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POSITIVE SEMIDEFINITE MAXIMUM NULLITY AND ZERO FORCING NUMBER*

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Abstract. The zero forcing number $\mathrm{Z}(G)$ is used to study the minimum rank/maximum nullity of the family of symmetric matrices described by a simple, undirected graph G. The positive semidefinite zero forcing number is a variant of the (standard) zero forcing number, which uses the same definition except with a different color-change rule. The positive semidefinite maximum nullity and zero forcing number for a variety of graph families are computed. In addition, field independence of the minimum rank of the hypercube is established, by showing there is a positive semidefinite matrix that is universally optimal.

Key words. Zero forcing number, Maximum nullity, Minimum rank, Positive semidefinite, Matrix, Graph.

AMS subject classifications. 05C50, 15A03, 15B57.

1. Introduction. The minimum rank/maximum nullity problem and zero forcing have been studied quite extensively. For many graphs and families of graphs, the maximum nullity and zero forcing number have been determined, and these are reported in the AIM graph catalog [2]. However, a great deal of work remains in determining the positive semidefinite maximum nullity and zero forcing number of many of these graphs and graph families. In Section 3, we use a variety of techniques to compute the positive semidefinite maximum nullity and zero forcing number for many of the graphs in the AIM graph catalog [2], and these results are summarized in Table 3.1. In Section 2, we determine the positive semidefinite maximum nullity and zero forcing number of all hypercubes by constructing a vector representation recursively. Our method produces a universally optimal matrix and establishes field independence of the minimum rank, answering an open question (see [8], [17]).

Every graph discussed in this paper is simple (no loops or multiple edges), undirected, and has a finite nonempty vertex set. Given a graph $G = (V, E)$, color the vertices of G either black or white. This is known as an *initial coloring* of G. Vertices change color according to the *positive semidefinite color-change rule*: Let S denote the set of black vertices of G. Identify the sets of vertices $W_1, \ldots, W_k$ of the k compo-

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nents of $G-S$. If $u \in S$ and $w \in W_i$ is the only white neighbor of u in $G[W_i \cup S]$, then change the color of w to black. Given an initial coloring of G, the *derived set* is the set of initial black vertices along with vertices that are colored black after repeated application of the positive semidefinite color-change rule, i.e., until no more changes are possible. A *positive semidefinite zero forcing set* of G is a subset $B \subseteq V(G)$ such that if initially the vertices of B are colored black and the remaining vertices are colored white, then the derived set is $V(G)$. The *positive semidefinite zero forcing number* $\mathrm{Z}_+(G)$ is defined as the minimum of $|B|$ over all positive semidefinite zero forcing sets $B \subseteq V(G)$. The positive semidefinite zero forcing number is a variant of the (standard) zero forcing number, which uses the same definition except with a different color-change rule: If u is black and w is the only white neighbor of u, then change the color of w to black. In [1], it was shown that the zero forcing number is an upper bound for the maximum nullity of a graph. Similarly, it was shown in [3] that the positive semidefinite zero forcing number is an upper bound for the maximum positive semidefinite nullity of a graph.

We denote the set of real symmetric $n \times n$ matrices by $S_n(\mathbb{R})$, and we denote the set of (possibly complex) Hermitian $n \times n$ matrices by H_n. Given a matrix $A \in H_n$, the *graph* of A, denoted $\mathcal{G}(A)$, is the graph with vertices $\{1, \ldots, n\}$ and edges $\{\{i,j\} : a_{ij} \neq 0, 1 \leq i < j \leq n\}$. Notice that the diagonal of A is ignored in determining $\mathcal{G}(A)$. We denote the set of real symmetric (real positive semidefinite) matrices of a graph by $\mathcal{S}(G)$ ($\mathcal{S}_+(G)$). The *minimum positive semidefinite rank* of G and the *minimum Hermitian positive semidefinite rank* of G are, respectively,

$$\mathrm{mr}_+^{\mathbb{R}}(G) = \min\{\mathrm{rank}(A) : A \in S_n \text{ is positive semidefinite and } \mathcal{G}(A) = G\}$$

and

$$\mathrm{mr}_+^{\mathbb{C}}(G) = \min\{\mathrm{rank}(A) : A \in H_n \text{ is positive semidefinite and } \mathcal{G}(A) = G\}.$$

The *maximum positive semidefinite nullity* of G and the *maximum Hermitian positive semidefinite nullity* of G are, respectively,

$$\mathrm{M}_+^{\mathbb{R}}(G) = \max\{\mathrm{null}(A) : A \in S_n \text{ is positive semidefinite and } \mathcal{G}(A) = G\}$$

and

$$\mathrm{M}_+^{\mathbb{C}}(G) = \max\{\mathrm{null}(A) : A \in H_n \text{ is positive semidefinite and } \mathcal{G}(A) = G\}.$$

Note that $\mathrm{mr}_+^{\mathbb{R}}(G) + \mathrm{M}_+^{\mathbb{R}}(G) = |G|$ and $\mathrm{mr}_+^{\mathbb{C}}(G) + \mathrm{M}_+^{\mathbb{C}}(G) = |G|$.

If $\mathrm{mr}_+^{\mathbb{R}}(G) = \mathrm{mr}_+^{\mathbb{C}}(G)$, then we denote the common value $\mathrm{mr}_+^{\mathbb{R}}(G) = \mathrm{mr}_+^{\mathbb{C}}(G)$ by $\mathrm{mr}_+(G)$. Similarly, we denote the common value $\mathrm{M}_+^{\mathbb{R}}(G) = \mathrm{M}_+^{\mathbb{C}}(G)$ by $\mathrm{M}_+(G)$. With

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the exception of the Möbius ladder, all of the graphs discussed in this paper have $\mathrm{M}_+^{\mathbb{R}}(G) = \mathrm{M}_+^{\mathbb{C}}(G) = \mathrm{Z}_+(G)$, and all have $\mathrm{M}_+^{\mathbb{R}}(G) = \mathrm{M}_+^{\mathbb{C}}(G)$.

Observation 1.1. If $\mathrm{M}_+^{\mathbb{R}}(G) = \mathrm{Z}_+(G)$, then $\mathrm{M}_+(G) = \mathrm{Z}_+(G)$ as $\mathrm{M}_+^{\mathbb{R}}(G) \le \mathrm{M}_+^{\mathbb{C}}(G) \le \mathrm{Z}_+(G)$ for every graph G.

To discuss field independence of the hypercube, we introduce minimum rank over fields other than $\mathbb{R}$ and $\mathbb{C}$. We denote the set of $n \times n$ symmetric matrices over a field F by $S_n(F)$. The *minimum rank over field* F of a graph G is

$$\mathrm{mr}^F(G) = \min\{\mathrm{rank}(A) : A \in S_n(F),\ \mathcal{G}(A) = G\},$$

and the *maximum nullity over* F of a graph G is

$$\mathrm{M}^F(G) = \max\{\mathrm{null}(A) : A \in S_n(F),\ \mathcal{G}(A) = G\}.$$

Clearly,

$$\mathrm{mr}^F(G) + \mathrm{M}^F(G) = |G|.$$

The minimum rank of a graph G is *field independent* if the minimum rank of G is the same over all fields. A matrix $A \in S_n(F)$ is *optimal* for a graph G over a field F if $\mathcal{G}(A) = G$ and $\mathrm{rank}^F(A) = \mathrm{mr}^F(G)$. If A is an integer matrix, then A can be viewed as a matrix with entries in $\mathbb{Z}_p$ for p a prime, and hence $A \in F^{n\times n}$ where $\mathbb{Z}_p \subseteq F$ or $\mathbb{Z} \subseteq F$. Note that the graph of A may depend on the characteristic of the field (e.g., $2 \equiv 0$ in $\mathbb{Z}_2$). But if all off-diagonal entries are 0, 1, or -1, the graph $\mathcal{G}(A)$ does not depend on the field. A *universally optimal matrix* is an integer matrix A such that every off-diagonal entry of A is 0, 1, or -1, and $\mathrm{rank}^F(A) = \mathrm{mr}^F(\mathcal{G}(A))$ for all fields F.

DeAlba et al. [8] explored universally optimal matrices and field independence of the minimum rank for a number of graph families. However, they were unable to verify field independence of the minimum rank of Q_n or to find a universally optimal matrix for Q_n. Huang et al. [17] later found universally optimal matrices for a subclass of the class of hypercubes. We improve upon these results by verifying that the minimum rank of the hypercube is field independent and finding a universally optimal matrix for every hypercube.

We will need some additional terminology. The subgraph $G[U]$ of $G = (V, E)$ *induced* by $U \subseteq V$ is the subgraph with vertex set U and edge set $\{\{i, j\} \in E : i, j \in U\}$. A subgraph G' of a graph G is a *clique* if there is an edge between every pair of vertices of G', i.e., $G' \cong K_{|G'|}$. A *clique covering* of G is a set of subgraphs of G that are cliques and such that every edge of G is contained in at least one clique. The *clique covering number* of G, $\mathrm{cc}(G)$, is the fewest number of cliques in a clique covering of G.

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Let $G = (V, E)$ be a graph with ordered set of vertices $V = \{v_1, v_2, \ldots, v_n\}$. We associate a vector $\vec{v}_i \in \mathbb{R}^d$ with each vertex v_i of G (for minimum Hermitian positive semidefinite rank, use $\vec{v}_i \in \mathbb{C}^d$) . If two vertices v_i and v_j are adjacent, then $\langle \vec{v}_i, \vec{v}_j \rangle \neq 0$, where $\langle \vec{v}_i, \vec{v}_j \rangle$ denotes the Euclidean inner product. If two vertices v_i and v_j are not adjacent, then $\langle \vec{v}_i, \vec{v}_j \rangle = 0$. We say $\vec{X} = \{\vec{v}_i\}_{i=1}^n$ is a *vector representation* of G. Let

$$X = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}.$$

Then $X^T X$ is a positive semidefinite matrix called the *Gram Matrix* of $\vec{X}$ with respect to the Euclidean inner product. The graph of $X^T X$ has vertices $1, 2, \ldots, n$ corresponding to the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ and edges corresponding to nonzero inner products among these vectors, i.e., $\mathcal{G}(X^T X) \cong G$. Furthermore, $\mathrm{mr}_+^{\mathbb{R}}(G) \leq d$ if and only if there is a vector representation of G in $\mathbb{R}^d$ (and analogously for $\mathrm{mr}_+^{\mathbb{C}}(G)$ and $\mathbb{C}^d$).

2. Hypercube. The *n-cube* Q_n, $n \geq 1$, is defined as the repeated Cartesian product of n complete graphs on two vertices. Specifically, $Q_1 = K_2$ and $Q_n = Q_{n-1} \square K_2$ for $n \geq 2$. The n-cube is often referred to as the *nth hypercube.* If $V(K_2) = \{0, 1\}$, then the vertex set of Q_n can be viewed as the set of n-tuples $(v_1, v_2, \ldots, v_n)$, where $v_i \in \{0, 1\}$. Moreover, two n-tuples share an edge if they differ in exactly one coordinate. The hypercubes Q_3 and Q_4 are shown in Figure 2.1.

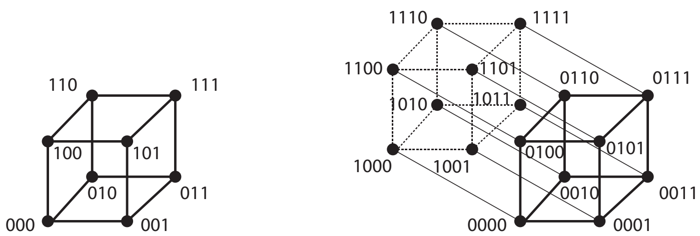

FIG. 2.1. *The hypercubes Q_3 and Q_4.*

Mitchell et al. [20] determined the minimum positive semidefinite rank of Q_3. In the next theorem, we determine the positive semidefinite maximum nullity and zero forcing number of all hypercubes. We construct a vector representation recursively. Note that the maximum nullity, maximum positive semidefinite nullity, zero forcing number, and positive semidefinite zero forcing number of the hypercube are all equal, and our method produces a universally optimal matrix, thereby establishing field independence. The technique of vector representation has not previously been used to establish field independence of minimum rank or to find a universally optimal

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matrix. We begin by illustrating the construction by example.

EXAMPLE 2.1. Let

$$X_3 = \begin{bmatrix} X_2 & C_2' \\ C_2 & X_2' \end{bmatrix} = \left[\begin{array}{cccc|cccc} 1 & 1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 0 & 1 & 0 & 1 & 1 \end{array}\right],$$

where the set of column vectors of X_2 is a vector representation of Q_2 and X_2' is obtained by interchanging the first and second columns of X_2 as well as the third and fourth columns of X_2. Furthermore,

$$C_2 = \left[\begin{array}{c|c} C_1 & 0 \\ \hline 0 & C_1' \end{array}\right] \text{ and } C_2' = \left[\begin{array}{c|c} C_1' & 0 \\ \hline 0 & C_1 \end{array}\right],$$

where $C_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}$ and $C_1' = \begin{bmatrix} -1 & 0 \end{bmatrix}$. Then the set of column vectors of X_3 is a vector representation of Q_3 because

$$X_3^T X_3 = \begin{bmatrix} 1 & 1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 3 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 3 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 3 \end{bmatrix}.$$

DEFINITION 2.2. Let $C_n = C_{n-1} \oplus C_{n-1}'$ and $C_n' = C_{n-1}' \oplus C_{n-1}$, where $C_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}$ and $C_1' = \begin{bmatrix} -1 & 0 \end{bmatrix}$. Let

$$X_n = \begin{bmatrix} X_{n-1} & C_{n-1}' \\ C_{n-1} & X_{n-1}' \end{bmatrix},$$

where $X_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}$ and the set of column vectors of X_{n-1} is a vector representation of Q_{n-1} and $X_{n-1}' = X_{n-1}P_{n-1}$, where $P_{n-1} = \underbrace{S_2 \oplus \cdots \oplus S_2}_{2^{n-2}}$ and $S_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

OBSERVATION 2.3. If the set of column vectors of X_n is a vector representation of Q_n and $X_n' = X_n P_n$, where $P_n = \underbrace{S_2 \oplus \cdots \oplus S_2}_{2^{n-1}}$ and $S_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then $\mathcal{G}((X_n')^T X_n') = Q_n$.

THEOREM 2.4. $\mathrm{M}_+(Q_n) = \mathrm{Z}_+(Q_n) = 2^{n-1}$ *and* $\mathrm{mr}_+(Q_n) = 2^{n-1}$, *and the set of column vectors of* X_n *is a vector representation of* Q_n.

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Proof. The proof is by inductively constructing a vector representation of Q_n in $\mathbb{R}^{2^{n-1}}$. We can extend the pattern illustrated in Example 2.1 in general. Let C_n and C_n' be defined as in Definition 2.2. Since $C_1^TC_1$ and $(C_1')^TC_1'$ are diagonal, so are $C_n^TC_n = C_{n-1}^TC_{n-1} \oplus (C_{n-1}')^TC_{n-1}'$ and $(C_n')^TC_n' = (C_{n-1}')^TC_{n-1}' \oplus C_{n-1}^TC_{n-1}$. Next, we give an equivalent formulation of C_n'. Let $P_n = \underbrace{S_2 \oplus \cdots \oplus S_2}_{2^{n-1}}$, where $S_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. We show by induction that $C_n' = -C_nP_n$. Clearly, $C_1' = -C_1P_1$. Suppose $C_{n-1}' = -C_{n-1}P_{n-1}$. Then

$$\begin{aligned} C_n' &= C_{n-1}' \oplus C_{n-1} \\ &= -C_{n-1}P_{n-1} \oplus C_{n-1} \\ &= -(C_{n-1} \oplus -C_{n-1}P_{n-1})P_n \\ &= -(C_{n-1} \oplus C_{n-1}')P_n \\ &= -C_nP_n. \end{aligned}$$

Let $X_n = \begin{bmatrix} X_{n-1} & C_{n-1}' \\ C_{n-1} & X_{n-1}' \end{bmatrix}$, as in Definition 2.2. First, we show by induction that $X_n^TC_n' + C_n^TX_n' = \underbrace{D_2 \oplus \cdots \oplus D_2}_{2^{n-1}}$, where $D_2 := \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Observe that $X_1^TC_1' + C_1^TX_1' = D_2$. Assume $X_{n-1}^TC_{n-1}' + C_{n-1}^TX_{n-1}' = \underbrace{D_2 \oplus \cdots \oplus D_2}_{2^{n-2}}$. Then

$$\begin{aligned} X_n^TC_n' + C_n^TX_n' &= X_n^TC_n' + C_n^TX_nP_n \\ &= \begin{bmatrix} X_{n-1}^T & C_{n-1}^T \\ (C_{n-1}')^T & (X_{n-1}')^T \end{bmatrix} \begin{bmatrix} C_{n-1}' & 0 \\ 0 & C_{n-1} \end{bmatrix} + \begin{bmatrix} C_{n-1}^T & 0 \\ 0 & (C_{n-1}')^T \end{bmatrix} \begin{bmatrix} X_{n-1} & C_{n-1}' \\ C_{n-1} & X_{n-1}' \end{bmatrix} P_n \\ &= \begin{bmatrix} X_{n-1}^TC_{n-1}' & C_{n-1}^TC_{n-1} \\ (C_{n-1}')^TC_{n-1}' & (X_{n-1}')^TC_{n-1} \end{bmatrix} + \begin{bmatrix} C_{n-1}^TX_{n-1}P_{n-1} & C_{n-1}^TC_{n-1}'P_{n-1} \\ (C_{n-1}')^TC_{n-1}P_{n-1} & (C_{n-1}')^TX_{n-1}'P_{n-1} \end{bmatrix} \\ &= \begin{bmatrix} X_{n-1}^TC_{n-1}' + C_{n-1}^TX_{n-1}' & 0 \\ 0 & (X_{n-1}')^TC_{n-1} + (C_{n-1}')^TX_{n-1} \end{bmatrix} \\ &= \begin{bmatrix} X_{n-1}^TC_{n-1}' + C_{n-1}^TX_{n-1}' & 0 \\ 0 & (X_{n-1}^TC_{n-1}' + C_{n-1}^TX_{n-1}')^T \end{bmatrix} \\ &= \big(D_2 \oplus \cdots \oplus D_2\big) \oplus \big(D_2 \oplus \cdots \oplus D_2\big). \end{aligned}$$

Finally, we show by induction that the set of column vectors of X_n is a vector

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representation of Q_n.

$$X_n^T X_n = \begin{bmatrix} X_{n-1}^T & C_{n-1}^T \\ (C'_{n-1})^T & (X'_{n-1})^T \end{bmatrix} \begin{bmatrix} X_{n-1} & C'_{n-1} \\ C_{n-1} & X'_{n-1} \end{bmatrix}$$
$$= \begin{bmatrix} X_{n-1}^T X_{n-1} + C_{n-1}^T C_{n-1} & X_{n-1}^T C'_{n-1} + C_{n-1}^T X'_{n-1} \\ (C'_{n-1})^T X_{n-1} + (X'_{n-1})^T C_{n-1} & (C'_{n-1})^T C'_{n-1} + (X'_{n-1})^T X'_{n-1} \end{bmatrix}.$$

By the induction hypothesis, the set of column vectors of X_{n-1} is a vector representation of Q_{n-1}, so $\mathcal{G}(X_{n-1}^T X_{n-1}) = Q_{n-1}$. Because $X'_{n-1} = X_{n-1}P_{n-1}$ and the set of column vectors of X_{n-1} is a vector representation of Q_{n-1}, $\mathcal{G}((X'_{n-1})^T X'_{n-1}) = Q_{n-1}$ by Observation 2.3. Since $C_{n-1}^T C_{n-1}$ and $(C'_{n-1})^T C'_{n-1}$ are diagonal, $X_{n-1}^T X_{n-1} + C_{n-1}^T C_{n-1} \in \mathcal{S}_+(Q_{n-1})$ and $(X'_{n-1})^T X'_{n-1} + (C'_{n-1})^T C'_{n-1} \in \mathcal{S}_+(Q_{n-1})$. Furthermore, $X_{n-1}^T C'_{n-1} + C_{n-1}^T X'_{n-1} = \underbrace{D_2 \oplus \cdots \oplus D_2}_{2^{n-2}}$. Thus, $\mathcal{G}(X_n^T X_n) = Q_n$.

We have $\mathrm{M}_+^{\mathbb{R}}(Q_n) \le \mathrm{Z}_+(Q_n) \le \mathrm{Z}(Q_n) = 2^{n-1}$ by [1], so $2^{n-1} = 2^n - 2^{n-1} \le \mathrm{mr}_+^{\mathbb{R}}(Q_n) \le 2^{n-1}$ as X_n is a representation of Q_n in $\mathbb{R}^{2^{n-1}}$. □

Theorem 2.5. *The minimum rank of Q_n is field independent and $X_n^T X_n$ is a universally optimal matrix for Q_n for the representation X_n in Definition* 2.2.

Proof. Given a graph G, $\mathrm{M}^F(G) \le \mathrm{Z}(G)$ for any field F, and $\mathrm{Z}(Q_n) = 2^{n-1}$ by [1]. Thus, if we assume $X_n^T X_n \in \mathcal{S}(F, Q_n)$, $2^n - 2^{n-1} = 2^{n-1} \le \mathrm{mr}^F(Q_n) \le \mathrm{rank}^F(X_n^T X_n) \le 2^{n-1}$. Hence, $\mathrm{mr}^F(Q_n) = 2^{n-1} = \mathrm{mr}(Q_n)$, establishing field independence of the minimum rank of Q_n. To show $X_n^T X_n$ is a universally optimal matrix for Q_n and thus that Q_n is field independent, it suffices to show that every off-diagonal entry of $X_n^T X_n$ is 0, 1, or -1. The proof is by induction. The off-diagonal entries of $X_1^T X_1$ are 1. Suppose every off-diagonal entry of $X_{n-1}^T X_{n-1}$ is 0, 1, or -1. Then the off-diagonal entries of $X_{n-1}^T X_{n-1} + C_{n-1}^T C_{n-1}$ and $(C'_{n-1})^T C'_{n-1} + (X'_{n-1})^T X'_{n-1}$ are 0, 1, and -1 as $C_{n-1}^T C_{n-1}$ and $(C'_{n-1})^T C'_{n-1}$ are diagonal matrices. We have already established in the proof of Theorem 2.4 that $X_{n-1}^T C'_{n-1} + C_{n-1}^T X'_{n-1} = \underbrace{D_2 \oplus \cdots \oplus D_2}_{2^{n-2}}$.

Thus, the off-diagonal entries of $X_n^T X_n$ are 0, 1, or -1. □

3. Graph families. In this section, we determine the positive semidefinite maximum nullity and zero forcing number of a variety of graph families. The results are summarized in Table 3.1. Many of these graph families appear in a graph catalog developed through the American Institute of Mathematics workshop "Spectra of Families of Matrices described by Graphs, Digraphs, and Sign Patterns" [2]. We use a variety of known results to determine the maximum positive semidefinite nullity.

The vertex connectivity $\kappa(G)$ of a connected graph G is the minimum size of $S \subseteq V(G)$ such that $G - S$ is disconnected or a single vertex. The following result

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is especially useful when the vertex connectivity and the positive semidefinite zero forcing number of a graph agree.

THEOREM 3.1. [18, 19] *For a graph G,* $\mathrm{M}_+^{\mathbb{R}}(G) \geq \kappa(G)$.

A *minor* of a graph G is obtained by performing a series of edge deletions, deletions of isolated vertices, and edge contractions. If $\beta(H) \leq \beta(G)$ for any minor H of G, then the graph parameter β is said to be *minor monotone*. Colin de Verdière ([7] in English) introduced a graph parameter $\mu(G)$ that was the first of several parameters that bound the maximum nullity from below and are minor monotone. The parameter $\nu(G)$ [6] is minor monotone and is the maximum nullity among matrices $A \in S_n$ satisfying:

- $\mathcal{G}(A) = G$.
- A is positive semidefinite.
- A satisfies the Strong Arnold Hypothesis.

Špacapan established the following result concerning the vertex connectivity of a Cartesian product. We use this result to determine the vertex connectivity of the Cartesian product of a complete graph and a path, thereby providing a lower bound for the maximum positive semidefinite nullity.

THEOREM 3.2. [22] *Let G and H be graphs of order at least two. Then*

$$\kappa(G \square H) = \min\{\kappa(G)|H|, \kappa(H)|G|, \delta(G) + \delta(H)\}.$$

PROPOSITION 3.3. *For $s \geq 2$ and $t \geq 2$,* $\mathrm{M}_+(K_s \square P_t) = \mathrm{Z}_+(K_s \square P_t) = s$ *and* $\mathrm{mr}_+(K_s \square P_t) = s(t-1)$.

Proof. By Theorem 3.2, $\kappa(K_s \square P_t) = \min\{\kappa(K_s)|P_t|, \kappa(P_t)|K_s|, \delta(K_s) + \delta(P_t)\} = \min\{(s-1)t, s, s\} = s$. Hence, $s = \kappa(K_s \square P_t) \leq \mathrm{M}_+^{\mathbb{R}}(K_s \square P_t) \leq \mathrm{Z}_+(K_s \square P_t) \leq \mathrm{Z}(K_s \square P_t) = s$ by Theorem 3.1 and [1]. Hence, all inequalities are equalities. $\square$

For the complete multipartite graph, the vertex connectivity and the positive semidefinite zero forcing number are equal.

PROPOSITION 3.4. *For $n_1 \geq n_2 \geq \cdots \geq n_k > 0$,*

$$\mathrm{M}_+(K_{n_1,n_2,\ldots,n_k}) = \mathrm{Z}_+(K_{n_1,n_2,\ldots,n_k}) = n_2 + n_3 + \cdots + n_k$$

and $\mathrm{mr}_+(K_{n_1,n_2,\ldots,n_k}) = n_1$.

If a graph has very low or very high maximum positive semidefinite nullity or positive semidefinite zero forcing number, then the two parameters are equal. In

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particular, $\mathrm{M}_+(G) = \mathrm{Z}_+(G)$ if $\mathrm{M}_+(G) \leq 2$ or $\mathrm{Z}_+(G) \geq |G| - 2$ [9]. The technique of using $\mathrm{M}_+(G) = \mathrm{Z}_+(G)$ is used to prove the next proposition.

PROPOSITION 3.5. *$\mathrm{M}_+(G) = \mathrm{Z}_+(G) = 2$ and $\mathrm{mr}_+(G) = |G|-2$ for G a polygonal path and G a unicyclic graph. Let T be a tree with $|T| = n$, $n \geq 4$, and $T \neq K_{1,n-1}$. Then $\mathrm{M}_+(\overline{T}) = \mathrm{Z}_+(\overline{T}) = n - 3$ and $\mathrm{mr}_+(\overline{T}) = 3$.*

COROLLARY 3.6. *For $s \geq 2$, $\mathrm{M}_+(P_s \square P_2) = \mathrm{Z}_+(P_s \square P_2) = 2$ and $\mathrm{mr}_+(P_s \square P_2) = 2s - 2$.*

Proof. Since $P_s \square P_2$ is a polygonal path, the result follows from Proposition 3.5. ☐

If we can find a minor H of G with $\nu(H) = \mathrm{Z}(G)$, then the maximum nullity, maximum positive semidefinite nullity, zero forcing number, and positive semidefinite zero forcing number of G are all equal, as stated in the next observation. We use this technique to prove Proposition 3.9, Proposition 3.10, and Corollary 3.13.

OBSERVATION 3.7. If G has a minor H with $\nu(H) = \mathrm{Z}(G)$, then $\nu(H) = \nu(G) = \mathrm{M}_+(G) = \mathrm{M}(G) = \mathrm{Z}_+(G) = \mathrm{Z}(G)$.

Hein van der Holst [16] showed that $\xi(Q_3) = 4$. Let X_3 and X_4 be the vector representations in Definition 2.2. One can verify by computation with software that $X_3^T X_3 \in \mathcal{S}_+(Q_3)$ satisfies the Strong Arnold Hypothesis. This result is useful because Q_3 is a minor of $C_s \square P_2$ for $s \geq 4$ and of $C_4 \square P_t$, and the standard zero forcing number of these graphs is equal to $\nu(Q_3)$. Note that $X_4^T X_4 \in \mathcal{S}_+(Q_4)$ does not satisfy the Strong Arnold Hypothesis.

PROPOSITION 3.8. *Let X_3 be the vector representation in Definition 2.2. Then the matrix $X_3^T X_3 \in \mathcal{S}_+(Q_3)$ satisfies the Strong Arnold Hypothesis, and so $\nu(Q_3) = 4$.*

PROPOSITION 3.9. *For $s \geq 4$, $\mathrm{M}_+(C_s \square P_2) = \mathrm{Z}_+(C_s \square P_2) = 4$, $\mathrm{mr}_+(C_s \square P_2) = 2s - 4$ and for $t \geq 2$, $\mathrm{M}_+(C_4 \square P_t) = \mathrm{Z}_+(C_4 \square P_t) = 4$ and $\mathrm{mr}_+(C_4 \square P_t) = 4t - 4$.*

Proof. Observe that Q_3 is a minor of $C_s \square P_2$ for $s \geq 4$ and of $C_4 \square P_t$. We know that $\nu(Q_3) = 4$ by Proposition 3.8, so $\nu(Q_3) = 4 = \mathrm{Z}(C_s \square P_2) = \mathrm{Z}(C_4 \square P_t)$ by [1]. By Observation 3.7, $\mathrm{M}_+(C_s \square P_2) = \mathrm{Z}_+(C_s \square P_2) = \mathrm{M}_+(C_4 \square P_t) = \mathrm{Z}_+(C_4 \square P_t) = 4$. ☐

The *sth half-graph* H_s is the graph constructed from the disjoint union of K_s and $\overline{K_s}$ by adding all edges $\{u_i, v_j\}$ such that $i + j \leq s + 1$, where $u_1, \ldots, u_s$ are the vertices of K_s and $v_1, \ldots, v_s$ are the vertices of $\overline{K_s}$. Although the next proposition uses minors, it is worth noting that the matrix A_s of rank s constructed in [8] whose graph is H_s is positive semidefinite. A *wheel graph* W_n is constructed from C_{n-1} by

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adding one vertex that is adjacent to each of the vertices of C_{n-1}.

PROPOSITION 3.10. *For* $n \geq 4$, $\mathrm{M}_+(W_n) = \mathrm{Z}_+(W_n) = 3$ *and* $\mathrm{mr}_+(W_n) = n-3$ *and* $\mathrm{M}_+(H_s) = \mathrm{Z}_+(H_s) = s$ *and* $\mathrm{mr}_+(H_s) = s$.

Proof. Observe that K_4 is a minor of W_n. Moreover, $\nu(K_4) = 3 = \mathrm{Z}(W_n)$ by [14]. By Observation 3.7, $\mathrm{M}_+(W_n) = \mathrm{Z}_+(W_n) = 3$.

Let $R = \{u_1, \ldots, u_s, v_1\} \subseteq V(H_s)$. Then $H_s[R] = K_{s+1}$ is a minor of H_s. Moreover, $\nu(K_{s+1}) = s = \mathrm{Z}(H_s)$ by [8]. By Observation 3.7, $\mathrm{M}_+(H_s) = \mathrm{Z}_+(H_s) = s$. ∎

Although the maximum positive semidefinite nullity and positive semidefinite zero forcing number have not been determined in general for the Cartesian product of two complete graphs and the Cartesian product of a cycle and a complete graph, we use vector representations to establish these values for the Cartesian product of a complete graph on three vertices with itself and the Cartesian product of a four cycle with a complete graph on three vertices. Since the maximum nullity, maximum positive semidefinite nullity, zero forcing number, and positive semidefinite zero forcing number are all equal for these particular graphs, we conjecture that the same is true in general for the Cartesian product of two complete graphs and the Cartesian product of a cycle and a complete graph. We number the vertices of $K_3 \square K_3$ and $C_4 \square K_3$ as in Figure 3.1.

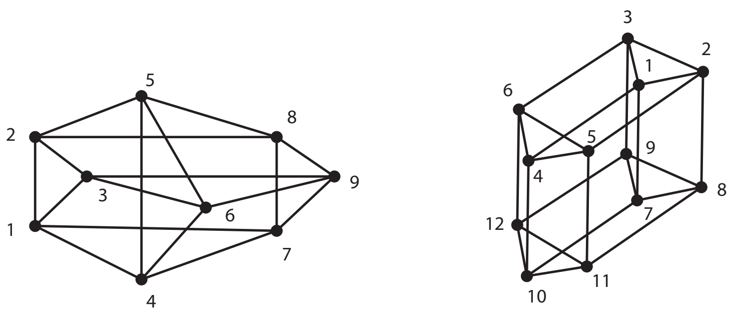

FIG. 3.1. *The Cartesian products* $K_3 \square K_3$ *and* $C_4 \square K_3$.

PROPOSITION 3.11. $\mathrm{M}_+(K_3 \square K_3) = \mathrm{Z}_+(K_3 \square K_3) = 5$ *and* $\mathrm{mr}_+(K_3 \square K_3) = 4$.

Proof. Let

$$X = \begin{bmatrix} 1 & 1/2 & -1/2 & 2 & 0 & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & -1 & 0 & 1/2 & 0 \\ 0 & 0 & -2 & 0 & 0 & -1 & 1 & 1/2 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

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Then the set of column vectors of X is a vector representation of $K_3 \square K_3$. ◻

PROPOSITION 3.12. *$\mathrm{M}_+(C_4 \square K_3) = \mathrm{Z}_+(C_4 \square K_3) = 6 = \nu(C_4 \square K_3)$ and $\mathrm{mr}_+(C_4 \square K_3) = 6$.*

Proof. Let

$$X = \begin{bmatrix} 1 & 1/2 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1/2 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1/2 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 1/2 & 1 \\ 0 & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Then the set of column vectors of X is a vector representation of $C_4 \square K_3$. One can verify by computation with software that $X^T X \in \mathcal{S}_+(C_4 \square K_3)$ satisfies the Strong Arnold Hypothesis, so $\nu(C_4 \square K_3) = 6$. ◻

COROLLARY 3.13. *For $s \geq 4$,*

$$\mathrm{M}_+(C_s \square K_3) = \mathrm{Z}_+(C_s \square K_3) = 6$$

and $\mathrm{mr}_+(C_s \square K_3) = 3s - 6$.

Proof. Observe that $C_4 \square K_3$ is a minor of $C_s \square K_3$ for $s \geq 4$. Thus, $\nu(C_4 \square K_3) = 6 = \mathrm{Z}(C_s \square K_3)$ by Proposition 3.12 and [1]. By Observation 3.7, $\mathrm{M}_+(C_s \square K_3) = \mathrm{Z}_+(C_s \square K_3) = 6$. ◻

Members of the AIM Minimum Rank - Special Graphs Work Group [1] used the technique in the next observation (using cliques) to determine the minimum positive semidefinite rank of the supertriangle T_n, the strong product of two paths $P_s \boxtimes P_t$, and the corona $K_t \circ K_s$. The (m,k)-pineapple (with $m \geq 3$, $k \geq 2$) is $P_{m,k} = K_m \cup K_{1,k}$ where $K_m \cap K_{1,k}$ is the degree k vertex of $K_{1,k}$. The *s-helm* $W(1,s)$ is constructed from an s-sun by adding a star vertex adjacent to each vertex on the s-cycle.

OBSERVATION 3.14. If $G = \cup_{i=1}^h G_i$, then $\mathrm{mr}_+^{\mathbb{R}}(G) \leq \sum_{i=1}^h \mathrm{mr}_+^{\mathbb{R}}(G_i)$ and $\mathrm{mr}_+^{\mathbb{C}}(G) \leq \sum_{i=1}^h \mathrm{mr}_+^{\mathbb{C}}(G_i)$.

PROPOSITION 3.15. *For $k \geq 2$, $m \geq 3$, $\mathrm{M}_+(K_m \cup K_{1,k}) = \mathrm{Z}_+(K_m \cup K_{1,k}) = m-1$ and $\mathrm{mr}_+(K_m \cup K_{1,k}) = k+1$, $\mathrm{M}_+(C_t \circ K_s) = \mathrm{Z}_+(C_t \circ K_s) = st - t + 2$ and $\mathrm{mr}_+(C_t \circ K_s) = 2t - 2$, and $\mathrm{M}_+(W(1,s)) = \mathrm{Z}_+(W(1,s)) = 3$ and $\mathrm{mr}_+(W(1,s)) = 2s - 2$.*

The *line graph* $L(G)$ of a graph $G = (V,E)$ is the graph with vertex set E, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G share an endpoint. For a graph $G = (V,E)$, an orientation G^τ assigns to each edge $\{u,v\}$

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exactly one of the two arcs $(u,v),(v,u)$. The *incidence matrix* of an orientation G^τ is the $|V|\times|E|$ $\{0,\pm1\}$-matrix $D(G^\tau)=[d_{ve}]$ having rows indexed by the vertices and columns indexed by the oriented edges of G, where $d_{ve}=\begin{cases} 0 & \text{if } v\notin e,\\ 1 & \text{if } \exists w, e=(w,v),\\ -1 & \text{if } \exists w, e=(v,w).\end{cases}$

If G is connected, $\operatorname{rank}(D(G^\tau))=|G|-1$ [11, Theorem 8.3.1].

PROPOSITION 3.16. *$\mathrm{M}_+(L(K_n))=\mathrm{Z}_+(L(K_n))=\frac{n(n-1)}{2}-n+2$ and $\mathrm{mr}_+(L(K_n))=n-2$.*

Proof. To show that $\mathrm{mr}(L(K_n))\le n-2$, the authors in [1] constructed a matrix in $\mathcal{S}(L(K_n))$ of rank at most $n-2$. We show that the matrix constructed in [1] is in fact positive semidefinite. Let D denote the incidence matrix of an orientation of K_{n-1}. Then $\operatorname{rank}(D)=n-2$. Let

$$M=\begin{bmatrix} B & D\\ D^T & D^TD\end{bmatrix},$$

where J_{n-1} is the matrix of ones and $B=I_{n-1}-\frac{1}{n-1}J_{n-1}$. Recall that the vertices of $L(K_n)$ are the edges of K_n. The matrix partition corresponds to the edges that are incident with vertex 1 of K_n and those that are not; thus B is $(n-1)\times(n-1)$. Observe that $M\in\mathcal{S}(L(K_n))$. Since $D^TJ_{n-1}=0$, we have

$$D^T(I_{n-1}-\frac{1}{n-1}J_{n-1})=D^T.$$

So

$$\begin{bmatrix} I & 0\\ -D^T & I\end{bmatrix}\begin{bmatrix} B & D\\ D^T & D^TD\end{bmatrix}=\begin{bmatrix} B & D\\ 0 & 0\end{bmatrix}.$$

The columns of B and of D are orthogonal to the all 1s vector, so $\operatorname{rank}(M)=\operatorname{rank}([B\ D])\le n-2$.

Observe that $B=B^T$. Computation shows that $B^TB=B$. Since $D^T(I_{n-1}-\frac{1}{n-1}J_{n-1})=D^T$, $(I_{n-1}-\frac{1}{n-1}J_{n-1})^TD=D$, that is $BD=D$. For any real matrix X, X^TX is positive semidefinite. Let

$$X=\begin{bmatrix} B & D\\ 0 & 0\end{bmatrix}.$$

Then

$$X^TX=\begin{bmatrix} B^T & 0\\ D^T & 0\end{bmatrix}\begin{bmatrix} B & D\\ 0 & 0\end{bmatrix}=\begin{bmatrix} B^TB & B^TD\\ D^TB & D^TD\end{bmatrix}=\begin{bmatrix} B & D\\ D^T & D^TD\end{bmatrix}=M$$

Thus, M is positive semidefinite and $\mathrm{mr}_+^{\mathbb{R}}(L(K_n))\le n-2$.

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Since K_n has a Hamiltonian path, $L(P_n) = P_{n-1}$ is an induced subgraph of $L(K_n)$. As a consequence of this fact and [1, Proposition 4.6], $\mathrm{Z}(L(K_n)) = \frac{n(n-1)}{2} - n + 2$. So

$$\frac{n(n-1)}{2} - n + 2 \le \mathrm{M}_+^{\mathbb{R}}(L(K_n)) \le \mathrm{Z}_+(L(K_n)) \le \mathrm{Z}(L(K_n)) = \frac{n(n-1)}{2} - n + 2.$$

Hence, all inequalities are equalities. □

It is known that if H is a subgraph of G (not necessarily induced), then $L(H)$ is an induced subgraph of $L(G)$. Any graph G of order n is a subgraph of K_n, so $L(G)$ is an induced subgraph of $L(K_n)$. Now, if G contains P_n as a subgraph, that is, G has a Hamiltonian path, then $L(G)$ contains $L(P_n) = P_{n-1}$ as an induced subgraph. As in [1], the next two results then follow from Proposition 3.16.

Corollary 3.17. *If* $|G| = n$, *then* $\mathrm{mr}_+(L(G)) \le n - 2$.

Corollary 3.18. *If G contains a Hamiltonian path and* $|G| = n \ge 2$, *then* $\mathrm{mr}_+(L(G)) = n - 2$.

The *necklace* N_s with s diamonds is a 3-regular graph formed from a $3s$-cycle by attaching s *diamond* vertices, where each diamond vertex is adjacent to three consecutive cycle vertices and distinct diamond vertices have disjoint neighborhoods.

Remark 3.19. [21] Observe that each diamond vertex is a duplicate vertex (u and v are said to be duplicate vertices if $N[u] = N[v]$, where $N[u]$ denotes the closed neighborhood of u) of the middle vertex of the three consecutive cycle vertices to which it is adjacent. The induced subgraph obtained by a sequential deletion of the s duplicate vertices of N_s is a $3s$-cycle. We know $\mathrm{mr}_+^{\mathbb{R}}(C_{3s}) = 3s - 2$. By [5, Corollary 2.3], $3s - 2 = \mathrm{mr}_+^{\mathbb{R}}(C_{3s}) = \mathrm{mr}_+^{\mathbb{R}}(N_s)$. Hence, $\mathrm{M}_+^{\mathbb{R}}(N_s) = s + 2$. By [9, Proposition 5.11], $2 = \mathrm{Z}_+(C_{3s}) = \mathrm{Z}_+(N_s) - s$, and so $\mathrm{Z}_+(N_s) = s + 2$.

A *block* of a graph is a maximal nonseparable induced subgraph. A graph is *block-clique* if every block is a clique. Recall that a graph is *chordal* if it does not contain an induced cycle on four or more vertices.

Remark 3.20. [21] Let G be a block-clique graph and let $b(G)$ denote the number of blocks of G. Since block-clique graphs are chordal, $\mathrm{mr}_+^{\mathbb{C}}(G) = \mathrm{cc}(G) = b(G)$ by [5, Theorem 3.6]. Furthermore, $\mathrm{OS}(G) = \mathrm{cc}(G)$ by [12, Proposition 3.6] (see [12] for a definition of the OS-number). As $\mathrm{Z}_+(G) + \mathrm{OS}(G) = |G|$ by [3, Theorem 3.6], $\mathrm{M}_+^{\mathbb{C}}(G) = \mathrm{Z}_+(G) = |G| - b(G)$. Note that $\mathrm{cc}(G) = \mathrm{mr}_+^{\mathbb{C}}(G) \le \mathrm{mr}_+^{\mathbb{R}}(G) \le \mathrm{cc}(G)$, and so we have equality throughout. It follows that $\mathrm{M}_+(G) = \mathrm{Z}_+(G) = |G| - b(G)$ and $\mathrm{mr}_+(G) = b(G)$.

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TABLE 3.1
Summary of positive semidefinite maximum nullity and zero forcing number results.

result #	G	order	$\mathrm{Z}_+(G)$	$\mathrm{M}_+(G)$	= [1]
[15]	P_n	n	1	1	Y
[15]	C_n	n	2	2	Y
[10]	K_n	n	$n-1$	$n-1$	Y
[15]	T (tree)	n	1	1	N
2.4	Q_n (hypercube)	2^n	2^{n-1}	2^{n-1}	Y
3.3	$K_s \Box P_t$	st	s	s	Y
3.4	$K_{n_1,n_2,\ldots,n_k}$, $n_1 \geq n_2 \geq \cdots \geq n_k > 0$	$n_1+n_2+\cdots+n_k$	$n_2+\cdots+n_k$	$n_2+\cdots+n_k$	N
3.5	polygonal path G	$\|G\|$	2	2	Y
3.5	unicyclic graph G	$\|G\|$	2	2	N
[1], 3.5	$\overline{T}$, T a tree, $\|T\|=n$, $n \geq 4$, $T \neq K_{1,n-1}$	n	$n-3$	$n-3$	Y
3.6	$P_s \Box P_2,\ s \geq 2$	$2s$	2	2	Y
Conj.	$P_s \Box P_t,\ s \geq t$	st	t	t	Y
3.9	$C_s \Box P_2,\ s \geq 4$	$2s$	4	4	Y
3.9	$C_4 \Box P_t,\ t \geq 2$	$4t$	4	4	Y
Conj.	$C_s \Box P_t$	st	$\min\{s,2t\}$	$\min\{s,2t\}$	Y
3.10	$W_n,\ n \geq 4$	n	3	3	Y
3.10	H_s (half-graph)	$2s$	s	s	Y
3.11	$K_3 \Box K_3$	9	5	5	Y
Conj.	$K_s \Box K_t$	st	$st-s-t+2$	$st-s-t+2$	Y
3.12, 3.13	$C_s \Box K_3,\ s \geq 4$	$3s$	6	6	Y
Conj.	$C_s \Box K_t,\ s \geq 4$	st	$2t$	$2t$	Y
3.15	$P_{m,k} = K_m \cup K_{1,k}$, $k \geq 2$, $m \geq 3$	$m+k$	$m-1$	$m-1$	N
3.15	$C_t \circ K_s$	$st+t$	$st-t+2$	$st-t+2$	N, $s=1$ Y, $s>1$
3.15	$W(1,s)$ (s-helm)	$2s+1$	3	3	N
3.16	$L(K_n)$	$\frac{n(n-1)}{2}$	$\frac{n(n-1)}{2}-n+2$	$\frac{n(n-1)}{2}-n+2$	Y

[1] $\mathrm{M}(G) = \mathrm{Z}(G) = \mathrm{M}_+(G) = \mathrm{Z}_+(G)$?

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result #	G	order	$\mathrm{Z}_+(G)$	$\mathrm{M}_+(G)$	= [1]
[1], [21]	$L(T)$, T a tree and $l = \#$ pendent vertices of T	$\lvert T\rvert - 1$	$l-1$	$l-1$	Y
[1]	T_n (supertriangle)	$\frac{1}{2}n(n+1)$	n	n	Y
[1]	$P_s \boxtimes P_t$	st	$s+t-1$	$s+t-1$	Y
[1]	$K_t \circ K_s,\ t \geq 2$	$st+t$	$st-1$	$st-1$	Y
[20]	Möbius ladder ML_{2n}	$2n$	$\begin{cases} 3 & \text{if } n=3, \\ 4 & \text{if } n=4, \\ 4 & \text{else.} \end{cases}$	$\begin{cases} 3 & \text{if } n=3, \\ 3 & \text{if } n=4, \\ 4 & \text{else.} \end{cases}$	N, $n=3,4$ [2] Y, $n>4$
3.19	N_s	$4s$	$s+2$	$s+2$	Y
3.20	block-clique (1-chordal) graph G, $b(G)= \#$ blocks of G	$\lvert G\rvert$	$\lvert G\rvert - b(G)$	$\lvert G\rvert - b(G)$	N
[4]	$\overline{C_n},\ n \geq 5$	n	$n-3$	$n-3$	Y
[13], [21]	complement of a 2-tree, $n \geq 3$, $l= \#$ dominating vertices	n	$\begin{cases} n & \lvert H\rvert = 3, \\ n-1 & \lvert H\rvert \geq 4,\ l=2, \\ n-3 & \lvert H\rvert \geq 5,\ l=1, \\ n-4 & \lvert H\rvert \geq 6,\ l=0 \end{cases}$	$\begin{cases} n & \lvert H\rvert = 3, \\ n-1 & \lvert H\rvert \geq 4,\ l=2, \\ n-3 & \lvert H\rvert \geq 5,\ l=1, \\ n-4 & \lvert H\rvert \geq 6,\ l=0 \end{cases}$	Y

[1] $\mathrm{M}(G) = \mathrm{Z}(G) = \mathrm{M}_+(G) = \mathrm{Z}_+(G)$?
[2] Note that $3 = \mathrm{M}_+(ML_8) < \mathrm{Z}_+(ML_8) = 4$.

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REFERENCES

[1] AIM Minimum Rank – Special Graphs Work Group. Zero forcing sets and the minimum rank of graphs. *Linear Algebra and its Applications*, 428:1628–1648, 2008.

[2] Webpage for the 2006 American Institute of Mathematics workshop *Spectra of families of matrices described by graphs, digraphs, and sign patterns.* Available at `http://aimath.org/pastworkshops/matrixspectrum.html`. This webpage has links to the AIM minimum rank graph catalog: Families of graphs (available directly at `http://aimath.org/pastworkshops/catalog2.html`).

[3] F. Barioli, et al. Zero forcing parameters and minimum rank problems. *Linear Algebra and its Applications*, 433:401–411, 2010.

[4] M. Booth, et al. On the minimum semidefinite rank of a simple graph. *Linear and Multilinear Algebra*, 59(5):483–506, 2011.

[5] M. Booth, et al. On the minimum rank among positive semidefinite matrices with a given graph. *SIAM Journal of Matrix Analysis and Applications*, 30:731–740, 2008.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 815-830, September 2012

[6] Y. Colin de Verdière. Multiplicities of eigenvalues and tree-width of graphs. *Journal of Combinatorial Theory, Series B*, 74:121–146, 1998.

[7] Y. Colin de Verdière. On a new graph invariant and a criterion for planarity. *Graph Structure Theory, Contemporary Mathematics*, American Mathematical Society, Providence, 147:137–147, 1993.

[8] L. DeAlba, et al. Universally optimal matrices and field independence of the minimum rank of a graph. *Electronic Journal of Linear Algebra*, 18:403–419, 2009.

[9] J. Ekstrand, et al. Positive semidefinite zero forcing. *Submitted*, available at `http://www.public.iastate.edu/%7Elhogben/Zplus.pdf`.

[10] S. Fallat and L. Hogben. The minimum rank of symmetric matrices described by a graph: A survey. *Linear Algebra and its Applications*, 426:558–582, 2007.

[11] C. Godsil and G. Royle. *Algebraic Graph Theory.* Springer-Verlag, New York, 2001.

[12] P. Hackney, et al. Linearly independent vertices and minimum semidefinite rank. *Linear Algebra and its Applications*, 431:1105–1115, 2009.

[13] L. Hogben. Orthogonal representations, minimum rank, and graph complements. *Linear Algebra and its Applications*, 428:2560–2568, 2008.

[14] L. Hogben. Spectral graph theory and the inverse eigenvalue problem of a graph. *Electronic Journal of Linear Algebra*, 14:12–31, 2005.

[15] H. van der Holst. Graphs whose positive semi-definite matrices have nullity at most two. *Linear Algebra and its Applications*, 375:1–11, 2003.

[16] H. van der Holst. Three-connected graphs whose maximum nullity is at most three. *Linear Algebra and its Applications*, 429:625–632, 2008.

[17] L.-H. Huang, G.J. Chang, and H.-G. Yeh. A note on universally optimal matrices and field independence of the minimum rank of a graph. *Linear Algebra and its Applications*, 433:585–594, 2010.

[18] L. Lovász, M. Saks, and A. Schrijver. Orthogonal representations and connectivity of graphs. *Linear Algebra and its Applications*, 114/115:439–454, 1989.

[19] L. Lovász, M. Saks, and A. Schrijver. A correction: "Orthogonal representations and connectivity of graphs". *Linear Algebra and its Applications*, 313:101–105, 2000.

[20] L. Mitchell, S. Narayan, and A. Zimmer. Lower bounds in minimum rank problems. *Linear Algebra and its Applications*, 432:430-440, 2010.

[21] T. Peters. *Positive Semidefinite Maximum Nullity and Zero Forcing Number.* Ph.D. Thesis, Iowa State University, 2012.

[22] S. Špacapan. Connectivity of Cartesian products of graphs. *Applied Mathematics Letters*, 21:682–685, 2008.