \right) = U + \sum_{[g] \in \Sigma / \sim} \mathfrak{I}_{[g]}.$$

Finally Proposition 3.1-2 gives us $\mathfrak{I}_{[g]} \star \mathfrak{I}_{[g']} = 0$ if $[g] \neq [g']$. $\square$

Let us denote by $\mathcal{Z}(\mathcal{M}_I(\mathcal{R})) = \{v \in \mathcal{M}_I(\mathcal{R}) : v \star \mathcal{M}_I(\mathcal{R}) + \mathcal{M}_I(\mathcal{R}) \star v = 0\}$ the center of $\mathcal{M}_I(\mathcal{R})$.

COROLLARY 3.4. *If $\mathcal{Z}(\mathcal{M}_I(\mathcal{R})) = 0$ and $\mathcal{M}_I(\mathcal{R})_0 = \sum_{g \in \Sigma} (\mathcal{M}_I(\mathcal{R})_g \star \mathcal{M}_I(\mathcal{R})_{-g})$. Then $(\mathcal{M}_I(\mathcal{R}), \star)$ is the direct sum of the graded ideals given in Theorem 3.2,*

$$\mathcal{M}_I(\mathcal{R}) = \bigoplus_{[g] \in \Sigma / \sim} \mathfrak{I}_{[g]},$$

being $\mathfrak{I}_{[g]} \star \mathfrak{I}_{[g']} = 0$ if $[g] \neq [g']$.

Proof. From $\mathcal{M}_I(\mathcal{R})_0 = \sum_{g \in \Sigma} (\mathcal{M}_I(\mathcal{R})_g \star \mathcal{M}_I(\mathcal{R})_{-g})$ it is clear that $\mathcal{M}_I(\mathcal{R}) = \sum_{[g] \in \Sigma / \sim} \mathfrak{I}_{[g]}$. The direct character of the sum follows from the facts $\mathfrak{I}_{[g]} \star \mathfrak{I}_{[g']} = 0$, if $[g] \neq [g']$, and $\mathcal{Z}(\mathcal{M}_I(\mathcal{R})) = 0$. $\square$

4. The simple components. In this section we consider an arbitrary graded hom-associativity algebra over an arbitrary base field. We are interesting in studying the conditions under which such an algebra decomposes as the direct sum of the family of its simple graded ideals. Hence, from now on $(\mathcal{M}, \star)$ will denote a hom-associativity algebra of arbitrary dimension and over an arbitrary base field $\mathbb{K}$, graded by means

of $(G, +_\sigma)$, the twist by an automorphism σ of an abelian group $(G, +)$. That is,

$$\mathcal{M} = \bigoplus_{g \in G} \mathcal{M}_g = \mathcal{M}_0 \oplus \left(\bigoplus_{g \in \Sigma} \mathcal{M}_g \right)$$

with

$$(4.1) \quad \mathcal{M}_g \star \mathcal{M}_{g'} \subset \mathcal{M}_{g+\sigma g'} = \mathcal{M}_{\sigma(g)+\sigma(g')}$$

for each $g, g' \in G$, and where $\Sigma := \{g \in G \setminus 0 : \mathcal{M}_g \neq 0\}$ denotes the support of the grading. Furthermore, the twisting map $\phi : \mathcal{M} \rightarrow \mathcal{M}$, (see Definition 1.1), is a graded automorphism of $(\mathcal{M}, \star)$ with respect to σ . That is,

$$(4.2) \quad \phi(\mathcal{M}_g) \subset \mathcal{M}_{\sigma(g)}$$

for each $g \in G$. Note that if $\sigma = Id_G$ and $\phi = Id_{\mathcal{M}}$, then we are dealing with the particular case of an associative algebra graded by means of an abelian group. Hence the results in this section extend those in [5]. We recall, see §1, that a *graded ideal* of $\mathcal{M}$ is a linear subspace $\mathfrak{I}$ of $\mathcal{M}$ satisfying $\mathfrak{I} \star \mathcal{M} + \mathcal{M} \star \mathfrak{I} \subset \mathfrak{I}$ and $\phi(\mathfrak{I}) = \mathfrak{I}$; and such that splits as $\mathfrak{I} = \bigoplus_{g \in G} \mathfrak{I}_g$ with each $\mathfrak{I}_g = \mathfrak{I} \cap \mathcal{M}_g$. We also recall that $\mathcal{M}$ is called *graded simple* if $\mathcal{M} \star \mathcal{M} \neq 0$ and its only graded ideals are $\{0\}$ and $\mathcal{M}$.

As usual, we will denote by $\mathcal{Z}(\mathcal{M}) = \{v \in \mathcal{M} : v \star \mathcal{M} + \mathcal{M} \star v = 0\}$ the *center* of $(\mathcal{M}, \star)$.

LEMMA 4.1. *Suppose $\mathcal{M}_0 = \sum_{g \in \Sigma} (\mathcal{M}_g \star \mathcal{M}_{-g})$. If $\mathfrak{I}$ is a graded ideal of $(\mathcal{M}, \star)$ such that $\mathfrak{I} \subset \mathcal{M}_0$, then $\mathfrak{I} \subset \mathcal{Z}(\mathcal{M})$.*

Proof. Let $\mathfrak{I}$ be a graded ideal of $(\mathcal{M}, \star)$ such that $\mathfrak{I} \subset \mathcal{M}_0$. By equation (4.1), we have

$$\mathfrak{I} \star \left(\bigoplus_{g \in \Sigma} \mathcal{M}_g \right) \subset \left(\bigoplus_{g \in \Sigma} \mathcal{M}_g \right) \cap \mathcal{M}_0 = 0$$

and so $\mathfrak{I} \star \left(\bigoplus_{g \in \Sigma} \mathcal{M}_g \right) = 0$. In a similar way $\left(\bigoplus_{g \in \Sigma} \mathcal{M}_g \right) \star \mathfrak{I} = 0$. Hence,

$$(4.3) \quad \mathfrak{I} \star \left(\bigoplus_{g \in \Sigma} \mathcal{M}_g \right) = \left(\bigoplus_{g \in \Sigma} \mathcal{M}_g \right) \star \mathfrak{I} = 0.$$

Taking into account $\phi(\mathfrak{I}) = \mathfrak{I}$, the hom-associativity of the product and equation (4.3) give us

$$\begin{aligned} \mathfrak{I} \star (\mathcal{M}_g \star \mathcal{M}_{-g}) &= \phi(\phi^{-1}(\mathfrak{I})) \star (\mathcal{M}_g \star \mathcal{M}_{-g}) = \\ &= (\phi^{-1}(\mathfrak{I}) \star \mathcal{M}_g) \star \phi(\mathcal{M}_{-g}) = 0 \end{aligned}$$

for each $g \in \Sigma$. In a similar way we get $(\mathcal{M}_g \star \mathcal{M}_{-g}) \star \mathfrak{I} = 0$ and so, since $\mathcal{M}_0 = \sum_{g \in \Sigma} (\mathcal{M}_g \star \mathcal{M}_{-g})$, we obtain

$$(4.4) \quad \mathfrak{I} \star \mathcal{M}_0 + \mathcal{M}_0 \star \mathfrak{I} = 0.$$

From equations (4.3) and (4.4) we finally get $\mathfrak{I} \subset \mathcal{Z}(\mathcal{M})$. $\square$

Let us introduce the concepts of Σ -multiplicativity and maximal length in the framework of graded hom-associative algebras, in a similar way to the ones for graded associative algebras, graded Lie algebras, split Lie color algebras and split Leibniz algebras (see [4, 5, 7, 8] for these notions and examples).

DEFINITION 4.2. We say that a graded, by means of $(G, +_\sigma)$, hom-associative algebra $(\mathcal{M}, \star)$ is Σ -multiplicative if given $g, g' \in \Sigma$ such that $g +_\sigma g' \in \Sigma \cup \{0\}$, then $\mathcal{M}_g \star \mathcal{M}_{g'} \neq 0$.

DEFINITION 4.3. It is said that a graded hom-associative algebra $(\mathcal{M}, \star)$ is of *maximal length* if $\dim \mathcal{M}_g = 1$ for each $g \in \Sigma$. We note that the above concepts appear in a natural way in the study of gradings of algebras. In fact, one may expect that there are problems which are naturally and more simply formulated exploiting bases dictated by Σ -multiplicative and of maximal length gradings. Let us illustrate this fact by considering the particular case of the (associative) matrix algebra $\mathcal{M}_n(\mathbb{K})$, in which the gradings induced by the Pauli matrices have been fundamental in the study of the mathematical physics problems in [16] and [19], (see also [25]), as consequence of being Σ -multiplicative and of maximal length. The general reason for that is the fact that such gradings decompose $\mathcal{M}_n(\mathbb{K})$ into the direct sum of n^2 subspaces of dimension 1 (i.e. defines a basis of $\mathcal{M}_n(\mathbb{K})$), and that the generators are all semisimple generators of the algebra. Let us describe these gradings in detail. Given the matrix algebra $\mathcal{M} = \mathcal{M}_2(\mathbb{C})$, we can consider a $\mathbb{Z}_2 \times \mathbb{Z}_2$ -grading on $\mathcal{M}$ associated to the *Pauli matrices*

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix};$$

given by

$$\mathcal{M}_{(0,0)} = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \right\}, \quad \mathcal{M}_{(1,0)} = \left\{ \begin{pmatrix} \beta & 0 \\ 0 & -\beta \end{pmatrix} \right\}$$

$$\mathcal{M}_{(0,1)} = \left\{ \begin{pmatrix} 0 & \gamma \\ \gamma & 0 \end{pmatrix} \right\}, \quad \mathcal{M}_{(1,1)} = \left\{ \begin{pmatrix} 0 & \delta \\ -\delta & 0 \end{pmatrix} \right\},$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{C}$.

If $\mathbb{K}$ contains a primitive n -th root of unit, then the Pauli matrices $-\sigma_3$ and σ_1 can be generalized to

$$X_a = \begin{pmatrix} \epsilon^{n-1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \epsilon^{n-2} & 0 & \cdots & 0 & 0 \\ \cdots & & & & & \\ 0 & 0 & 0 & \cdots & \epsilon & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \text{ and } X_b = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & & & & & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Since $X_a X_b = \epsilon X_b X_a$ and $X_a^n = X_b^n = I$, we get the $\mathbb{Z}_n \times \mathbb{Z}_n$ -grading on the matrix algebra $\mathcal{M} = \mathcal{M}_n(\mathbb{K})$ given by $\mathcal{M}_{(k,l)} = \mathbb{K} X_a^k X_b^l$ for all $(k, l) \in \mathbb{Z}_n \times \mathbb{Z}_n$, being easy to verify that the above grading on $\mathcal{M} = \mathcal{M}_n(\mathbb{K})$ is Σ -multiplicative and of maximal length. We note that in the paper [10] the term *fine* is used for each grading on $\mathcal{M} = \mathcal{M}_n(\mathbb{K})$ such that $\dim \mathcal{M}_g \leq 1$ for each $g \in G$ and it is justified the interest of its study, being clear that each grading of maximal length is fine in the above sense. In fact, it is proved in [2] for the zero characteristic case, (and extended later to arbitrary characteristic), that each grading on $\mathcal{M}_n(\mathbb{K})$ can be obtained by combining elementary gradings and Σ -multiplicative and of maximal length gradings.

REMARK 4.4. Observe that the definitions, arguments and results given in Sections 2 and 3 for graded hom-associative algebras of the type $\mathcal{M}_I(\mathcal{R})$ also hold for our graded hom-associative algebras $(\mathcal{M}, \star)$.

THEOREM 4.5. *Let $(\mathcal{M}, \star)$ be a graded hom-associative algebra of maximal length and Σ -multiplicative. Then $(\mathcal{M}, \star)$ is graded simple if and only if $\mathcal{Z}(\mathcal{M}) = 0$, $\mathcal{M}_0 = \sum_{g \in \Sigma} (\mathcal{M}_g \star \mathcal{M}_{-g})$ and Σ has all of its elements connected.*

Proof. Suppose $(\mathcal{M}, \star)$ is graded simple. It is easy to verify that $\mathcal{Z}(\mathcal{M})$ is a graded ideal of $\mathcal{M}$ and so $\mathcal{Z}(\mathcal{M}) = 0$. Now, Theorem 3.2-2. and Remark 4.4 complete the proof of the first implication. To prove the converse, consider $\mathfrak{I} = \bigoplus_{g \in G} \mathfrak{I}_g$, where $\mathfrak{I}_g = \mathfrak{I} \cap \mathcal{M}_g$, a nonzero graded ideal of $(\mathcal{M}, \star)$. By the maximal length of $(\mathcal{M}, \star)$, if we denote by $\Sigma_{\mathfrak{I}} := \{g \in \Sigma : \mathfrak{I}_g \neq 0\}$, we can write $\mathfrak{I} = \mathfrak{I}_0 \oplus (\bigoplus_{g \in \Sigma_{\mathfrak{I}}} \mathcal{M}_g)$, being also $\Sigma_{\mathfrak{I}} \neq \emptyset$ as consequence of Lemma 4.1. Hence, we can take $g_0 \in \Sigma_{\mathfrak{I}}$ being so

$$0 \neq \mathcal{M}_{g_0} \subset \mathfrak{I}.$$

Since the fact $\phi(\mathfrak{I}) = \mathfrak{I}$ and equation (4.2) give us that

$$(4.5) \quad \text{if } g \in \Sigma_{\mathfrak{I}} \text{ then } \{\sigma^z(g) : z \in \mathbb{Z}\} \subset \Sigma_{\mathfrak{I}},$$

we get

$$(4.6) \quad \{\mathcal{M}_{\sigma^z(g_0)} : z \in \mathbb{Z}\} \subset \mathfrak{I}.$$

Now consider a $g' \in \Sigma$ such that $g' \notin \{\pm\sigma^z(g_0) : z \in \mathbb{Z}\}$. The fact that g_0 and g' are connected gives us a connection $\{g_1, g_2, \dots, g_k\}$, $k \geq 2$, from g_0 to g' such that

$$\begin{aligned} g_1 &= \sigma^n(g_0) \text{ for some } n \in \mathbb{N}. \\ \sigma(g_1) + \sigma(g_2) &\in \Sigma \\ \sigma^2(g_1) + \sigma^2(g_2) + \sigma(g_3) &\in \Sigma \\ &\dots\dots\dots \\ \sigma^i(g_1) + \sigma^i(g_2) + \sigma^{i-1}(g_3) + \dots + \sigma(g_{i+1}) &\in \Sigma \\ &\dots\dots\dots \\ \sigma^{k-2}(g_1) + \sigma^{k-2}(g_2) + \sigma^{k-3}(g_3) + \dots + \sigma^{k-i}(g_i) + \dots + \sigma(g_{k-1}) &\in \Sigma \\ \sigma^{k-1}(g_1) + \sigma^{k-1}(g_2) + \sigma^{k-2}(g_3) + \dots + \sigma^{k-i+1}(g_i) + \dots + \sigma(g_k) &= \epsilon\sigma^m(g') \\ &\text{for some } m \in \mathbb{N} \text{ and } \epsilon \in \{\pm\}. \end{aligned}$$

Since $g_1, g_2 \in \Sigma$ and also $g_1 +_\sigma g_2 = \sigma(g_1) + \sigma(g_2) \in \Sigma$, the Σ -multiplicativity and maximal length of $\mathcal{M}$ give us $0 \neq \mathcal{M}_{g_1} \star \mathcal{M}_{g_2} = \mathcal{M}_{\sigma(g_1) + \sigma(g_2)}$. Now taking into account that $0 \neq \mathcal{M}_{g_1} \subset \mathfrak{I}$ as consequence of equation (4.6) we get

$$0 \neq \mathcal{M}_{\sigma(g_1) + \sigma(g_2)} \subset \mathfrak{I}.$$

We can argue in a similar way from $\sigma(g_1) + \sigma(g_2)$, g_3 and $(\sigma(g_1) + \sigma(g_2)) +_\sigma g_3 = \sigma^2(g_1) + \sigma^2(g_2) + \sigma(g_3)$ to get

$$0 \neq \mathcal{M}_{\sigma^2(g_1) + \sigma^2(g_2) + \sigma(g_3)} \subset \mathfrak{I}.$$

Following this process with the connection $\{g_1, g_2, \dots, g_k\}$ we obtain that

$$0 \neq \mathcal{M}_{\sigma^{k-1}(g_1) + \sigma^{k-1}(g_2) + \sigma^{k-2}(g_3) + \dots + \sigma^{k-i+1}(g_i) + \dots + \sigma(g_k)} \subset \mathfrak{I}$$

and so either $\mathcal{M}_{\sigma^m(g')} \subset \mathfrak{I}$ or $\mathcal{M}_{-\sigma^m(g')} \subset \mathfrak{I}$.

Now taking into account equations (4.5) and (4.6), we get

$$(4.7) \quad \text{either } \{\mathcal{M}_{\sigma^z(g)} : z \in \mathbb{Z}\} \subset \mathfrak{I} \text{ or } \{\mathcal{M}_{\sigma^z(-g)} : z \in \mathbb{Z}\} \subset \mathfrak{I} \text{ for each } g \in \Sigma.$$

Observe that equation (4.7) can be reformulated by asserting that given any $g \in \Sigma$ either $\{\sigma^z(g) : z \in \mathbb{Z}\}$ or $\{\sigma^z(-g) : z \in \mathbb{Z}\}$ is contained in $\Sigma_{\mathfrak{I}}$. Since $\mathcal{M}_0 = \sum_{g \in \Sigma} (\mathcal{M}_g \star \mathcal{M}_{-g})$ we have as a consequence that

$$(4.8) \quad \mathcal{M}_0 \subset \mathfrak{I}.$$

We also have that in case $g, -g \in \Sigma_{\mathfrak{I}}$ for some $g \in \Sigma$, then $\Sigma_{\mathfrak{I}} = \Sigma$. Indeed, given any $g' \in \Sigma$, $g' \notin \{\pm\sigma^z(g) : z \in \mathbb{Z}\}$, there exists a connection

$$\{g_1, g_2, \dots, g_k\} \subset \Sigma,$$

$k \geq 2$, from g' to g . So

$$g_1 = \sigma^n(g'),$$

for some $n \in \mathbb{N}$ and

$$\sigma^{k-1}(g_1) + \sigma^{k-1}(g_2) + \sigma^{k-2}(g_3) + \cdots + \sigma^{k-i+1}(g_i) + \cdots + \sigma(g_k) = \epsilon \sigma^m(g)$$

for some $m \in \mathbb{N}$ and $\epsilon \in \{\pm\}$. From here, we have that the set

$$\{\sigma^m(\epsilon g), -\sigma(g_k), -\sigma^3(g_{k-1}), \dots, -\sigma^{2i+1}(g_{k-i}), \dots, -\sigma^{2k-3}(g_2)\} \subset \Sigma$$

is a connection from ϵg to g' , (see equation (2.7) in the proof of Proposition 2.7 and Remark 4.4), and by arguing with the Σ -multiplicativity and maximal length of $\mathcal{M}$ as above we get that

$$(\cdots ((\mathcal{M}_{\sigma^m(\epsilon g)} \star \mathcal{M}_{-\sigma(g_k)}) \star \mathcal{M}_{-\sigma^3(g_{k-1})}) \cdots) \star \mathcal{M}_{-\sigma^{2k-3}(g_2)} = \mathcal{M}_{\sigma^{2k+n-2}(g')}.$$

From here, taking into account $\mathcal{M}_g \oplus \mathcal{M}_{-g} \subset \mathfrak{I}$ and so, by equation (4.5), $\{\mathcal{M}_{\pm \sigma^z(g)} : z \in \mathbb{Z}\} \subset \mathfrak{I}$ we get $\mathcal{M}_{\sigma^{2k+n-2}(g')} \subset \mathfrak{I}$ and, taking again into account equation (4.5), that $\mathcal{M}_{g'} \subset \mathfrak{I}$. Hence, $\Sigma_{\mathfrak{I}} = \Sigma$. This fact joint to equation (4.8) let us assert:

$$(4.9) \quad \text{If } g, -g \in \Sigma_{\mathfrak{I}} \text{ for some } g \in \Sigma, \text{ then } \mathfrak{I} = \mathcal{M}.$$

Since $0 \neq \mathcal{M}_0 = \sum_{g \in \Sigma} (\mathcal{M}_g \star \mathcal{M}_{-g})$ and $\mathcal{Z}(\mathcal{M}) = 0$, and taking also into account the hom-associativity of $\mathcal{M}$, there exist $g, g' \in \Sigma$ such that

$$(4.10) \quad \text{either } (\mathcal{M}_g \star \mathcal{M}_{-g}) \star \mathcal{M}_{g'} \neq 0 \text{ or } \mathcal{M}_{g'} \star (\mathcal{M}_g \star \mathcal{M}_{-g}) \neq 0.$$

From here, equation (4.7) and the maximal length of $\mathcal{M}$ give us

$$0 \neq \mathcal{M}_{g'} \subset \mathfrak{I}.$$

This fact, joint with equation (4.10) and the hom-associativity and maximal length of $\mathcal{M}$, implies that

$$(4.11) \quad \text{either } 0 \neq \mathcal{M}_{\sigma(-g)+g'} \subset \mathfrak{I} \text{ or } 0 \neq \mathcal{M}_{g'+\sigma(g)} \subset \mathfrak{I}.$$

We also know that either $g \in \Sigma_{\mathfrak{I}}$ or $-g \in \Sigma_{\mathfrak{I}}$, (equation (4.7)). Suppose we have the case $g \in \Sigma_{\mathfrak{I}}$. Equation (4.11) and the Σ -multiplicativity and maximal length of $\mathcal{M}$ give us either

$$\mathcal{M}_{\sigma(-g)+g'} \star \mathcal{M}_{-g'} = \mathcal{M}_{\sigma^2(-g)} \subset \mathfrak{I}$$

or

$$\mathcal{M}_{\sigma(g)} \star \mathcal{M}_{-g'-\sigma(g)} = \mathcal{M}_{\sigma(-g')} \subset \mathfrak{I},$$

(the last equality uses the fact that $g \in \Sigma_{\mathfrak{J}}$ implies $\sigma(g) \in \Sigma_{\mathfrak{J}}$). From here, equation (4.5) gives us either $\mathcal{M}_{-g} \subset \mathfrak{J}$ or $\mathcal{M}_{-g'} \subset \mathfrak{J}$. Now, equation (4.9) let us conclude $\mathfrak{J} = \mathcal{M}$. In the case $-g \in \Sigma_{\mathfrak{J}}$, we have in a similar way that either $\mathcal{M}_{-g} \star \mathcal{M}_{g-\sigma(g')} = \mathcal{M}_{\sigma^2(-g')} \subset \mathfrak{J}$ or $\mathcal{M}_{g'+\sigma(g)} \star \mathcal{M}_{-g'} = \mathcal{M}_{\sigma^2(g)} \subset \mathfrak{J}$ and so equation (4.9) let us also conclude $\mathfrak{J} = \mathcal{M}$. Consequently $\mathcal{M}$ is graded simple. $\square$

THEOREM 4.6. *Let $(\mathcal{M}, \star)$ be a graded hom-associative algebra of maximal length, Σ -multiplicative, with $\mathcal{Z}(\mathcal{M}) = 0$ and satisfying $\mathcal{M}_0 = \sum_{g \in \Sigma} (\mathcal{M}_g \star \mathcal{M}_{-g})$. Then*

$$\mathcal{M} = \bigoplus_{[g] \in \Sigma/\sim} \mathfrak{J}_{[g]},$$

where each $\mathfrak{J}_{[g]}$ is a graded simple (graded) ideal having its support, $\Sigma_{\mathfrak{J}_{[g]}}$, with all of its elements $\Sigma_{\mathfrak{J}_{[g]}}$ -connected.

Proof. By Corollary 3.4 and Remark 4.4,

$$\mathcal{M} = \bigoplus_{[g] \in \Sigma/\sim} \mathfrak{J}_{[g]}$$

is the direct sum of the ideals

$$\mathfrak{J}_{[g]} = \mathfrak{J}_{0,[g]} \oplus V_{[g]} = \text{span}_{\mathbb{K}}\{\mathcal{M}_{g'} \star \mathcal{M}_{-g'} : g' \in [g]\} \oplus \bigoplus_{g' \in [g]} \mathcal{M}_{g'},$$

having any $\mathfrak{J}_{[g]}$ as support $\Sigma_{\mathfrak{J}_{[g]}} = [g]$. In order to apply Theorem 4.5 to each $\mathfrak{J}_{[g]}$, observe that the Σ -multiplicativity of $\mathcal{M}$ and Proposition 3.1-2 show that $\Sigma_{\mathfrak{J}_{[g]}}$ has all of its elements $\Sigma_{\mathfrak{J}_{[g]}}$ -connected, that is, connected through connections contained in $\Sigma_{\mathfrak{J}_{[g]}}$. We also have that each of the $\mathfrak{J}_{[g]}$ is Σ -multiplicative as consequence of the Σ -multiplicativity of $\mathcal{M}$. Clearly $\mathfrak{J}_{[g]}$ is of maximal length, and finally $\mathcal{Z}_{\Sigma_{\mathfrak{J}_{[g]}}}(\mathfrak{J}_{[g]}) = 0$, (where $\mathcal{Z}_{\Sigma_{\mathfrak{J}_{[g]}}}(\mathfrak{J}_{[g]})$ denotes de center of $\mathfrak{J}_{[g]}$ in $\mathfrak{J}_{[g]}$), as consequence of $\mathfrak{J}_{[g]} \star \mathfrak{J}_{[g']} = 0$ if $[g] \neq [g']$, (Theorem 3.3), and $\mathcal{Z}(\mathcal{M}) = 0$. We can apply Theorem 4.5 to any $\mathfrak{J}_{[g]}$ so as to conclude $\mathfrak{J}_{[g]}$ is graded simple. It is clear that the decomposition $\mathcal{M} = \bigoplus_{[g] \in \Sigma/\sim} \mathfrak{J}_{[g]}$

satisfies the assertions of the theorem. $\square$

Acknowledgments. We would like to thank the referee for his detailed reading of this work.

REFERENCES

- [1] F. Ammar, and A.Makhlouf. Hom-Lie superalgebras and Hom-Lie admissible superalgebras. *J. Algebra* 324:1513-1528, 2010
- [2] Y. Bathurin, S. Sehgal, and M. Zaicev. Group gradings on associative algebras. *J. Algebra* 241:677-298, 2011.

- [3] Y. Bathurin, and M. Zaicev. Involutions on graded matrix algebras. *J. Algebra* 315:527-540, 2007.
- [4] A.J. Calderón. On the structure of graded Lie algebras. *J. Math. Phys.* 50:103513, 8 pp, 2009.
- [5] A.J. Calderón. Graded associative algebras. *Rep. Math. Phys.* 69:75–86, 2012.
- [6] A.J. Calderón, C. Draper, and C. Martin. Gradings on the real forms of the Albert algebra, of $\mathfrak{g}_2$, and of $\mathfrak{f}_4$. *J. Math. Phys.* 51:053516, 21 pp., 2010.
- [7] A.J. Calderón, and J.M. Sánchez. On the structure of split Lie color algebras. *Linear Algebra Appl.* 436:307–315, 2012.
- [8] A.J. Calderón, and J.M. Sánchez. On split Leibniz algebras. *Linear Algebra Appl.* 436: 1648–1660, 2012.
- [9] A.A. Chasov. Isomorphisms of graded matrix algebras. (Russian) *Vestnik Moskov. Univ. Ser. I Mat. Mekh.* 70:12–18, 2004. Translation in *Moscow Univ. Math. Bull.* 59, 2005.
- [10] M. Chaves, and D. Singleton. Phantom energy from graded algebras. *Modern Phys. Lett. A* 22:29-40, 2007.
- [11] Y.S. Cheng, and Y.C. Su. (Co)homology and universal central extension of Hom-Leibniz algebras. *Acta Math. Sin. (Engl. Ser.)* 27:813-830, 2011.
- [12] R. Coquereaunx, G. Esposito-Farese and F. Scheck. Noncommutative geometry and graded algebras in electroweak interactions. *Internat. J. Modern Phys. A* 7:6555-6593, 1992.
- [13] G. Dahl. The doubly graded matrix cone and Ferrers matrices. *Linear Algebra Appl.* 368:171-190, 2003.
- [14] C. Draper, and C. Martin, Gradings on the Albert algebra and on $\mathfrak{f}_4$. *Rev. Mat. Iberoam.* 25:841–908, 2009.
- [15] A. Eli, H. Darrell and M. Natapov. Graded identities of matrix algebras and the universal graded algebra. *Trans. Amer. Math. Soc.* 362:3125-3147, 2010.
- [16] D.S. Freed, and C. Vafa. Global anomalies on orbifolds. *Commun. Math. Phys.* 110:107–119, 1987.
- [17] Y. Fregier, and A. Gohr. On Hom-type algebras. *J. Gen. Lie Theory Appl.* 4:G101001, 16 pp., 2010.
- [18] Y. Fregier, A. Gohr, and S. Silvestrov. Unital algebras of Hom-associative type and surjective or injective twistings. *J. Gen. Lie Theory Appl.* 3:285-295, 2009.
- [19] G.V. Gehlen, V. Rittenberg, and H. Ruegg. Conformal invariance and finite-dimensional quantum chains. *J. Phys. A.* 29:107–119, 1986.
- [20] D. Geir. The doubly graded matrix cone and Ferrers matrices. *Linear Algebra Appl.* 368:171-190, 2003.
- [21] A. Gohr. On hom-algebras with surjective twisting. *J. Algebra* 324:1483-1491, 2010.
- [22] J. Hartwig, S. Larsson and S. Silvestrov. Deformations of Lie algebras using σ -derivations. *J. Algebra* 295:314–361, 2006.
- [23] M. Havlicek, J. Patera, E. Pelantova, and J. Tolar. On fine gradings and their symmetries. *Czechoslovak J. Phys.* 51:383-391, 2001.
- [24] Q. Jin, and X. Li. Hom-Lie algebra structures on semi-simple Lie algebras. *J. Algebra* 319:1398–1408, 2008.
- [25] M. Kovhetov. Gradings on finite-dimensional simple Lie algebras. *Acta Appl. Math.* 108:101-127, 2009.
- [26] D. Larsson, and S. Silvestrov. Quasi-hom-Lie algebras, central extensions and 2-cocycle-like identities. *J. Algebra* 288:321-344, 2005.
- [27] A. Makhlouf. Hom-alternative algebras and Hom-Jordan algebras. *Int. Electron. J. Algebra* 8:177-190, 2010.
- [28] A. Makhlouf, and S. Silvestrov. Hom-algebra structures. *J. Gen. Lie Theory Appl.* 2:51–64, 2008.
- [29] A. Makhlouf, and S. Silvestrov. Notes on 1-parameter formal deformations of Hom-associative and Hom-Lie algebras. *Forum Math.* 22:715–739, 2010.

- [30] A. Makhlouf, and S. Silvestrov. Hom-algebras and Hom-coalgebras. *J. Algebra Appl.* 9:553–589.
- [31] J. Patera, E. Pelantova, and M. Svobodova. Fine gradings of $\mathfrak{o}(4, \mathbb{C})$. *J. Math. Phys.* 45:2188–2198, 2004.
- [32] J. Sandra Mara Alves. On graded central polynomials of the graded algebra $M_2(E)$. *Comm. Algebra* 38:2184–2198, 2010.
- [33] A. Verbovetsky. Lagrangian formalism over graded algebras. *J. Geom. Phys.* 18:195–214, 1996.
- [34] D. Yau. Hom-algebras and homology. *J. Lie Theory* 19:409–421, 2009.
- [35] D. Yau. Hom-Novikov algebras. *J. Phys. A* 44:085202, 20 pp., 2011.