

ON GENERALIZED SCHUR COMPLEMENT OF NONSTRICTLY DIAGONALLY DOMINANT MATRICES AND GENERAL H -MATRICES*

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Abstract. This paper proposes the definition of the generalized Schur complement on nonstrictly diagonally dominant matrices and general H -matrices by using a particular generalized inverse, and then, establishes some significant results on heredity, nonsingularity and the eigenvalue distribution for these generalized Schur complements.

Key words. Generalized Schur complement, Nonstrictly diagonally dominant matrices, General H -matrices.

AMS subject classifications. 15A15, 15F10.

1. Introduction. As a useful tool, the (generalized) Schur complements have various important applications in many aspects of matrix theory, applied math, and statistics. A great deal of work on the topic has been done by a number of authors (see e.g. [1, 2, 5, 4, 8, 9, 11, 13, 14, 15, 20, 21, 22, 25, 33]). Recently, considerable interest in the work on Schur complements of strictly or irreducibly diagonally dominant matrices and generalized strictly diagonally dominant matrices (nonsingular H -matrices) has been witnessed and some properties such as diagonal dominance and the eigenvalue distribution on Schur complements of these matrices have been proposed. Readers are referred to [7, 10, 12, 16, 17, 18, 19, 24, 26, 28, 29, 30, 31, 32, 34, 35]. But, little attention is paid to the work on generalized Schur complements of nonstrictly

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diagonally dominant matrices and general H -matrices. Does the generalized Schur complement of these matrices have the same properties?

This paper proposes the definition of the generalized Schur complement on non-strictly diagonally dominant matrices and general H -matrices by using a particular generalized inverse, and devotes to study heredity, nonsingularity and the eigenvalue distribution of these generalized Schur complements. Furthermore, the following results are established:

Let an $n \times n$ matrix A be a nonstrictly diagonally dominant matrix (or a general H -matrix) and the subset $\alpha \subseteq N = \{1, 2, \dots, n\}$, and define the matrix $\hat{A} = (\hat{a}_{ij}) \in \mathbb{C}^{n \times n}$ and the matrix $A_\alpha = (\check{a}_{ij}) \in \mathbb{C}^{n \times n}$ by

$$\hat{a}_{ij} = \begin{cases} \operatorname{Re}(a_{ii}), & \text{if } i = j, \\ a_{ij}, & \text{otherwise} \end{cases} \quad \text{and} \quad \check{a}_{ij} = \begin{cases} \operatorname{Re}(a_{ii}), & i = j \in N - \alpha, \\ a_{ij}, & \text{otherwise,} \end{cases}$$

respectively.

- (i) Then the generalized Schur complement matrix $A/_0\alpha$ of the matrix A is diagonally dominant (or a general H -matrix) and $A/_0\alpha$ is nonsingular if and only if $\dim[N(A(\alpha))] = \dim[N(A)]$;
- (ii) If $\hat{A}$ is nonstrictly diagonally dominant (or a general H -matrix) and $\dim[N(\hat{A}(\alpha))] = \dim[N(\hat{A})]$, then the Schur complement matrix $A/_0\alpha$ of the matrix A has $|J_{R_+}(A)| - |J_{R_+}^\alpha(A)|$ eigenvalues with positive real part and $|J_{R_-}(A)| - |J_{R_-}^\alpha(A)|$ eigenvalues with negative real part;
- (iii) If A_α is nonstrictly diagonally dominant (or a general H -matrix) and $\dim[N(A(\alpha))] = \dim[N(A_\alpha)]$, then the Schur complement matrix $A/_0\alpha$ of the matrix A has $|J_{R_+}(A)| - |J_{R_+}^\alpha(A)|$ eigenvalues with positive real part and $|J_{R_-}(A)| - |J_{R_-}^\alpha(A)|$ eigenvalues with negative real part.

The paper is organized as follows. Some notations and preliminary results about special matrices are given in Section 2. The definition of generalized Schur complement for nonstrictly diagonally dominant matrices and general H -matrices is proposed in Section 3. Some results on heredity and nonsingularity on the generalized Schur complement of nonstrictly diagonally dominant matrices and general H -matrices are then presented in Section 4. The main results of this paper are given in Section 5, where we give the different conditions for the matrix A with complex diagonal entries and the subset $\alpha \subseteq N$ such that the generalized Schur complement matrix $A/_0\alpha$ has $|J_{R_+}(A)| - |J_{R_+}^\alpha(A)|$ eigenvalues with positive real part and $|J_{R_-}(A)| - |J_{R_-}^\alpha(A)|$ eigenvalues with negative real part.

2. Preliminaries. In this section, we give some notation and preliminary results about special matrices that are used in this paper. The set of all $m \times n$ complex (real) matrices is denoted by $\mathbb{C}^{m \times n}$ ($\mathbb{R}^{m \times n}$). Let $|\alpha|$ denote the cardinality of the set

$\alpha \subseteq \langle n \rangle = \{1, 2, \dots, n\}$. For nonempty index sets $\alpha, \beta \subseteq \langle n \rangle$, $A(\alpha, \beta)$ is the submatrix of $A \in \mathbb{C}^{n \times n}$ with row indices in α and column indices in β . The submatrix $A(\alpha, \alpha)$ is abbreviated to $A(\alpha)$. Let $A \in \mathbb{C}^{m \times n}$, $N(A) = \{x \in \mathbb{C}^n \mid Ax = 0\}$ is the null space of A . The dimension of the null space $N(A)$ is denoted $\dim[N(A)]$.

Let $A = (a_{ij}) \in \mathbb{R}^{m \times n}$ and $B = (b_{ij}) \in \mathbb{R}^{m \times n}$, we write $A \geq B$, if $a_{ij} \geq b_{ij}$ holds for all $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$. A matrix $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is a Z -matrix if $a_{ij} \leq 0$ for all $i \neq j$. We use Z_n to denote the set of all $n \times n$ Z -matrices. A matrix $A = (a_{ij}) \in Z_n$ is an M -matrix if A can be expressed in the form $A = sI - B$, where $B \geq 0$, and $s \geq \rho(B)$, the spectral radius of B . If $s > \rho(B)$, A is a nonsingular M -matrix; if $s = \rho(B)$, A is a singular M -matrix. M_n , $M_n^\bullet$ and M_n^0 will be used to denote the set of all $n \times n$ M -matrices, the set of all $n \times n$ nonsingular M -matrices and the set of all $n \times n$ singular M -matrices, respectively. It is easy to see that

$$(2.1) \quad M_n = M_n^\bullet \cup M_n^0 \quad \text{and} \quad M_n^\bullet \cap M_n^0 = \emptyset.$$

The comparison matrix of a given matrix $A = (a_{ij}) \in \mathbb{C}^{n \times n}$, denoted by $\mu(A) = (\mu_{ij})$, is defined by

$$\mu_{ij} = \begin{cases} |a_{ii}|, & \text{if } i = j, \\ -|a_{ij}|, & \text{if } i \neq j. \end{cases}$$

It is clear that $\mu(A) \in Z_n$ for a matrix $A \in \mathbb{C}^{n \times n}$. A matrix $A = a_{ij} \in \mathbb{C}^{n \times n}$ is a general H -matrix if $\mu(A) \in M_n$ (see [3]). If $\mu(A) \in M_n^\bullet$, A is an invertible H -matrix; if $\mu(A) \in M_n^0$ with $a_{ii} = 0$ for at least one $i \in \langle n \rangle$, A is a singular H -matrix; if $\mu(A) \in M_n^0$ with $a_{ii} \neq 0$ for all $i \in \langle n \rangle$, A is a mixed H -matrix. H_n , H_n^I , H_n^S and H_n^M will denote the set of all $n \times n$ general H -matrices, the set of all $n \times n$ invertible H -matrices, the set of all $n \times n$ singular H -matrices and the set of all $n \times n$ mixed H -matrices, respectively (see [6]). Similar to (2.1), we have

$$H_n = H_n^I \cup H_n^S \cup H_n^M \quad \text{and} \quad H_n^I \cap H_n^S \cap H_n^M = \emptyset.$$

For $n \geq 2$, an $n \times n$ complex matrix A is reducible if there exists an $n \times n$ permutation matrix P such that

$$PAP^T = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

where A_{11} is an $r \times r$ submatrix and A_{22} is an $(n-r) \times (n-r)$ submatrix, where $1 \leq r < n$. If no such permutation matrix exists, then A is called irreducible. If A is a 1×1 complex matrix, then A is irreducible if its single entry is nonzero, and reducible otherwise.

Given a matrix $A = (a_{ij}) \in \mathbb{C}^{n \times n}$ and a set $\alpha \subseteq \langle n \rangle$, we define the matrix $\hat{A} = (\hat{a}_{ij}) \in \mathbb{C}^{n \times n}$ and the matrix $A_\alpha = (\check{a}_{ij}) \in \mathbb{C}^{n \times n}$ by

$$(2.2) \quad \hat{a}_{ij} = \begin{cases} \operatorname{Re}(a_{ii}), & \text{if } i = j, \\ a_{ij}, & \text{otherwise} \end{cases} \quad \text{and} \quad \check{a}_{ij} = \begin{cases} \operatorname{Re}(a_{ii}), & i = j \in \langle n \rangle - \alpha, \\ a_{ij}, & \text{otherwise,} \end{cases}$$

respectively.

DEFINITION 2.1. A matrix $A \in \mathbb{C}^{n \times n}$ is *diagonally dominant by row* if

$$(2.3) \quad |a_{ii}| \geq \sum_{j=1, j \neq i}^n |a_{ij}|$$

holds for all $i \in \langle n \rangle$. If inequality in (2.3) holds strictly for all $i \in \langle n \rangle$, A is *strictly diagonally dominant by row*. If A is irreducible and the inequality in (2.3) holds strictly for at least one $i \in \langle n \rangle$, A is *irreducibly diagonally dominant by row*. If (2.3) holds with equality for all $i \in \langle n \rangle$, A is *diagonally equipotent by row*.

$D_n(SD_n, ID_n)$ and DE_n will be used to denote the sets of all $n \times n$ (strictly, irreducibly) diagonally dominant matrices and the set of all $n \times n$ diagonally equipotent matrices, respectively.

DEFINITION 2.2. A matrix $A \in \mathbb{C}^{n \times n}$ is *generalized diagonally dominant* if there exist positive constants α_i , $i \in \langle n \rangle$, such that

$$(2.4) \quad \alpha_i |a_{ii}| \geq \sum_{j=1, j \neq i}^n \alpha_j |a_{ij}|$$

holds for all $i \in \langle n \rangle$. If inequality in (2.4) holds strictly for all $i \in \langle n \rangle$, A is *generalized strictly diagonally dominant*. If (2.4) holds with equality for all $i \in \langle n \rangle$, A is *generalized diagonally equipotent*.

We will denote the sets of all $n \times n$ generalized (strictly) diagonally dominant matrices and the set of all $n \times n$ generalized diagonally equipotent matrices by $GD_n(GSD_n)$ and GDE_n , respectively.

DEFINITION 2.3. A matrix A is *nonstrictly diagonally dominant*, if either (2.3) or (2.4) holds with equality for at least one $i \in \langle n \rangle$.

REMARK 2.4. Let $A = (a_{ij}) \in \mathbb{C}^{n \times n}$ be nonstrictly diagonally dominant and $\alpha = \langle n \rangle - \alpha' \subset \langle n \rangle$. If $A(\alpha)$ is a (generalized) diagonally equipotent principal submatrix of A , then the following hold:

- $A(\alpha, \alpha') = 0$;
- $A(i_1) = (a_{i_1 i_1})$ being (generalized) diagonally equipotent implies $a_{i_1 i_1} = 0$.

REMARK 2.5. Definition 2.2 and Definition 2.3 show that

$$D_n \subset GD_n \quad \text{and} \quad GSD_n \subset GD_n.$$

The following properties of diagonally dominant matrices and H -matrices will be used in the rest of the paper.

LEMMA 2.6. [29, 31] *A matrix $A \in D_n(GD_n)$ is singular if and only if the matrix A has at least either one zero principal submatrix or one irreducible and (generalized) diagonally equipotent principal submatrix $A_k = A(i_1, i_2, \dots, i_k)$, $1 < k \leq n$, which satisfies the condition that there exists a $k \times k$ unitary diagonal matrix U_k such that*

$$(2.5) \quad U_k^{-1} D_{A_k}^{-1} A_k U_k = \mu(D_{A_k}^{-1} A_k),$$

where $D_{A_k} = \text{diag}(a_{i_1 i_1}, a_{i_2 i_2}, \dots, a_{i_k i_k})$.

LEMMA 2.7. [29, 30] *Let $A \in D_n(GD_n)$. Then A is singular if and only if A has at least one singular principal submatrix.*

LEMMA 2.8. [27, 29, 30, 31] *Let $A \in D_n(GD_n)$. Then $A \in H_n^I$ if and only if A has no (generalized) diagonally equipotent principal submatrices. Furthermore, if $A \in D_n \cap Z_n(GD_n \cap Z_n)$, then $A \in M_n^\bullet$ if and only if A has no (generalized) diagonally equipotent principal submatrices.*

LEMMA 2.9. [3] $H_n^I = GSD_n$

LEMMA 2.10. [6] $GD_n \subset H_n$.

The following counterexample shows that $H_n \subseteq GD_n$ is not true. Let

$$A = \begin{bmatrix} 0 & 0 & 100 & 100 \\ 0 & 0 & 100 & 100 \\ 0 & 0 & 10 & 10 \\ 0 & 0 & 10 & 10 \end{bmatrix}.$$

It is obvious that A is an H -matrix. But there is not a positive diagonal matrix D such that $AD \in D_4$. Thus, $A \notin GD_4$. Consequently, in general, $GD_n = H_n$ doesn't hold. But, under the condition of "irreducibility", the equality holds.

LEMMA 2.11. [6] *Let $A \in \mathbb{C}^{n \times n}$ be irreducible. Then $A \in H_n$ if and only if $A \in GD_n$.*

Under the condition of "reducibility", we have the following conclusion.

LEMMA 2.12. Let $A \in \mathbb{C}^{n \times n}$ be reducible. Then $A \in H_n$ if and only if in the Frobenius normal form of A

$$(2.6) \quad PAP^T = \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1s} \\ 0 & R_{22} & \cdots & R_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_{ss} \end{bmatrix},$$

each irreducible diagonal square block R_{ii} is generalized diagonally dominant, where P is a permutation matrix and R_{ii} , $i = 1, 2, \dots, s$, is either a 1×1 zero matrix or an irreducible square matrix.

The proof of this theorem follows from Lemma 2.11 and Theorem 5 in [6].

3. Generalized inverse and generalized Schur complement. In this section, the notion of $\star$ -generalized inverse is introduced and used to propose a definition of generalized Schur complement for some matrices.

DEFINITION 3.1. A generalized inverse X of a matrix $A \in \mathbb{C}^{m \times n}$ is a matrix $X \in \mathbb{C}^{n \times m}$ which satisfies one or more of the equations: (1) $AXA = A$; (2) $XAX = X$; (3) $(AX)^* = AX$; (4) $(XA)^* = XA$, where A^* denotes the conjugate transpose of A . If $\{i, j, k\}$ is a subset of $\{1, 2, 3, 4\}$, then any matrix satisfying (i), (j), and (k) will be called an (i, j, k) -inverse of A , denoted by $M^{(i,j,k)}$. Similarly, we define an (i, j) -inverse of A , denoted by $M^{(i,j)}$, and an (i) -inverse of A , denoted by $M^{(i)}$. For each A , there is a unique matrix satisfying (1), (2), (3), and (4), which is called the Moore-Penrose inverse of A and is denoted by A^+ .

DEFINITION 3.2. Let $A \in \mathbb{C}^{(m+n) \times (m+n)}$ and assume that there exists an $(m+n) \times (m+n)$ permutation matrix P such that $P^TAP = \begin{bmatrix} B & 0 \\ C & D \end{bmatrix}$, where P^T denotes the transpose of the matrix P , $B \in \mathbb{C}^{m \times m}$ is singular and $D \in \mathbb{C}^{n \times n}$ is nonsingular. The $\star$ -generalized inverse of the matrix A is defined by

$$A^* = P \begin{bmatrix} B^+ & 0 \\ -D^{-1}CB^+ & D^{-1} \end{bmatrix} P^T,$$

where B^+ denotes the Moore-Penrose inverse of B .

THEOREM 3.3. Let $A \in \mathbb{C}^{(m+n) \times (m+n)}$ satisfies the conditions of Definition 3.2. Then the following conclusions hold:

1. A^* is a $(1, 2, 3)$ -generalized inverse of A ;
2. $A^* = A^+$ if and only if either the matrix equation $YB = C$ (where Y is unknown) is solvable or the order of the matrix D is zero.

Proof. The first conclusion follows from the direct computations. Again, the second conclusion follows Corollary 2.6 in [23]. $\square$

Based on the notion of the $\star$ -generalized inverse, the definition of generalized Schur complement is introduced.

DEFINITION 3.4. Let $A \in \mathbb{C}^{n \times n}$, $\alpha \subset N = \{1, 2, \dots, n\}$ and $\alpha' = N - \alpha$ in which indices in both α and α' are arranged in increasing order. If $A(\alpha)$ is nonsingular, the matrix

$$A/\alpha = A(\alpha') - A(\alpha', \alpha)[A(\alpha)]^{-1}A(\alpha, \alpha')$$

is called the Schur complement with respect to $A(\alpha)$; if $A(\alpha)$ is singular, the matrix

$$(3.1) \quad A/_0\alpha = A(\alpha') - A(\alpha', \alpha)x A(\alpha, \alpha')$$

is called the generalized Schur complement with respect to $A(\alpha)$, where $x = [A(\alpha)]^+$ when $A(\alpha)$ is irreducible and $x = [A(\alpha)]^*$ when $A(\alpha)$ is reducible. We shall confine ourselves to the singular $A(\alpha)$ as far as $A/_0\alpha$ is concerned.

If the generalized Schur complement of a nonstrictly diagonally dominant matrix or general H -matrix exists, we have the following conclusions.

THEOREM 3.5. Let $A \in D_n(GD_n)$ and assume that $A/_0\alpha$ exists for some $\alpha \subset N$. Then, A is singular.

Proof. Since $A/_0\alpha$ exists for some $\alpha \subset N$, $A(\alpha)$ is singular. Thus, Lemma 2.7 shows that A is singular. $\square$

THEOREM 3.6. Let $A \in H_n$ and assume that $A/_0\alpha$ exists for some $\alpha \subset N$. Then, A is singular.

Proof. Since $A/_0\alpha$ exists for some $\alpha \subset N$, $A(\alpha)$ is singular. If $A(\alpha)$ is generalized diagonally dominant, then it is either one of diagonal square blocks in (2.6) or the principal submatrix of the diagonal square block. Thus, Lemma 2.7 and Lemma 2.12 show that A is singular. If $A(\alpha)$ isn't generalized diagonally dominant, $A(\alpha)$ must have at least one singular generalized diagonally equipotent principal submatrix $A(\beta)$ for $\beta \subseteq \alpha$ since $A(\alpha) \in H_{|\alpha|}$ for $A \in H_n$. It follows from Lemma 2.7 and Lemma 2.12 that A is singular. $\square$

4. Some properties of generalized Schur complements of nonstrictly diagonally dominant matrices and general H -matrices. It is well known that Schur complements of diagonally dominant matrices are diagonally dominant (see, e.g., [10, 16, 17, 28]); the same is true of generalized diagonally dominant matrices and H -matrices (see, e.g., [7, 12, 17, 19, 30]). This property is called the *heredity* of the Schur complement. In [7, 10, 12, 16, 17, 18, 19, 28, 29, 30, 32, 34], some properties

such as heredity and nonsingularity of the Schur complement of nonstrictly diagonally dominant matrices and general H -matrices are studied and some significant results have been presented. However, a problem arises, i.e., do these properties still hold for the generalized Schur complements of nonstrictly diagonally dominant matrices and general H -matrices?

In what follows, we will present some interesting results on heredity and nonsingularity of the generalized Schur complements for nonstrictly diagonally dominant matrices and general H -matrices.

THEOREM 4.1. *Let $A \in D_n(GD_n)$, $\alpha \subset N$ and $\alpha' = N - \alpha$. If $A(\alpha)$ is singular and irreducible, then $A/_0\alpha = A(\alpha') \in D_{n-|\alpha|}(GD_{n-|\alpha|})$.*

Proof. Since $A \in D_n(GD_n)$, $A(\alpha) \in D_{|\alpha|}(GD_{|\alpha|})$ and $A(\alpha') \in D_{n-|\alpha|}(GD_{n-|\alpha|})$. Again, since $A(\alpha)$ is singular and irreducible, it follows from Remark 2.4 that $A(\alpha)$ is an irreducible (generalized) diagonally equipotent principal submatrix and $A(\alpha, \alpha') = 0$. Thus, (3.1) gives the conclusion of this theorem. $\square$

THEOREM 4.2. *Let $A \in H_n$, $\alpha \subset N$ and $\alpha' = N - \alpha$. If $A(\alpha)$ is singular and irreducible, then $A/_0\alpha = A(\alpha') \in H_{n-|\alpha|}$.*

Proof. Since $A \in H_n$ and $A(\alpha)$ is irreducible, it follows from Lemma 2.12 that $A(\alpha)$ is one of irreducible diagonal square blocks in the Frobenius normal form (2.6) of A . Obviously, we have $A/_0\alpha = A(\alpha') \in H_{n-|\alpha|}$. $\square$

LEMMA 4.3. [16, 29, 30] *Let $A \in D_n(GD_n)$ and $\alpha \subset N$. If $A(\alpha)$ is nonsingular, then $A/\alpha \in D_{n-|\alpha|}(GD_{n-|\alpha|})$.*

THEOREM 4.4. *Given a matrix $A \in D_n(GD_n)$ and a set $\alpha = N - \alpha' \subseteq N$, if $A(\alpha)$ is singular and $A(\gamma)$ is the largest (generalized) diagonally equipotent principal submatrix of $A(\alpha)$ for $\gamma = \alpha - \gamma' \subseteq \alpha$, then $A/_0\alpha = A(\alpha' \cup \gamma')/\gamma' \in D_{|\alpha'|}(GD_{|\alpha'|})$, where*

$$(4.1) \quad A(\alpha' \cup \gamma') = \begin{bmatrix} A(\gamma') & A(\gamma', \alpha') \\ A(\alpha', \gamma') & A(\alpha') \end{bmatrix}.$$

Proof. If $\gamma = \alpha$, i.e., $A(\alpha)$ is diagonally equipotent and singular, then it follows from Remark 2.5 that $A(\alpha, \alpha') = 0$. As a result,

$$A/_0\alpha = A(\alpha') - A(\alpha', \alpha)[A(\alpha)]^+ A(\alpha, \alpha') = A(\alpha') \in D_{|\alpha'|}(GD_{|\alpha'|}).$$

Now we consider the case when $\gamma \subset \alpha$. Since $A \in D_n$ and $A(\gamma)$ is diagonally equipotent, we have with Remark 2.5 that $A(\gamma, \gamma') = 0$ and $A(\gamma, \alpha') = 0$. Thus, there

exists an $|\alpha| \times |\alpha|$ permutation matrix P_α such that

$$P_\alpha A(\alpha) P_\alpha^T = \begin{bmatrix} A(\gamma) & 0 \\ A(\gamma', \gamma) & A(\gamma') \end{bmatrix},$$

correspondingly

$$(4.2) \quad \begin{aligned} A(\alpha', \alpha) P_\alpha^T &= (A(\alpha', \gamma), A(\alpha', \gamma')) \quad \text{and} \\ P_\alpha A(\alpha, \alpha') &= \begin{pmatrix} A(\gamma, \alpha') \\ A(\gamma', \alpha') \end{pmatrix} = \begin{pmatrix} 0 \\ A(\gamma', \alpha') \end{pmatrix}. \end{aligned}$$

Since $A(\gamma)$ is the largest (generalized) diagonally equipotent principal submatrix of the singular matrix $A(\alpha)$ for $\gamma = \alpha - \gamma' \subseteq \alpha$, $A(\gamma')$ has no diagonally equipotent principal submatrices, it follows from Lemma 2.8 that $A(\gamma') \in H_{|\gamma'|}^I$. Hence, $A(\gamma')$ is nonsingular. However, $A(\gamma)$ is singular. Therefore, we have

$$(4.3) \quad [A(\alpha)]^* = P_\alpha^T \begin{bmatrix} [A(\gamma)]^+ & 0 \\ -[A(\gamma')]^{-1} A(\gamma', \gamma) [A(\gamma)]^+ & [A(\gamma')]^{-1} \end{bmatrix} P_\alpha.$$

Let $\mathcal{B} = \begin{bmatrix} [A(\gamma)]^+ & 0 \\ -[A(\gamma')]^{-1} A(\gamma', \gamma) [A(\gamma)]^+ & [A(\gamma')]^{-1} \end{bmatrix} P_\alpha$. Then, $[A(\alpha)]^* = P_\alpha^T \mathcal{B} P_\alpha$.

As a consequence, it follows from (4.2) and (4.3) that

$$\begin{aligned} A/0\alpha &= A(\alpha') - A(\alpha', \alpha) [A(\alpha)]^+ A(\alpha, \alpha') \\ &= A(\alpha') - A(\alpha', \alpha) P_\alpha^T \mathcal{B} P_\alpha A(\alpha, \alpha') \\ &= A(\alpha') - (A(\alpha', \gamma), A(\alpha', \gamma')) \mathcal{B} \begin{bmatrix} 0 \\ A(\gamma', \alpha') \end{bmatrix} \\ &= A(\alpha') - A(\alpha', \gamma') [A(\gamma')]^{-1} A(\gamma', \alpha') \\ &= A(\alpha' \cup \gamma') / \gamma'. \end{aligned}$$

Since $A \in D_n(GD_n)$, $A(\alpha' \cup \gamma') \in D_{|\alpha' \cup \gamma'|}(GD_{|\alpha' \cup \gamma'|})$. Again, $A(\gamma')$ is nonsingular. Lemma 4.3 yields that $A/0\alpha = A(\alpha' \cup \gamma') / \gamma' \in D_{|\alpha'|}(GD_{|\alpha'|})$. $\square$

THEOREM 4.5. *Given a matrix $A \in H_n$ and a set $\alpha = N - \alpha' \subseteq N$, assume that $A(\alpha)$ is singular and $A(\beta_j)$, $j = 1, 2, \dots, t$ with $t < |\alpha|$, are all generalized diagonally equipotent principal submatrices of $A(\alpha)$, let $\gamma = \bigcup_{j=1}^t \beta_j = \alpha - \gamma' \subseteq \alpha$, then $A/0\alpha = A(\alpha' \cup \gamma') / \gamma' \in H_{|\alpha'|}$, where $A(\alpha' \cup \gamma')$ is defined by (4.1).*

Proof. Similar to the proof of Theorem 4.4, we can obtain the conclusion of this theorem by Lemma 2.12. $\square$

Theorem 4.4 and Theorem 4.5 not only show the heredity of generalized Schur complements of nonstrictly diagonally dominant matrices and general H -matrices, but also reveal the relationship between generalized Schur complements and Schur complements of nonstrictly diagonally dominant matrices and general H -matrices.

In the following, we study nonsingularity of generalized Schur complements of nonstrictly diagonally dominant matrices and general H -matrices. We will need the following lemma.

LEMMA 4.6. *Let $A \in D_n(GD_n)$ and $\alpha = N - \alpha' \subseteq N$, assume that $A(\alpha)$ is singular and $A(\gamma)$ is the largest (generalized) diagonally equipotent principal submatrix of $A(\alpha)$ for $\gamma = \alpha - \gamma' \subseteq \alpha$. Then $A(\alpha' \cup \gamma')$ in (4.1) is nonsingular if and only if $\dim[N(A(\alpha))] = \dim[N(A)]$.*

Proof. $A \in D_n(GD_n)$ indicates $A(\alpha) \in D_{|\alpha|}(GD_{|\alpha|})$. Again, $A(\alpha)$ is singular and $A(\gamma)$ is the largest singular (generalized) diagonally equipotent principal submatrix of $A(\alpha)$ for $\gamma = \alpha - \gamma' \subseteq \alpha$. Assume that $\dim[N(A(\alpha))] = \dim[N(A)]$, all singular principal submatrices of A are all singular principal submatrices of $A(\gamma)$. Thus, $A(\alpha' \cup \gamma')$ is nonsingular. Conversely, assume that $A(\alpha' \cup \gamma')$ is nonsingular. Since $A(\alpha)$ is singular and $A(\gamma)$ is the largest (generalized) diagonally equipotent principal submatrix of $A(\alpha)$ for $\gamma = \alpha - \gamma' \subseteq \alpha$. This implies that all singular principal submatrices of A are all singular principal submatrices of $A(\alpha)$. Thus, $\dim[N(A(\alpha))] = \dim[N(A)]$. $\square$

THEOREM 4.7. *Given a matrix $A \in D_n(GD_n)$ and a set $\alpha \subseteq N$, if $A(\alpha)$ is singular, then $A/_{0\alpha}$ is nonsingular if and only if $\dim[N(A(\alpha))] = \dim[N(A)]$.*

Proof. Assume that $A \in D_n$, then $A(\alpha) \in D_{|\alpha|}$. Since $A(\alpha)$ is singular, it follows from Lemma 2.6 that $A(\alpha)$ has some singular diagonally equipotent principal submatrices. Let $A(\gamma)$ be the largest diagonally equipotent principal submatrix of $A(\alpha)$ for $\gamma = \alpha - \gamma' \subseteq \alpha$ and $A(\gamma')$ is nonsingular. Theorem 4.4 and Lemma 3.13 in [30] show that $A/_{0\alpha}$ is nonsingular if and only if $A(\alpha' \cup \gamma')$ in (4.1) is nonsingular for $\alpha' = N - \alpha$. Lemma 4.6 shows that $A(\alpha' \cup \gamma')$ is nonsingular if and only if $\dim[N(A(\alpha))] = \dim[N(A)]$. If $A \in GD_n$, the proof can be obtained by a similar approach. $\square$

LEMMA 4.8. *Let $A \in H_n$ and $\alpha = N - \alpha' \subseteq N$, assume that $A(\alpha)$ is singular and $A(\beta_j)$, $j = 1, 2, \dots, t$ with $t < |\alpha|$, are all generalized diagonally equipotent principal submatrices of $A(\alpha)$, let $\gamma = \bigcup_{j=1}^t \beta_j = \alpha - \gamma' \subseteq \alpha$. Then $A(\alpha' \cup \gamma')$ in (4.1) is nonsingular if and only if $\dim[N(A(\alpha))] = \dim[N(A)]$.*

Proof. The proof is similar to the proof of Lemma 4.6. $\square$

THEOREM 4.9. *Given a matrix $A \in H_n$ and a set $\alpha \subseteq N$, if $A(\alpha)$ is singular, then $A/_{0\alpha}$ is nonsingular if and only if $\dim[N(A(\alpha))] = \dim[N(A)]$.*

Proof. The proof is similar to the proof of Theorem 4.7. $\square$

5. The eigenvalue distribution of the generalized Schur complement.

The eigenvalue distribution of the Schur complements of some special matrices in-

cluding strictly diagonally dominant matrices and nonsingular H -matrices has been studied and there several significant results to be proposed in [12, 17, 18, 19, 24, 26, 29, 31, 35]. Later these results were extended to Schur complements of nonstrictly diagonally dominant matrices and general H -matrices with complex diagonal entries.

THEOREM 5.1. [32] *Given a matrix $A = (a_{ij}) \in \mathbb{C}^{n \times n}$ and a set $\alpha \subset N$, if $\hat{A} \in D_n(GD_n)$ and is nonsingular, where $\hat{A} = (\hat{a}_{ij}) \in \mathbb{C}^{n \times n}$ is defined by (2.2), then A/α has $|J_{R_+}(A)| - |J_{R_+}^\alpha(A)|$ eigenvalues with positive real part and $|J_{R_-}(A)| - |J_{R_-}^\alpha(A)|$ eigenvalues with negative real part, where $J_{R_+}(A) = \{i \mid \operatorname{Re}(a_{ii}) > 0, i \in N\}$, $J_{R_-}(A) = \{i \mid \operatorname{Re}(a_{ii}) < 0, i \in N\}$, $J_{R_+}^\alpha(A) = \{i \mid \operatorname{Re}(a_{ii}) > 0, i \in \alpha\}$, and $J_{R_-}^\alpha(A) = \{i \mid \operatorname{Re}(a_{ii}) < 0, i \in \alpha\}$.*

THEOREM 5.2. [32] *Given a matrix $A = (a_{ij}) \in \mathbb{C}^{n \times n}$ and a set $\alpha \subset N$, if $\hat{A} \in H_n$ and is nonsingular, where $\hat{A} = (\hat{a}_{ij}) \in \mathbb{C}^{n \times n}$ is defined by (2.2), then A/α has $|J_{R_+}(A)| - |J_{R_+}^\alpha(A)|$ eigenvalues with positive real part and $|J_{R_-}(A)| - |J_{R_-}^\alpha(A)|$ eigenvalues with negative real part.*

THEOREM 5.3. [32] *Given a matrix $A \in \mathbb{C}^{n \times n}$ and a set $\alpha \subset N$, if $A_\alpha \in D_n(GD_n)$ and is nonsingular, where $A_\alpha = (\check{a}_{ij}) \in \mathbb{C}^{n \times n}$ is defined by (2.2), then A/α has $|J_{R_+}(A)| - |J_{R_+}^\alpha(A)|$ eigenvalues with positive real part and $|J_{R_-}(A)| - |J_{R_-}^\alpha(A)|$ eigenvalues with negative real part.*

THEOREM 5.4. [32] *Given a matrix $A \in \mathbb{C}^{n \times n}$ and a set $\alpha \subset N$, if $A_\alpha \in H_n$ and is nonsingular, where $A_\alpha = (\check{a}_{ij}) \in \mathbb{C}^{n \times n}$ is defined by (2.2), then A/α has $|J_{R_+}(A)| - |J_{R_+}^\alpha(A)|$ eigenvalues with positive real part and $|J_{R_-}(A)| - |J_{R_-}^\alpha(A)|$ eigenvalues with negative real part.*

In the rest of this paper, the results on the eigenvalue distribution are extended to generalized Schur complements of nonstrictly diagonally dominant matrices and general H -matrices.

THEOREM 5.5. *Given a matrix $A = (a_{ij}) \in \mathbb{C}^{n \times n}$ with $\hat{A} \in D_n(GD_n)$, where $\hat{A} = (\hat{a}_{ij}) \in \mathbb{C}^{n \times n}$ is defined by (2.2), and a subset $\alpha \subseteq N$, if $A(\alpha)$ is singular and $\dim[N(\hat{A}(\alpha))] = \dim[N(\hat{A})]$, then the generalized Schur complement matrix $A/_{0\alpha}$ of the matrix A has $|J_{R_+}(A)| - |J_{R_+}^\alpha(A)|$ eigenvalues with positive real part and $|J_{R_-}(A)| - |J_{R_-}^\alpha(A)|$ eigenvalues with negative real part.*

Proof. Since $\hat{A} \in D_n(GD_n)$, $A \in D_n(GD_n)$. Again, since $A(\alpha)$ is singular, it follows from Lemma 2.6 that $A(\alpha)$ has (generalized) diagonally equipotent principal submatrices. Assume that $A(\gamma)$ is the largest (generalized) diagonally equipotent principal submatrix of $A(\alpha)$ for $\gamma = \alpha - \gamma' \subseteq \alpha$. Thus, Theorem 4.4 shows that

$A/_0\alpha = A(\alpha' \cup \gamma')/\gamma' \in D_{|\alpha'|}(GD_{|\alpha'|})$, where

$$(5.1) \quad A(\alpha' \cup \gamma') = \begin{bmatrix} A(\gamma') & A(\gamma', \alpha') \\ A(\alpha', \gamma') & A(\alpha') \end{bmatrix}.$$

Furthermore,

$$(5.2) \quad \begin{aligned} |J_{R_+}^{\alpha' \cup \gamma'}(A)| - |J_{R_+}^{\gamma'}(A)| &= |J_{R_+}(A)| - |J_{R_+}^\alpha(A)| \quad \text{and} \\ |J_{R_-}^{\alpha' \cup \gamma'}(A)| - |J_{R_-}^{\gamma'}(A)| &= |J_{R_-}(A)| - |J_{R_-}^\alpha(A)|. \end{aligned}$$

Because of $\hat{A} \in D_n(GD_n)$ and $\dim[N(\hat{A}(\alpha))] = \dim[N(\hat{A})]$, it follows from Lemma 4.6 that $\hat{A}(\alpha' \cup \gamma') \in D_{|\alpha' \cup \gamma'|}(GD_{|\alpha' \cup \gamma'|})$ and is nonsingular. Following Theorem 5.1, the Schur complement matrix $A(\alpha' \cup \gamma')/\gamma'$ of the matrix $A(\alpha' \cup \gamma')$ has $|J_{R_+}^{\alpha' \cup \gamma'}(A)| - |J_{R_+}^{\gamma'}(A)|$ eigenvalues with positive real part and $|J_{R_-}^{\alpha' \cup \gamma'}(A)| - |J_{R_-}^{\gamma'}(A)|$ eigenvalues with negative real part. As a result, the generalized Schur complement matrix $A/_0\alpha$ of the matrix A has $|J_{R_+}(A)| - |J_{R_+}^\alpha(A)|$ eigenvalues with positive real part and $|J_{R_-}(A)| - |J_{R_-}^\alpha(A)|$ eigenvalues with negative real part. $\square$

THEOREM 5.6. *Given a matrix $A = (a_{ij}) \in \mathbb{C}^{n \times n}$ with $\hat{A} \in H_n$, where $\hat{A} = (\hat{a}_{ij}) \in \mathbb{C}^{n \times n}$ is defined by (2.2), and a subset $\alpha \subseteq N$, if $A(\alpha)$ is singular and $\dim[N(\hat{A}(\alpha))] = \dim[N(\hat{A})]$, then the Schur complement matrix $A/_0\alpha$ of the matrix A has $|J_{R_+}(A)| - |J_{R_+}^\alpha(A)|$ eigenvalues with positive real part and $|J_{R_-}(A)| - |J_{R_-}^\alpha(A)|$ eigenvalues with negative real part.*

Proof. Similar to the proof of Theorem 5.5, we can obtain the proof by Theorem 4.5, Lemma 4.8 and Theorem 5.2. $\square$

THEOREM 5.7. *Given a matrix $A = (a_{ij}) \in \mathbb{C}^{n \times n}$ with $A_\alpha \in D_n(GD_n)$, where $A_\alpha = (\check{a}_{ij}) \in \mathbb{C}^{n \times n}$ is defined by (2.2), and a subset $\alpha \subseteq N$, if $A(\alpha)$ is singular and $\dim[N(A(\alpha))] = \dim[N(A_\alpha)]$, then the Schur complement matrix $A/_0\alpha$ of the matrix A has $|J_{R_+}(A)| - |J_{R_+}^\alpha(A)|$ eigenvalues with positive real part and $|J_{R_-}(A)| - |J_{R_-}^\alpha(A)|$ eigenvalues with negative real part.*

Proof. Similar to the proof of Theorem 5.5, by Theorem 4.4, $A/_0\alpha = A(\alpha' \cup \gamma')/\gamma' \in D_{|\alpha'|}(GD_{|\alpha'|})$, where $A(\alpha' \cup \gamma')$ is as in (5.1) and $A(\gamma)$ is the largest (generalized) diagonally equipotent principal submatrix of $A(\alpha)$ for $\gamma = \alpha - \gamma' \subseteq \alpha$. Furthermore, (5.2) is true. Since $A_\alpha \in D_n(GD_n)$ and $\dim[N(A(\alpha))] = \dim[N(A_\alpha)]$, it follows from Lemma 4.6 that $\bar{A}_{\gamma'} = A_\alpha(\alpha' \cup \gamma') \in D_{|\alpha' \cup \gamma'|}(GD_{|\alpha' \cup \gamma'|})$ and is nonsingular. Theorem 5.3 shows that the Schur complement matrix $A(\alpha' \cup \gamma')/\gamma'$ of the matrix $A(\alpha' \cup \gamma')$ has $|J_{R_+}^{\alpha' \cup \gamma'}(A)| - |J_{R_+}^{\gamma'}(A)|$ eigenvalues with positive real part and $|J_{R_-}^{\alpha' \cup \gamma'}(A)| - |J_{R_-}^{\gamma'}(A)|$ eigenvalues with negative real part. As a result, the generalized Schur complement matrix $A/_0\alpha$ of the matrix A has $|J_{R_+}(A)| - |J_{R_+}^\alpha(A)|$ eigenvalues

with positive real part and $|J_{R-}(A)| - |J_{R-}^{\alpha}(A)|$ eigenvalues with negative real part. $\square$

THEOREM 5.8. *Given a matrix $A = (a_{ij}) \in \mathbb{C}^{n \times n}$ with $A_{\alpha} \in H_n$, where $A_{\alpha} = (\check{a}_{ij}) \in \mathbb{C}^{n \times n}$ is defined by (2.2), and a subset $\alpha \subseteq N$, if $A(\alpha)$ is singular and $\dim[N(A(\alpha))] = \dim[N(A_{\alpha})]$, then the Schur complement matrix $A/_{\alpha}$ of the matrix A has $|J_{R+}(A)| - |J_{R+}^{\alpha}(A)|$ eigenvalues with positive real part and $|J_{R-}(A)| - |J_{R-}^{\alpha}(A)|$ eigenvalues with negative real part.*

Proof. Similar to the proof of Theorem 5.7, we can obtain the the proof by Theorem 4.5, Lemma 4.8 and Theorem 5.4. $\square$

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