A NOTE ON BLOCK REPRESENTATIONS OF THE GROUP INVERSE OF LAPLACIAN MATRICES

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Abstract. Let $G$ be a weighted graph with Laplacian matrix $L$ and signless Laplacian matrix $Q$. In this note, block representations for the group inverse of $L$ and $Q$ are given. The resistance distance in a graph can be obtained from the block representation of the group inverse of $L$.

Key words. Group inverse, Laplacian matrix, Signless Laplacian matrix, Resistance distance.

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1. Introduction. For an $n \times n$ matrix $A$, the group inverse of $A$ is the unique $n \times n$ matrix $X$ satisfying the matrix equations $AXA = A$, $XAX = X$ and $AX =XA$. It is well known that the group inverse of $A$ exists if and only if $\text{rank}(A) = \text{rank}(A^2)$ (see [28,29]). If the group inverse of $A$ exists, it is unique, which is denoted by $A^\#$. The matrix $A$ is group invertible if $A^\#$ exists. For a matrix $B$, let $B^+$ denote the Moore-Penrose inverse of $B$. Some representations for the group inverse of block matrices (operators) are given in [2–8,11,12,14,16,17,28]. More details for the theory of generalized inverse can be found in [9].

Let $G$ be a undirected weighted graph without loops or multiple edges, and each edge of $G$ has been labeled by a positive real number, which is called the weight of the edge. The adjacency matrix $A$ of $G$ is the matrix whose $(i, j)$-entry equals 0 if there is no edge joining vertices $i$ and $j$ and equals the weight of the edge joining vertices $i$ and $j$ otherwise. Let $D$ be the diagonal matrix whose $i$-th diagonal entry equals the sum of the weights of the edges incident to the vertex $i$ in $G$. The matrices $D - A$ and $D + A$ are called the Laplacian matrix and signless Laplacian matrix of $G$, respectively. It is known that $D - A$ and $D + A$ are positive semidefinite.

Let $L$ be the Laplacian matrix of a weighted graph $G$. Since $L$ is symmetric,
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$L^#$ exists and $L^# = L^+$ (see [9]). In [20], Kirkland et al. gave a representation for the group inverse of irreducible Laplacian matrices in terms of bottleneck matrix and all-ones matrix. In [18], Ho and van Dooren used an SVD (singular value decomposition) approach to calculate the Moore-Penrose (group) inverse of the Laplacian of a bipartite graph. In this note, we give a block representation for the group inverse of (signless) Laplacian matrices. Applying this block representation, we give a formula for the resistance distance in a graph.

2. Some lemmas. For a group invertible matrix $S$, let $S^\pi$ denote the projection matrix $I - SS^#$, where $I$ is the identity matrix.

**Lemma 2.1.** [25] Let $M$ be a Hermitian positive semidefinite matrix, which is partitioned as $M = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$. Then $A\pi B = 0$, $BC\pi = 0$.

**Lemma 2.2.** [3] Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A$ is nonsingular, and $S = D - CA^{-1}B$ is group invertible. Then $M^#$ exists if and only if $R = A^2 + BS\pi C$ is nonsingular. If $M^#$ exists, then

$$M^# = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix},$$

where

$$X = AR^{-1}(A + BS\pi C)R^{-1}A,$$

$$Y = AR^{-1}(A + BS\pi C)R^{-1}BS\pi - AR^{-1}BS^#,$$

$$Z = S\pi CR^{-1}(A + BS\pi C)R^{-1}A - S\pi CR^{-1}A,$$

$$W = S\pi CR^{-1}(A + BS\pi C)R^{-1}BS\pi - S\pi CR^{-1}BS\pi - S\pi CR^{-1}BS^# + S^#.$$

Klein and Randić introduced the concept of resistance distance in [21]. A graph $G$ can be viewed as an electrical network $N$ by replacing each edge of $G$ with a resistor. For two vertices $i$ and $j$ in $G$, the resistance distance between them is defined to be the effective resistance between them in the electrical network $N$ (see [21]). The resistance distance is a distance function in graphs, it has important applications in chemical graph theory. Some results on resistance distance can be found in [13,21,24,26,27].

For a matrix $M$, let $M_{ij}$ denote the $(i,j)$-entry of $M$. Let $G$ be a connected weighted graph with Laplacian matrix $L$. Let $\Omega_{ij}$ denote the resistance distance between vertices $i$ and $j$ in $G$. It is known that $\Omega_{ij} = L^\pi_{ii} + L^\pi_{jj} - L^\pi_{ij} - L^\pi_{ji}$ (see [1]). Note that $L$ is symmetric, $L^# = L^+$. Hence, we have the following lemma.

**Lemma 2.3.** Let $G$ be a connected weighted graph with vertex set $\{1, 2, \ldots, n\}$ and Laplacian matrix $L$. Then $\Omega_{ij} = L^\pi_{ii} + L^\pi_{jj} - L^\pi_{ij} - L^\pi_{ji}$.
3. Main results. Some expressions for the Moore-Penrose inverse of a $2 \times 2$ block matrix are given in \[19,22\]. But the expressions in \[19,22\] are very complicated. We first give a new expression for the group inverse of Laplacian matrices as follow.

**Theorem 3.1.** Let $G$ be a weighted graph with Laplacian matrix $L$. If $L$ is partitioned as $L = \begin{pmatrix} L_1 & L_2 \\ L_2^\top & L_3 \end{pmatrix}$ ($L_1$ is square), then

$$L^\# = \begin{pmatrix} X & Y \\ Y^\top & Z \end{pmatrix},$$

where

$$X = L_1 R^\# K R^\# L_1,$$

$$Y = L_1 R^\# K R^\# L_2 S^\top - L_1 R^\# L_2 S^\#,$$

$$Z = S^\top L_2^\top R^\# K R^\# L_2 S^\top - S^\# L_2^\top R^\# L_2 S^\# + S^\#,$$

$$R = L_1^\top + L_2 S^\# L_2^\top,$$

$$K = L_1 + L_2 S^\# L_2^\top,$$

$$S = L_3 - L_2^\top L_1^\# L_2.$$

**Proof.** Since $L_1$, $L_3$ are real symmetric, there exist orthogonal matrices $P_1$, $P_2$ such that

$$L_1 = P_1 \begin{pmatrix} \Delta_1 & 0 \\ 0 & 0 \end{pmatrix} P_1^\top, \quad L_3 = P_2 \begin{pmatrix} \Delta_2 & 0 \\ 0 & 0 \end{pmatrix} P_2^\top,$$

where $\Delta_1$, $\Delta_2$ are nonsingular diagonal matrices, the zero blocks can be vacuous.

Then we have

$$L_1^\# = P_1 \begin{pmatrix} \Delta_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} P_1^\top, \quad L_3^\# = P_2 \begin{pmatrix} \Delta_2^{-1} & 0 \\ 0 & 0 \end{pmatrix} P_2^\top.$$

Suppose that $L_2 = P_1 \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} P_2^\top$. By Lemma 2.1, we have $L_1^\top L_2 = 0$, $L_2 L_3^\top = 0$. Hence, $M_2 = 0$, $M_3 = 0$, $M_4 = 0$. Then

$$L^\# = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} \begin{pmatrix} \Delta_1 & 0 & M_1 & 0 \\ 0 & 0 & 0 & 0 \\ M_1^\top & 0 & \Delta_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^\# \begin{pmatrix} P_1^\top & 0 \\ 0 & P_2^\top \end{pmatrix} = U \begin{pmatrix} M^\# & 0 \\ 0 & 0 \end{pmatrix} U^{-1},$$
where \( M = \begin{pmatrix} \Delta_1 & M_1 \\ M_1^T & \Delta_2 \end{pmatrix} \), \( U = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \). Recall that \( \Delta_1 \) is a nonsingular diagonal matrix. Since \( \Delta_2 - M_1^T \Delta_1^{-1} M_1 \), the Schur complement of \( M \), is real symmetric, it is group invertible. By Lemma 2.2, we have
\[
M^\# = \begin{pmatrix} \tilde{X} & \tilde{Y} \\ \tilde{Y}^T & \tilde{W} \end{pmatrix},
\]
where
\[
\tilde{X} = \Delta_1 \tilde{R}^{-1} \tilde{K} \tilde{R}^{-1} \Delta_1,
\tilde{Y} = \Delta_1 \tilde{R}^{-1} \tilde{K} \tilde{R}^{-1} M_1 \tilde{S}^\pi - \Delta_1 \tilde{R}^{-1} M_1 \tilde{S}^\#,
\tilde{W} = \tilde{S} \tilde{M}_1^T \tilde{R}^{-1} \tilde{K} \tilde{R}^{-1} M_1 \tilde{S}^\pi - \tilde{S}^\# \tilde{M}_1^T \tilde{R}^{-1} M_1 \tilde{S}^\pi - \tilde{S}^\pi \tilde{M}_1^T \tilde{R}^{-1} M_1 \tilde{S}^\# + \tilde{S}^\#,
\tilde{R} = \Delta_1^2 + M_1 \tilde{S}^\pi M_1^T,
\tilde{K} = \Delta_1 + M_1 \tilde{S}^\# M_1^T,
\tilde{S} = \Delta_2 - M_1^T \Delta_1^{-1} M_1.
\]
By \( L^\# = U \begin{pmatrix} M^\# & 0 \\ 0 & 0 \end{pmatrix} U^{-1} \), we can obtain the representation of \( L^\# \). \( \square \)

\[\text{Figure 1: Weighted graph } G.\]

Let \( F_1, F_2 \) be two subsets of the set \( \{1, 2, \ldots, n\} \). The complement of \( F_1 \) and \( F_2 \) in \( \{1, 2, \ldots, n\} \) are denoted by \( \overline{F_1} \) and \( \overline{F_2} \), respectively. For a matrix \( L \) of order \( n \), let \( L[F_1|F_2] \) denote the submatrix of \( L \) determined by the rows whose index is in \( F_1 \) and the columns whose index is in \( F_2 \). Here we give an example for Theorem 3.1.
By Theorem 3.1, we get

\[ L = \begin{pmatrix}
2 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 4 & -1 & 0 & -1 & -2 & 0 \\
0 & 0 & -1 & 7 & -3 & -1 & -2 & 0 \\
0 & 0 & 0 & -3 & 5 & -2 & 0 & 0 \\
0 & 0 & -1 & -1 & -2 & 7 & -3 & 0 \\
0 & 0 & -2 & -2 & 0 & -3 & 7 & 0 \\
-2 & 0 & 0 & 0 & 0 & 0 & 0 & 2
\end{pmatrix}. \]

Considering the weighted graph \( G \) shown in Figure 1. The Laplacian matrix of \( G \) is

\[ L_1 = L[\{1\}][\{1\}], \quad L_2 = L[\{1\}][\{1\}], \quad L_3 = L[\{1\}][\{1\}] \] Then

\[ S = L_3 - L_2^T L_1^\# L_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 4 & -1 & 0 & -1 & -2 & 0 \\
0 & -1 & 7 & -3 & -1 & -2 & 0 \\
0 & 0 & -3 & 5 & -2 & 0 & 0 \\
0 & -1 & -1 & -2 & 7 & -3 & 0 \\
0 & -2 & -2 & 0 & -3 & 7 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \]

\[ S^\# = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2052/11725 & -548/11725 & -139/1675 & -438/11725 & -93/11725 & 0 \\
0 & -548/11725 & 1152/11725 & 11/1675 & -363/11725 & -318/11725 & 0 \\
0 & -139/1675 & 11/1675 & 261/1675 & -34/1675 & -99/1675 & 0 \\
0 & -438/11725 & -363/11725 & -34/1675 & -1122/11725 & -83/11725 & 0 \\
0 & -93/11725 & -318/11725 & -99/1675 & -83/11725 & 1187/11725 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \]

\[ S^\pi = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 0 \\
0 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 0 \\
0 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 0 \\
0 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \]

\[ K = L_1 + L_2 S^\# L_2^T = 2, \quad R = L_2^2 + L_2 S^\pi L_2^T = 8, \quad R^\# = 1/8. \]

By Theorem 3.1, we get \( L^\# = \begin{pmatrix} X & Y \\ Y^T & Z \end{pmatrix} \), where

\[ X = 1/8, \quad Y = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1/8 \end{pmatrix} \]
If we let $L_1 = L[[1,2][1,2]]$, $L_2 = L[[1,2][1,2]]$, $L_3 = L[[1,2][1,2]]$, then

$$S = L_3 - L_2^T L_1^T L_2 = \begin{pmatrix}
4 & -1 & 0 & -1 & -2 & 0 \\
-1 & 7 & -3 & -1 & -2 & 0 \\
0 & -3 & 5 & -2 & 0 & 0 \\
-1 & -1 & -2 & 7 & -3 & 0 \\
-2 & -2 & 0 & -3 & 7 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$S^\# = \begin{pmatrix}
2052/11725 & -548/11725 & -139/1675 & -438/11725 & -93/11725 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-548/11725 & 1152/11725 & 11/1675 & -363/11725 & -318/11725 & 0 \\
0 & -139/1675 & 11/1675 & 261/1675 & -34/1675 & -99/1675 \\
-438/11725 & -363/11725 & -34/1675 & 1122/11725 & -83/11725 & 0 \\
0 & -93/11725 & -318/11725 & -99/1675 & -83/11725 & 1187/11725 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

$$S^\pi = \begin{pmatrix}
1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 0 \\
1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 0 \\
1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 0 \\
1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},$$

$$R = L_1^2 + L_2 S^\pi L_2^T = \begin{pmatrix} 8 & 0 \\ 0 & 0 \end{pmatrix}, \quad R^\# = \begin{pmatrix} 1/8 & 0 \\ 0 & 0 \end{pmatrix}, \quad K = L_1 + L_2 S^\# L_2^T = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.$$

By Theorem 3.1, we get $L^\# = \begin{pmatrix} X^T & Y \\ Y & Z \end{pmatrix}$, where

$$X = \begin{pmatrix} 1/8 & 0 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1/8 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}.$$
Let \( L \) be the Laplacian matrix of a weighted graph, and \( L \) is partitioned as \( L = \begin{pmatrix} L_1 & L_2 \\ L_2^\top & L_3 \end{pmatrix} \), where \( L_1 \) is square. The group inverse of generalized Schur complement \( S = L_3 - L_1^\top L_1^# L_2 \) plays the key role in the block representation of \( L^# \) (cf. Theorem 3.1). The Laplacian matrix is an M-matrix. It is known that the Schur complement of an M-matrix is an M-matrix (see [15]). Hence, \( S \) is an M-matrix. Clearly, we have \( Le = 0 \), where \( e \) denotes an all-ones column vector with suitable dimension. By \( Le = 0 \), we get \( L_1 e + L_2 e = 0 \), \( L_2^\top e + L_3 e = 0 \). Then we have

\[
Se = L_3 e - L_2^\top L_1^# L_2 e = -L_2^\top L_1^# L_1 e = -L_1^\top L_1^# L_1 e = -(L_1^# L_2)^\top e.
\]

Lemma 2.1 implies that \( Se = 0 \). Clearly, \( S \) is symmetric. Since \( S \) is an M-matrix and \( Se = 0 \), \( S \) is the Laplacian matrix of a weighted graph. Hence, we can obtain a block representation for \( S^# \) from Theorem 3.1. We give a algorithm for \( L^# \) as follows.

**Step 1.** Let \( L = \begin{pmatrix} L_1 & L_2 \\ L_2^\top & L_3 \end{pmatrix} \), where \( L_1 \) is square. Apply Theorem 3.1 to represent \( L^# \).

**Step 2.** Let \( S = L_3 - L_2^\top L_1^# L_2 = \begin{pmatrix} S_1 & S_2 \\ S_2^\top & S_3 \end{pmatrix} \), where \( S_1 \) is square. Go to step 1 to calculate \( S^# \).

The group inverse of matrices has numerous applications in singular differential equations, Markov chains and iterative methods etc (see [9,10,23]). Here we give a new application for the group inverses of 2 \( \times \) 2 block matrices.

**Theorem 3.2.** Let \( G \) be a weighted graph with Laplacian matrix \( L \). Let \( i \) and \( j \) be two vertices of \( G \), and \( i \) and \( j \) belong to the same component of \( G \). Then the resistance distance between \( i \) and \( j \) is \( \Omega_{ij} = eXe^\top \), where

\[
e = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad X = L_1^# KR^# L_1, \quad R = L_1^2 + L_2 S^\top L_2^\top, \quad K = L_1 + L_2 S^# L_2^\top,
\]

\[
S = L_3 - L_2^\top L_1^# L_2, \quad L_1 = L([i,j]|[i,j]), \quad L_2 = L([i,j]|[i,j]), \quad L_3 = L([i,j]|[i,j]).
\]
Lemma 2.3 implies that $\Omega_{ij}$ where $L$ and 6 in the weighted graph $G$. Let $L_1 = L[i,j][i,j]$. By Theorem 3.1, we have

$$L^\# = \begin{pmatrix} X & Y \\ Y^T & Z \end{pmatrix},$$

where

$$X = L_1 R^# KR^# L_1, \quad Y = L_1 R^# KR^# L_2 S^\pi - L_1 R^# L_2 S^#, \quad Z = S^\pi L_1^2 R^# KR^# L_2 S^\pi - S^\pi L_1^2 R^# L_2 S^# + S^#,$$

$$R = L_2^2 + L_2 S^\pi L_2^T, \quad K = L_1 + L_2 S^# L_2^T, \quad S = L_3 - L_2^T L_1^# L_2.$$

Lemma 2.3 implies that $\Omega_{ij} = (1 - 1) X (1 - 1)^T$. □

Now we use Theorem 3.2 to calculate the resistance distance between vertices 4 and 6 in the weighted graph $G$ shown in Figure 1. Let $L$ be the Laplacian matrix of $G$. Let $L_1 = L[[4,6][4,6]], L_2 = L[[4,6][4,6]], L_3 = L[[4,6][4,6]].$ Then

$$S = L_3 - L_2^T L_1^# L_2 = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 11/3 & -5/6 & -17/6 & 0 \\ 0 & 0 & -5/6 & 137/48 & -97/48 & 0 \\ 0 & 0 & -17/6 & -97/48 & 233/48 & 0 \\ -2 & 0 & 0 & 0 & 0 & 2 \end{pmatrix},$$

$$S^# = \begin{pmatrix} 1/8 & 0 & 0 & 0 & 0 & -1/8 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 188/1407 & -142/1407 & -46/1407 & 0 \\ 0 & 0 & -142/1407 & 227/1407 & -85/1407 & 0 \\ 0 & 0 & -46/1407 & -85/1407 & 131/1407 & 0 \\ -1/8 & 0 & 0 & 0 & 0 & 1/8 \end{pmatrix},$$

$$S^\pi = \begin{pmatrix} 1/2 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/3 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 1/2 \end{pmatrix},$$

$$K = L_1 + L_2 S^# L_2^T = \begin{pmatrix} 3516/469 & -372/469 \\ -372/469 & 3420/469 \end{pmatrix}, \quad R = L_2^2 + L_2 S^\pi L_2^T = \begin{pmatrix} 62 & -2 \\ -2 & 62 \end{pmatrix}.$$

Proof. There exists a permutation matrix $P$ such that $L = P \begin{pmatrix} L_1 & L_2 \\ L_2 & L_3 \end{pmatrix} P^T,$ where $L_1 = L[i,j][i,j]$. By Theorem 3.1, we have

$$L^\# = \begin{pmatrix} X & Y \\ Y^T & Z \end{pmatrix},$$

where

$$X = L_1 R^# KR^# L_1, \quad Y = L_1 R^# KR^# L_2 S^\pi - L_1 R^# L_2 S^#, \quad Z = S^\pi L_1^2 R^# KR^# L_2 S^\pi - S^\pi L_1^2 R^# L_2 S^# + S^#,$$

$$R = L_2^2 + L_2 S^\pi L_2^T, \quad K = L_1 + L_2 S^# L_2^T, \quad S = L_3 - L_2^T L_1^# L_2.$$
By Theorem 3.2, the resistance distance between vertices 4 and 6 is

\[ \Omega_{46} = (1 - 1) \begin{pmatrix} 1152/11275 & -363/11725 \\ -363/11725 & 1122/11725 \end{pmatrix} (1 - 1) = 120/469. \]

For a connected graph \( G \), let \( d(u, v) \) denote the distance between two vertices \( u, v \) in \( G \).

**Theorem 3.3.** Let \( G \) be a connected bipartite graph with signless Laplacian matrix \( Q \). Let \( u \) and \( v \) be two vertices of \( G \). Then

\[ \Omega_{uv} = \begin{cases} Q_{uu}^\# + Q_{vv}^\# + Q_{uv}^\# + Q_{vu}^\# & \text{if } d(u, v) \text{ is odd}, \\ Q_{uu}^\# + Q_{vv}^\# - Q_{uv}^\# - Q_{vu}^\# & \text{if } d(u, v) \text{ is even}. \end{cases} \]

**Proof.** Since \( G \) is a bipartite graph, its adjacency matrix can be written as \( A = \begin{pmatrix} 0 & B \\ B^\top & 0 \end{pmatrix} \), where two zero sub-blocks of \( A \) correspond to the two color classes of \( G \). Suppose that \( Q = \begin{pmatrix} D_1 & B \\ B^\top & D_2 \end{pmatrix} \) is the signless Laplacian matrix of \( G \). Then \( L = \begin{pmatrix} D_1 & -B \\ -B^\top & D_2 \end{pmatrix} \) is the Laplacian matrix of \( G \). Clearly, we have

\[ Q = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} D_1 & -B \\ -B^\top & D_2 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \]

\[ Q^\# = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} D_1 & -B \\ -B^\top & D_2 \end{pmatrix}^\# \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \]

If \( d(u, v) \) is odd, then \( u, v \) belong to different color classes of \( G \). Lemma 2.3 implies that \( \Omega_{uv} = Q_{uu}^\# + Q_{vv}^\# + Q_{uv}^\# + Q_{vu}^\# \). If \( d(u, v) \) is even, then \( u, v \) belong to the same color class of \( G \). Lemma 2.3 implies that \( \Omega_{uv} = Q_{uu}^\# + Q_{vv}^\# - Q_{uv}^\# - Q_{vu}^\#. \]
Theorem 3.4. Let $G$ be a weighted graph with signless Laplacian matrix $Q$. If $Q$ is partitioned as $Q = \begin{pmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{pmatrix}$ ($Q_1$ is square), then

\[ Q^# = \begin{pmatrix} X & Y \\ Y^T & Z \end{pmatrix}, \]

where

\[ X = Q_1 R^# K R^# Q_1, \quad Y = Q_1 R^# K R^# Q_2 S^T - Q_1 R^# Q_2 S^#, \]
\[ Z = S^T Q_1 R^# K R^# Q_2 S^T - S^# Q_1 R^# Q_2 S^T - S^T Q_2 R^# S^# + S^#, \]
\[ R = Q_1^T + Q_2 S^T Q_2^T, \quad K = Q_1 + Q_2 S^# Q_2^T, \quad S = Q_3 - Q_2^T Q_1 Q_2. \]

Proof. The proof is similar to the proof of Theorem 3.1. \[ \square \]

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REFERENCES


