

OPINION DYNAMICS WITH STUBBORN VERTICES*

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Abstract. Consider a social network where each person holds an opinion represented by a numerical value. Whenever a member of the social network is given a chance, the member updates his/her opinion according to a certain convex combination of the opinions of all network members. The influence digraph of the network has network members as vertices, and there is an arc from a vertex v to a vertex u if and only if, in the opinion update formula for v , the coefficient of u 's opinion is positive. The sink vertices in the influence digraph correspond to those stubborn people who never change their opinions. Assuming network members update their opinions one by one according to a given sequence, this note provides a description of the resulting opinion dynamics when every vertex can reach some sink vertex in the influence digraph.

Key words. Harmonic function, Infinite matrix product, Influence digraph, Inhomogeneous absorbing Markov chain.

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1. Introduction. Due to the growing importance of evolving social networks, the quantitative analysis of the dynamical behavior of some abstract model of discrete time dynamical systems has attracted the attention of many research groups [4, 5, 7, 10, 13, 19, 21, 22]. We consider a simple model of *opinion dynamics* [7, 10, 13, 19] in this note. Let V be a finite community where everyone holds a numerical opinion. The opinion distribution can be represented as a column vector of real numbers indexed by V , called the *opinion profile*. Whenever a community member v is allowed to update his/her opinion, he/she will need to know the current opinion profile $\alpha \in \mathbb{R}^V$ and always change his/her opinion according to a certain time-invariant convex combination of the opinions of all members of the community, say $\sum_{u \in V} a_v(u)\alpha(u)$, where $a_v(u) \geq 0$ and $\sum_{u \in V} a_v(u) = 1$. This means that the new opinion profile will be $A_v\alpha$, where A_v is the row-stochastic matrix obtained from the identity matrix, namely the Kronecker delta function δ , on V by replacing the row indexed by v to be $a_v^\top$, the transpose of the vector a_v . We call A_v the *influence matrix* for v . The *influence matrix for the opinion dynamics*, denoted by A , is the matrix indexed by V whose v th row is $a_v^\top$ for each $v \in V$. To establish a dynamics, besides the time-invariant

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information exchange mechanism specified by the influence matrix A , we need a map $w \in V^{\mathbb{N}}$, called the *opinion update sequence*, and an initial opinion profile, say α_1 , the opinion profile at time $t = 1$. Now, for each $t \in \mathbb{N}$, the opinion profile at time t is $\alpha_t = A_{w(t-1)} A_{w(t-2)} \cdots A_{w(1)} \alpha_1$ and this results in an evolving profile.

We are interested in two questions: First, to which extent can one expect some loss of memory, also called a merging property [23], from our model of opinion dynamics? Second, how stable will the opinion formation process be? Note that if the initial opinion profile takes a constant value, then the opinion profile will stay the same forever, and thus, there will be no memory loss at all. Also, when the opinion update sequence takes different constant values, usually we expect some different types of dynamical behaviors. So, to make our model more reasonable, we should restrict our discussion to some suitable setting.

We define the *influence digraph* of the above opinion dynamics model to be the digraph D with V as the vertex set and an arc from v to u if and only if $a_v(u) > 0$, i.e., u has direct influence on v . We can say that the social relations in the community V are represented analytically by A and combinatorially by D . Let V_0 represent the set of stubborn people, namely $V_0 = \{v \in V : a_v(v) = 1\}$ and let $V_1 = V \setminus V_0$. Surely, V_0 consists of those who stick with their original opinions, and thus, the best we can expect is that the process is ergodic among V_1 [17, 26], namely, everyone in V_1 will have his/her own limiting opinion which is irrelevant with his/her original opinion. Recall that a *path* in a digraph is a sequence of distinct vertices such that there is an arc from u to w in the digraph for any two vertices u and w appearing in the sequence consecutively in that order. If there is a vertex $v \in V_1$ such that no path in D can go from v to a vertex in V_0 , we can set the initial opinion of a vertex u to be 1 if u is reachable in D from v and 0 otherwise. Then one can check easily that the process does remember the history of those vertices reachable from v . This suggests us to call the influence digraph D *memoryless* provided that every vertex can reach a sink vertex in D . We next turn to discuss opinion update sequence. A typical way of obtaining an opinion update sequence w is to let $w(t), t \in \mathbb{N}$, be independent identically distributed random variables which have positive probability for attaining any value inside V_1 . For this case, the second Borel-Cantelli lemma guarantees that almost surely $w^{-1}(v)$ is an infinite set for each $v \in V_1$. This motivates us to define a map w from $\mathbb{N}$ to a set S to be *typical with respect to* $S_1 \subseteq S$ when $w^{-1}(v)$ is an infinite set for each $v \in S_1$.

THEOREM 1.1. *Assume that the influence digraph D is memoryless. Let the dynamics be driven by an opinion update sequence w which is typical with respect to V_1 . Then there exists $\alpha \in \mathbb{R}^V$ such that $\alpha = \lim_{t \rightarrow \infty} \alpha_t$ and this limiting opinion profile α is totally determined by $\alpha_1|_{V_0}$ and the influence matrix A —that is, α is irrelevant of the choice of $\alpha_1|_{V_1}$ and the typical sequence w .*

For $w \in V^{\mathbb{N}}$ and any two positive integers $s \leq t$, let $w[s, t]$ refer to the word $w(s)w(s+1)\cdots w(t) \in V^{t-s+1}$. For example, for $w = (123)^{\infty}$, $w[5, 8] = 2312$. If $W = W_1W_2\cdots W_n$ is a word on the alphabet V , then its *reversal* is $\overleftarrow{W} := W_nW_{n-1}\cdots W_1$ and A_W refers to the matrix product $A_{W_1}A_{W_2}\cdots A_{W_n}$. Given $w \in V^{\mathbb{N}}$, for any natural numbers $s \leq t$, we can form the *forward accumulation* $A_w(s, t) := A_{w[s, t]} = A_{w(s)}A_{w(s+1)}\cdots A_{w(t)}$ and the *backward accumulation* $A_w(t, s) := A_{\overleftarrow{w[s, t]}} = A_{w(t)}A_{w(t-1)}\cdots A_{w(s)}$. To understand how the opinions spread in a network under our opinion dynamics model, we need to understand the convergence property of the backward accumulation $A_w(t, 1)$ for $t \in \mathbb{N}$ as well as how stable it is under the perturbation of the sequence w . Recall that a Markov chain is *absorbing* if it has at least one absorbing state and if, from any state, it is possible to reach at least one absorbing state [12, p. 26]. So, if we let the matrices act on probabilistic row vectors from the right, those stubborn people will then correspond to absorbing states and the study of the forward accumulations $A_w(1, t)$ for $t \in \mathbb{N}$ will naturally correspond to the study of an absorbing inhomogeneous Markov chain [12, 16, 17, 18, 20, 23, 25] as long as D is memoryless. Though inhomogeneous Markov chains are important in applications and many results on them have been developed [18], there are much fewer precise quantitative results for inhomogeneous Markov chains than for their homogeneous counterparts. It is even noted in [23] that “almost nothing is known about the quantitative behavior of time inhomogeneous chains”. It is thus worth mentioning that our work on opinion dynamics here has some natural quantitative consequences on the corresponding inhomogeneous Markov chains. (See Theorem 3.3, Remark 3.4, (3.8), and (3.9).)

A set S of square matrices of the same size is an RCP (right-convergent product set) [9, 11, 27] if $\lim_{t \rightarrow \infty} w(1)w(2)\cdots w(t)$ exists for each $w \in S^{\mathbb{N}}$. Similarly, the set S is an LCP (left-convergent product set) [2, 9, 11] if $\lim_{t \rightarrow \infty} w(t)w(t-1)\cdots w(1)$ exists for each $w \in S^{\mathbb{N}}$. Due to various kinds of background, convergence of infinite matrix products has been vastly studied [2, 8, 9, 11, 17, 18, 21, 22, 24, 25, 26, 27]. The most common approach to obtain the convergence result is to establish a certain contraction property and usually, the limit of the matrix products is a positive matrix of rank 1. Under the condition that D is memoryless, we will show that $\{A_v : v \in V\}$ is both an RCP and an LCP. Furthermore, both limiting infinite products may have arbitrary ranks. (See Theorem 3.3 and Remark 3.5.)

Our work relies on some basic properties of memoryless digraphs, which can be found in the context of absorbing Markov chains in the literature. To make the note self-contained, we present these basic facts in Section 2. In Section 3 we show that the dynamical structure of the opinion dynamics with memoryless influence digraph is quite simple. We will see there how the beliefs of those people in V_0 propagate all around the community and contribute to the opinion formation in V_1 . The stability of the opinion dynamics without stubborn vertices may worth further study.

Finally, we mention that this work indeed arises from our effort to understand and generalize some computer simulation on the *energy division game* on a connected graph. At the beginning, each vertex of the graph collects a certain amount of energy and at each time step one vertex fires to distribute its energy equally among its neighbors. The problem is to see if there is always a firing sequence which causes the energy distribution approaching to the uniform distribution on the whole graph. This game was proposed by Baiting Xu [28] as a variant of the lit-only σ -game [14], and we do not know much on this new game yet.

2. Harmonic functions. For any set S , write the $S \times S$ identity matrix as δ_S . For any $v \in S$ and any $T \subseteq S$, let $\mathbf{i}_v \in \mathbb{R}^S$ be the v th column of δ_S and let $\mathbf{i}_T := \sum_{v \in T} \mathbf{i}_v$. In this notation, we omit the reference to the set S when it is clear from the context. Corresponding to the partition $V = V_0 \cup V_1$, we can write the influence matrix A in block form as

$$A = \begin{bmatrix} \delta_{V_0} & \mathbf{0} \\ B & C \end{bmatrix}.$$

Note that the rows of $\begin{bmatrix} B & C \end{bmatrix}$ consist of $a_v^\top$ for $v \in V_1$. For any $n \in \mathbb{N}$, it is not hard to see that there is $B_n \in \mathbb{R}^{V_1 \times V_0}$ such that

$$A^n = \begin{bmatrix} \delta_{V_0} & \mathbf{0} \\ B_n & C^n \end{bmatrix}.$$

For any real matrix M , we write $\rho(M)$ for the spectral radius of M and let $r(M)$ and $R(M)$ stand for the minimal row sum and maximal row sum of M , respectively.

LEMMA 2.1. [16, Theorem 11.4] *Let the influence digraph D be memoryless. Then $\rho(C) < 1$, and $\delta_{V_1} - C$ is nonsingular.*

Proof. It is straightforward that $B_n + C^n B = B_{n+1}$ for each $n \in \mathbb{N}$. Note that A^n is row-stochastic for each $n \in \mathbb{N}$. Since each bounded monotone sequence of real numbers converges, the existence of $\lim_{n \rightarrow \infty} B_n$ is guaranteed. Since D is memoryless, $\lim_{n \rightarrow \infty} B_n$ cannot have a zero row. Thus, there exists $n \in \mathbb{N}$ such that $r(B_n) > 0$ and hence $R(C^n) = 1 - r(B_n) < 1$. By the Geršgorin Disc Theorem, $\rho(C^n) \leq R(C^n) < 1$. This implies that $\rho(C) = \lim_{n \rightarrow \infty} (\rho(C^n))^{\frac{1}{n}} < 1$, and thus, $\delta_{V_1} - C$ is nonsingular. $\square$

Following [20, p. 13, p. 116], we define the space of *harmonic functions* on V with respect to the row-stochastic matrix A as $\ker(A - \delta_V) = \{x \in \mathbb{R}^V : Ax = x\}$. In a narrow sense, one talks about a harmonic function on a graph G with set of poles V_0 which is a function whose value at every vertex $v \in V(G) \setminus V_0$ coincides with the arithmetic average of its values in the neighbour vertices of v [1]. The harmonic functions in our current more general setting has some close connection with the asymmetric Laplacian [3, 6].

By Lemma 2.1, when D is memoryless, we can define

$$\hat{B} := \begin{bmatrix} \delta_{V_0} \\ (\delta_{V_1} - C)^{-1}B \end{bmatrix}.$$

In the setting of absorbing Markov chain, the matrix $(\delta_{V_1} - C)^{-1}$ is called the *fundamental matrix* ([12, p. 27], [25, p. 122]) and many basic descriptive quantities of the Markov process can be calculated based on it. The next lemma shows that a harmonic function is uniquely determined by its boundary value, namely by its restriction on V_0 [12, p. 29].

LEMMA 2.2. [12, p. 29] *Suppose the influence digraph D is memoryless. Then $\ker(A - \delta_V) = \cap_{v \in V_1} \ker(A_v - \delta_v)$ is exactly the column space of $\hat{B}$. Especially, $\alpha \in \ker(A - \delta_V)$ if and only if $\alpha|_{V_1} = (\delta_{V_1} - C)^{-1}B\alpha|_{V_0}$.*

Proof. It is easy to check that $A\hat{B} = \hat{B}$. By Lemma 2.1, $\dim(\ker(A - \delta_V)) = |V_0| = \text{rank}(\hat{B})$, from which the lemma follows. $\square$

We construct the $V \times V$ matrix

$$(2.1) \quad M := \begin{bmatrix} \hat{B} & \mathbf{0} \end{bmatrix}$$

and call it the *consensus matrix* of the opinion dynamics. Let

$$\tilde{B} := \begin{bmatrix} \delta_{V_0} & \mathbf{0} \\ (\delta_{V_1} - C)^{-1}B & \delta_{V_1} \end{bmatrix},$$

which is surely an invertible matrix. This tells us that the column vectors of $\hat{B}$ together with $\mathbf{i}_v$, $v \in V_1$, form a basis of $\mathbb{R}^V$. It is easy to see that

$$(2.2) \quad M\tilde{B} = M.$$

We are ready to have the following results.

LEMMA 2.3. *The map $x \mapsto Mx$ for $x \in \mathbb{R}^V$ is the projection from $\mathbb{R}^V$ onto the column space of $\hat{B}$ along the subspace spanned by $\{\mathbf{i}_v : v \in V_1\}$.*

THEOREM 2.4. [12, p. 28] *If D is memoryless, then $\lim_{t \rightarrow \infty} A^t = M$.*

Proof. By Lemma 2.1, we have $\rho(C) < 1$. Thus, $\lim_{t \rightarrow \infty} C^t = \mathbf{0}$ which implies that $\lim_{t \rightarrow \infty} A^t \mathbf{i}_v = \mathbf{0}$ for each $v \in V_1$. This together with Lemma 2.2 gives $\lim_{t \rightarrow \infty} A^t \tilde{B} = M$. By (2.2), we obtain $\lim_{t \rightarrow \infty} A^t = M\tilde{B}^{-1} = M$. $\square$

3. Infinite matrix product. Let U and W be two finite words. We say that U is a *subword* of W provided U can be obtained from W by striking out 0 or more symbols. For example, 225 is a subword of 12345265. Let W be a word over

the alphabet V_1 . We say that W displays V_1 k times provided W has a subword $W_1W_2\cdots W_k$ where each W_i is a word in which each letter from V_1 appears exactly once.

LEMMA 3.1. *Let W be a word which displays V_1 at least $|V_1|$ times. Let $P = v_0, v_1, v_2, \dots, v_n$ be a path in the influence digraph D with v_n being the only vertex in both the path and V_0 . Then $v_0v_1\cdots v_{n-1}$ is a subword of both W and $\overleftarrow{W}$.*

Proof. It is clear that $n \leq |V_1|$. This guarantees that $v_0v_1\cdots v_{n-1}$ must be a subword of both W and $\overleftarrow{W}$ as both of them display V_1 at least $|V_1|$ times. $\square$

For any path $v_0, v_1, \dots, v_n$ in D , we define its *weight* to be

$$a_{v_0}(v_1)a_{v_1}(v_2)\cdots a_{v_{n-1}}(v_n).$$

By the assumption that D is memoryless, for any $v \in V_1$, there exist paths connecting v to vertices in V_0 . We use $wt(v)$ to denote the maximum weight of such a path. Let

$$(3.1) \quad \eta := \min_{v \in V_1} wt(v),$$

which must satisfy $0 < \eta \leq 1$.

LEMMA 3.2. *Suppose the influence digraph D is memoryless. Let η be the number given by (3.1) and W be a word which displays V_1 at least $|V_1|$ times. Then, for any probability vector $\gamma \in \mathbb{R}^V$, we have $\gamma^\top A_W \mathbf{i}_{V_1} \leq \gamma^\top \mathbf{i}_{V_1} (1 - \frac{\eta}{|V_1|})$ and $\gamma^\top A_{\overleftarrow{W}} \mathbf{i}_{V_1} \leq \gamma^\top \mathbf{i}_{V_1} (1 - \frac{\eta}{|V_1|})$.*

Proof. Take a vertex $v \in V_1$ such that

$$(3.2) \quad g := \gamma^\top \mathbf{i}_v \geq \frac{\gamma^\top \mathbf{i}_{V_1}}{|V_1|}.$$

Set

$$(3.3) \quad \beta^\top := \gamma^\top - g \mathbf{i}_v^\top.$$

By (3.1), there is a sequence of pairwise distinct vertices $v_0, v_1, \dots, v_n$ such that $v_n \in V_0$, $v = v_0$ and

$$(3.4) \quad a_{v_0}(v_1)a_{v_1}(v_2)\cdots a_{v_{n-1}}(v_n) = wt(v) \geq \eta.$$

By Lemma 3.1, we can pick up a sequence of positive integers $T_n < T_{n-1} < \cdots < T_1 \leq T = |W|$ so that for any $s \in [n]$, T_s is the minimum integer such that $W[1, T_s]$ contains $v_0v_1\cdots v_{n-s}$ as a subword. This allows us to obtain

$$(3.5) \quad \begin{cases} \mathbf{i}_v^\top A_{[1, T_n]} \mathbf{i}_{v_1} & \geq a_{v_0}(v_1); \\ \mathbf{i}_v^\top A_{W[1, T_{n-1}]} \mathbf{i}_{v_2} & \geq a_{v_0}(v_1)a_{v_1}(v_2); \\ & \vdots \\ \mathbf{i}_v^\top A_{W[1, T_1]} \mathbf{i}_{v_n} & \geq a_{v_0}(v_1)a_{v_1}(v_2)\cdots a_{v_{n-1}}(v_n). \end{cases}$$

Therefore,

$$\begin{aligned}
 \gamma^\top \mathbf{i}_{V_1} - \gamma^\top A_W \mathbf{i}_{V_1} &= \gamma^\top A_{W[1,T]} \mathbf{i}_{V_0} - \gamma^\top \mathbf{i}_{V_0} \\
 &= (g \mathbf{i}_v^\top A_{W[1,T]} \mathbf{i}_{V_0} - g \mathbf{i}_v^\top \mathbf{i}_{V_0}) \\
 &\quad + (\beta^\top A_{W[1,T]} \mathbf{i}_{V_0} - \beta^\top \mathbf{i}_{V_0}) \quad (\text{By (3.3)}) \\
 &\geq g \mathbf{i}_v^\top A_{W[1,T]} \mathbf{i}_{V_0} - g \mathbf{i}_v^\top \mathbf{i}_{V_0} \\
 &= g \mathbf{i}_v^\top A_{W[1,T]} \mathbf{i}_{V_0} \\
 &\geq g \mathbf{i}_v^\top A_{W[1,T_1]} \mathbf{i}_{V_0} \\
 &\geq g \mathbf{i}_v^\top A_{W[1,T_1]} \mathbf{i}_{v_n} \\
 &\geq g a_{v_0}(v_1) a_{v_1}(v_2) \cdots a_{v_{n-1}}(v_n) \quad (\text{By (3.5)}) \\
 &\geq \frac{\gamma^\top \mathbf{i}_{V_1}}{|V_1|} \eta, \quad (\text{By (3.2) and (3.4)})
 \end{aligned}$$

which proves the first inequality. The second inequality can be established by symmetry. $\square$

THEOREM 3.3. *Let the influence digraph D be memoryless and let $w \in V^{\mathbb{N}}$ be a typical sequence with respect to V_1 . Then $\lim_{t \rightarrow \infty} A_w(1, t) = \lim_{t \rightarrow \infty} A_w(t, 1) = M$, where M is the consensus matrix of the opinion dynamics as specified in (2.1).*

Proof. Since $\tilde{B}$ is invertible, our task is to prove that

$$\lim_{t \rightarrow \infty} A_w(1, t) \tilde{B} = \lim_{t \rightarrow \infty} A_w(t, 1) \tilde{B} = M \tilde{B}.$$

By Lemmas 2.2 and 2.3, it suffices to show $\lim_{t \rightarrow \infty} A_w(1, t) \mathbf{i}_v = \lim_{t \rightarrow \infty} A_w(t, 1) \mathbf{i}_v = \mathbf{0}$ for all $v \in V_1$. Taking into account the nonnegativity of $A_w(1, t)$, we just need to show $\lim_{t \rightarrow \infty} \gamma^\top A_w(1, t) \mathbf{i}_{V_1} = \lim_{t \rightarrow \infty} \gamma^\top A_w(t, 1) \mathbf{i}_{V_1} = 0$ for any probability vector γ . Take any number $\epsilon > 0$. We aim to find a T , such that for any $t > T$,

$$\max(\gamma^\top A_{w[1,t]} \mathbf{i}_{V_1}, \gamma^\top A_{w[t,1]} \mathbf{i}_{V_1}) < \epsilon.$$

Let η be the positive number given by (3.1). Choose $n \in \mathbb{N}$ so that $(1 - \frac{\eta}{|V_1|})^n < \epsilon$. As w is a typical sequence, there is $T \in \mathbb{N}$ such that for any $t \geq T$ the word $w[1, t]$ displays V_1 at least $n|V_1|$ times. Note that the product of a probability row vector and a row-stochastic matrix is still a probability row vector. Applying Lemma 3.2 repeatedly, for any $t \geq T$, we obtain

$$(3.6) \quad \max(\gamma^\top A_{w[1,t]} \mathbf{i}_{V_1}, \gamma^\top A_{w[t,1]} \mathbf{i}_{V_1}) \leq \gamma^\top \mathbf{i}_{V_1} (1 - \frac{\eta}{|V_1|})^n \leq (1 - \frac{\eta}{|V_1|})^n < \epsilon,$$

from which the proof is complete. $\square$

Proof of Theorem 1.1. In light of Theorem 3.3 and (2.1), we have $\lim_{t \rightarrow \infty} \alpha_t = \lim_{t \rightarrow \infty} A_w(t, 1) \alpha_1 = M \alpha_1 = \hat{B} \alpha_1|_{V_0}$. $\square$

REMARK 3.4. If we view the sequence $w \in V^{\mathbb{N}}$ as a sequence of random variables satisfying some reasonable assumptions, then we often can estimate the number $T \in \mathbb{N}$

used in the proof of Theorem 3.3 such that for any $t \geq T$ the word $w[1, t]$ displays V_1 at least $n|V_1|$ times, and thus, we can make use of (3.6) to quantitatively estimate the speed of convergence of the evolving opinion profile to its limiting distribution.

REMARK 3.5. Assume that the influence digraph D is memoryless and $w \in V_1^{\mathbb{N}}$ is any opinion update sequence. Let $U = \{v \in V_1 : w^{-1}(v) \text{ is an infinite set}\}$ and let $k = \min\{t \in \mathbb{N} : w(s) \in U, \forall s \geq t\}$. Theorem 3.3 shows that $\lim_{t \rightarrow \infty} A_w(k, t) = \lim_{t \rightarrow \infty} A_w(t, k)$ is a matrix whose v th column is zero for each $v \in U$. Note that

$$\lim_{t \rightarrow \infty} A_w(t, 1) = \left(\lim_{t \rightarrow \infty} A_w(t, k) \right) A_w(k, 1)$$

and

$$\lim_{t \rightarrow \infty} A_w(1, t) = A_w(1, k) \left(\lim_{t \rightarrow \infty} A_w(k, t) \right).$$

In the framework of inhomogeneous Markov chain, this says that, regardless of the initial probability distribution, the process will be running away from those states in U asymptotically.

By comparing Theorem 2.4 with Theorem 3.3, we can see that determining the limiting opinion profile is equivalent to calculating the probabilities of absorption in an absorbing homogeneous Markov chain with A as the transition matrix. Many methods for the latter task have been developed in the literature. (See [15] and references therein.)

As an example, let us address a special opinion dynamics model, which we refer to as the *path model*; note that a corresponding simpler Markov chain model is the so-called random walk between two absorbing barriers [25, p. 123] studied in the context of the Gambler's Ruin Problem. Let n be a positive integer and $[n]$ be the set $\{1, 2, \dots, n\}$. Assume that $n+2$ agents, represented by $[n+2]$, are estimating the price of an item and agent 1 and agent $n+2$ will never update their opinions. Let the network underlying the information exchange be a path on these $n+2$ agents such that given the current opinion profile α and if it is the turn for an agent i ($2 \leq i \leq n+1$) to update his/her opinion he/she will change it from $\alpha(i)$ to $x_{i-1}\alpha(i-1) + y_{i-1}\alpha(i+1) + (1 - x_{i-1} - y_{i-1})\alpha(i)$, where x_{i-1} , y_{i-1} and $1 - x_{i-1} - y_{i-1}$ are all nonnegative numbers.

For any $j \in [n+1]$, set

$$X_{j-1} := \prod_{k=1}^{j-1} x_k \quad \text{and} \quad Y_j := \prod_{k=j}^n y_k;$$

while for any $i \in [n+2]$, let

$$z_i := \sum_{j=1}^{i-1} X_{j-1} Y_j.$$

Note that we adopt the standard convention that an empty sum (resp. product) of numbers is 0 (resp. 1). For instance, when $n = 2$ we have $z_1 = 0, z_2 = y_1 y_2, z_3 = y_1 y_2 + x_1 y_2, z_4 = y_1 y_2 + x_1 y_2 + x_1 x_2$. It is clear that $0 = z_1 \leq z_2 \leq \dots \leq z_{n+1} \leq z_{n+2}$. We assume that the influence digraph for this path model is memoryless, which amounts to asserting $z_{n+2} > 0$.

To proceed, let us consider two special vectors $\mathbf{k} := \sum_{i=1}^{n+2} \frac{z_i}{z_{n+2}} \mathbf{i}_i$ and $\mathbf{j} := \mathbf{i}_{[n+2]}$, which satisfy

$$(3.7) \quad A_i \mathbf{k} = \mathbf{k} \quad \text{and} \quad A_i \mathbf{j} = \mathbf{j}$$

for any $i \in [n+2]$. The second half of (3.7) is quite obvious. To obtain the first half, we need to show that the equation $x_i z_i + (1 - x_i - y_i) z_{i+1} + y_i z_{i+2} = z_{i+1}$ holds for all $i \in [n]$. This can be seen after performing the following simple calculation:

$$\begin{aligned} (x_i z_i + (1 - x_i - y_i) z_{i+1} + y_i z_{i+2}) - z_{i+1} &= y_i (z_{i+2} - z_{i+1}) - x_i (z_{i+1} - z_i) \\ &= y_i X_i Y_{i+1} - x_i X_{i-1} Y_i \\ &= 0. \end{aligned}$$

By Lemma 2.2, (2.1) and (3.7), we now see that, up to a simultaneous permutations of rows and columns, the consensus matrix M for our path model is the $(n+2) \times (n+2)$ matrix with $M \mathbf{i}_1 = \mathbf{j} - \mathbf{k}$, $M \mathbf{i}_{n+2} = \mathbf{k}$ and $M \mathbf{i}_t = \mathbf{0}$ for all $2 \leq t \leq n+1$. As an illustration, when $x_1 = \dots = x_n = y_1 = \dots = y_n = \frac{1}{3}$, we have

$$(3.8) \quad M = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ \frac{n}{n+1} & 0 & 0 & \dots & 0 & \frac{1}{n+1} \\ \frac{n-1}{n+1} & 0 & 0 & \dots & 0 & \frac{2}{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{n+1} & 0 & 0 & \dots & 0 & \frac{n}{n+1} \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}.$$

Moreover, when $n = 1$, the matrix A_2 can be diagonalized by appealing to the basis $\{\mathbf{j}, \mathbf{k}, \mathbf{i}_2\}$, and thus, the powers of A_2 can be easily calculated. This leads to another solution to [25, Appendix B, Example].

To conclude this paper, we mention that one may generalize the path model a little further to the star-like tree model. Let $G := S(m; k_1, k_2, \dots, k_m)$ be a star-like tree with m branches of lengths $k_1, k_2, \dots, k_m$, respectively. That is, $V(G) = \{v_{i,j} : i \in [m] \text{ and } j \in [k_i]\} \cup \{v\}$ and $E(G) = \{v_{i,j} v_{i,j+1} : i \in [m] \text{ and } j \in [k_i - 1]\} \cup \{v v_{i,1} : i \in [m]\}$. Suppose that v_{i,k_i} are all stubborn vertices and that all other vertices will always update their opinions to the average of the opinions of their closed neighbors when their turns to change opinions come and they have such chances infinitely often.

Assume that the opinion profile at time 1 is α_1 . We state here without proof that each vertex $v_{i,j}$ has a limiting opinion

$$(3.9) \quad \frac{k + j(m - \frac{k}{k_i})}{mk_i} \alpha_1(v_{i,k_i}) + \frac{k_i - j}{mk_i} \sum_{i' \in [m] \setminus \{i\}} \frac{k}{k_{i'}} \alpha_1(v_{i',k_{i'}}),$$

where k is the harmonic average of $k_1, k_2, \dots, k_m$, namely, $k = (\frac{k_1^{-1} + \dots + k_m^{-1}}{m})^{-1}$.

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