

MAXIMA OF THE Q -INDEX: ABSTRACT GRAPH PROPERTIES*

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Abstract. Let $q(G)$ denote the spectral radius of the signless Laplacian matrix of a graph G , also known as the Q -index of G . The aim of this note is to study a general extremal problem:

How large can $q(G)$ be when G belongs to an abstract graph property?

Even knowing very little about the graph property, this paper shows that useful conclusions about the asymptotics of $q(G)$ can be made, which turn out to be efficient in concrete applications.

Key words. Signless Laplacian, Monotone property, Hereditary property, Largest eigenvalue, Eigenvalue bounds.

AMS subject classifications. 05C50.

1. Introduction and main results. Given a graph G , write A for its adjacency matrix and let D be the diagonal matrix of the row-sums of A , i.e., the degrees of G . The matrix $Q(G) = A + D$, called the *signless Laplacian* or the Q -matrix of G , has been intensively studied recently; see, e.g., the survey of Cvetković [8] and its references.

Let $q(G)$ denote the largest eigenvalue of the Q -matrix of a graph G . The aim of this note is to investigate $q(G)$ when G belongs to some abstract graph property.

1.1. Graph properties. Recall that a *graph property* \mathcal{P} is a class of graphs closed under isomorphisms. A property \mathcal{P} is called *monotone* if it is closed under taking subgraphs, and is called *hereditary* if it is closed under taking induced subgraphs. Clearly, every monotone property is also hereditary.

An easy way to construct monotone and hereditary properties is by forbidding certain subgraphs. Indeed, given a family of graphs \mathcal{F} , the class of graphs with no subgraph belonging to \mathcal{F} is a monotone property, and likewise, the class of graphs with no *induced* subgraph belonging to \mathcal{F} is a hereditary property, which hereafter we shall denote by $\mathcal{P}^*(\mathcal{F})$. In fact, it is easy to see that every monotone or hereditary

*Received by the editors on April 11, 2012. Accepted for publication on June 28, 2012. Handling Editor: Bryan L. Shader.

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property arises by forbidding some, possibly infinitely many, subgraphs.

For several decades, graph properties have been intensively investigated in traditional graph theory; see, e.g., [2], [3], [6], and [4]. Particular attention has been given to hereditary and monotone properties, but there are other interesting types of property also, some of which will be discussed below.

On the other hand, little is known about the spectra of various graph properties, although recently graph properties have been studied in connection to spectral extrema of the adjacency matrices. In the present paper we are making first steps to study the Q -spectra of graph properties. Thus, given a graph property \mathcal{P} , we write \mathcal{P}_n for the set of graphs of order n belonging to \mathcal{P} , and set

$$q(\mathcal{P}_n) = \max_{G \in \mathcal{P}_n} q(G).$$

In the spirit of classical extremal graph theory, a natural problem arises.

PROBLEM 1.1. *Given a hereditary property \mathcal{P} , find $q(\mathcal{P}_n)$.*

Much research has been dedicated to the solution of this problem for particular graph properties \mathcal{P} . For instance, it has been proved (see, e.g., [7], [12], and [17]) that if \mathcal{P} is the class of all r -chromatic graphs, then

$$q(\mathcal{P}_n) = q(T_r(n)) = \frac{2r-2}{r}n + o(n),$$

where $T_r(n)$ is the r -partite Turán graph of order n . In fact, it has been shown recently ([1], [13]) that the same equality holds if \mathcal{P} is the class of graphs with clique number at most r .

It seems unlikely that we could determine precisely $q(\mathcal{P}_n)$ for every property \mathcal{P} , and therefore it is of interest to search for asymptotic solutions of the above problem. Some ground for such approach is provided by the following theorem, proved in Section 2.

THEOREM 1.2. *If \mathcal{P} is a hereditary graph property, then the limit*

$$\lim_{n \rightarrow \infty} \frac{q(\mathcal{P}_n)}{n}$$

exists.

Furthermore, for a hereditary property \mathcal{P} , write $\nu(\mathcal{P})$ for the limit established in Theorem 1.2. Now the problem of finding $q(\mathcal{P}_n)$ can be replaced by the presumably easier problem of determining $\nu(\mathcal{P})$, thus giving the asymptotics

$$q(\mathcal{P}_n) = (\nu(\mathcal{P}) + o(1))n.$$

As it turns out, for a vast class of hereditary graph properties, the constant $\nu(\mathcal{P})$ can be found, as stated in Theorem 1.3 below. A nontrivial analog to the Erdős-Stone-Bollobás theorem ([11], [5]) is proved in [16], which in particular implies a theorem about monotone graph properties, strengthening the Erdős-Simonovits theorem [10].

THEOREM 1.3. *Let \mathcal{F} be a family of graphs with $r = \min \{\chi(F) : F \in \mathcal{F}\}$. If $r \geq 3$, then*

$$\nu(\mathcal{P}^*(\mathcal{F})) = \frac{2r-4}{r-1}.$$

Note that $T_{r-1}(n)$, the $(r-1)$ -partite Turán graph of order n , contains no copy of any graph $F \in \mathcal{F}$, while

$$q(T_{r-1}(n)) > \frac{2r-4}{r-1}n - 2.$$

Hence, Theorem 1.3 shows that if \mathcal{F} is a family of graphs with minimum chromatic number $r \geq 3$ and G is a graph of order n , with no copy of a graph $F \in \mathcal{F}$, then

$$(1.1) \quad q(G) \leq \frac{2r-4}{r-1}n + o(n).$$

Furthermore, in view of the known fact $q(G) \geq 4e(G)/|G|$, for $r \geq 3$ we obtain the Erdős-Simonovits theorem itself.

THEOREM 1.4. *Let \mathcal{F} be a family of graphs with $r = \min \{\chi(F) : F \in \mathcal{F}\}$. Let $ex(n, \mathcal{F})$ be the maximum number of edges in a graph of order n , that does not contain any graph belonging to \mathcal{F} . Then*

$$\lim_{n \rightarrow \infty} \frac{ex(n, \mathcal{F})}{n^2} = \frac{r-2}{2(r-1)}.$$

Theorem 1.3 is quite extensive, but one question remains: How relevant is (1.1) when \mathcal{F} consists of nontrivial bipartite graphs? A short argument shows that the Theorem 1.3 does not apply in this case. Indeed, if F is a connected bipartite graph which is not a star, then all stars belong to $\mathcal{P}^*(F)$ and so $\nu(\mathcal{P}^*(F)) \geq 1$. The following proposition, proved in Section 2, shows that for such a graph F we have $\nu(\mathcal{P}^*(F)) = 1$.

PROPOSITION 1.5. *Let F be a bipartite graph. Then there exists $c = c(F) > 0$ such that if G is an F -free graph of order n , then*

$$q(G) \leq n + n^{1-c}.$$

This shows that Theorem 1.2 could not provide enough precision for $q(\mathcal{P}_n^*(F))$, when F is a connected bipartite graph. A similar situation is present in the classical extremal graph theory, where obtaining precise bounds in extremal problems for forbidden bipartite graphs proves to be notoriously difficult in general.

1.2. Multiplicative graph properties. At first glance Theorem 1.2 seems too general to be of much use, but for a wide class of graph properties it may provide further useful knowledge. Thus, we shall describe a natural collection of graph properties that, by the way, does not seem well studied for their own sake in traditional graph theory.

For any graph G and integer $r \geq 1$, write $G^{(r)}$ for the graph obtained by replacing each vertex u of G by a set of r independent vertices and each edge uv of G by a complete bipartite graph $K_{r,r}$. This construction is known as a “blow-up” of G . Note that the adjacency matrix of $G^{(r)}$ is obtained as the Kronecker product of the adjacency matrix of G and the square all one matrix J_k of size k .

A graph property \mathcal{P} is called *multiplicative* if $G \in \mathcal{P}$ implies that $G^{(r)} \in \mathcal{P}$ for all $r \geq 1$.

Multiplicative graph properties seem to be important in spectral graph theory; see [15] for example. This is mostly due to the fact that the Kronecker product of matrices fits well with spectra.

Let us emphasize that multiplicative properties are of a totally new type, independent of the hereditary or monotone types. For example, “all graphs”, “ r -partite graphs”, “ K_r -free graphs”, “graphs with no odd cycle shorter than k ” are both multiplicative and hereditary, while “Hamiltonian”, “non-planar”, “ k -connected”, “having a 1-factor” are multiplicative, but not hereditary properties.

In Section 2, we shall prove the following theorem, which strengthens Theorem 1.2.

THEOREM 1.6. *If \mathcal{P} is a hereditary and multiplicative graph property, then*

$$q(G) \leq \nu(\mathcal{P}) |G|$$

for every $G \in \mathcal{P}$.

Note that if \mathcal{P} is not hereditary, the above theorem may not hold. Indeed, define a multiplicative property \mathcal{P} by setting

$$\mathcal{P}_n = \begin{cases} \text{the set of all graphs of order } n, & \text{if } n \text{ is a composite number;} \\ \text{the cycle of length } n, & \text{if } n \text{ is a prime.} \end{cases}$$

Clearly, \mathcal{P} is multiplicative, but note that

$$q(\mathcal{P}_n) = \begin{cases} 2n - 2, & \text{if } n \text{ is a composite number;} \\ 4, & \text{if } n \text{ is a prime.} \end{cases}$$

Hence, $\nu(\mathcal{P})$ does not exist.

Also, Theorem 1.6 may not hold if \mathcal{P} is not multiplicative. To see this, take the property \mathcal{P} of all C_4 -free graphs, which is hereditary, but not multiplicative. As already noted in Proposition 1.5, we have $\nu(\mathcal{P}) = 1$, but the friendship graph of odd order n shows that $q(\mathcal{P}_n) > n$, so the conclusion of Theorem 1.6 does not hold in this case.

Theorem 1.6 may be useful. Consider, for instance, the following general application: Suppose that \mathcal{P} is a hereditary and multiplicative graph property. Suppose also that for some sufficiently large graphs $G \in \mathcal{P}$ we are able to prove that

$$q(G) \leq c|G| + f(|G|),$$

where $f(x)$ is a function satisfying $f(x) = o(x)$. Then, in view of Theorem 1.6, we can conclude that

$$q(G) \leq c|G|$$

for every $G \in \mathcal{P}$. This approach has been used in [1].

2. Proofs.

Proof of Theorem 1.2. Choose a graph $G \in \mathcal{P}_n$ with $q(G) = q(\mathcal{P}_n)$. Let $\mathbf{x} = (x_1, \dots, x_n)$ be a unit eigenvector to $q(G)$ and set $x = \min\{x_1, \dots, x_n\}$. Recall first that

$$q(G) = \langle Q\mathbf{x}, \mathbf{x} \rangle = \sum_{ij \in E(G)} (x_i + x_j)^2.$$

Let u be a vertex for which $x_u = x$, and write $G - u$ for the graph obtained by removing the vertex u . Also, write $\Gamma(v)$ for the set of neighbors of a vertex v . We have,

$$\begin{aligned} q(\mathcal{P}_n) &= \sum_{ij \in E(G)} (x_i + x_j)^2 = \sum_{ij \in E(G-u)} (x_i + x_j)^2 + \sum_{j \in \Gamma(u)} (x + x_j)^2 \\ &= \sum_{ij \in E(G-u)} (x_i + x_j)^2 + d(u)x^2 + 2x \sum_{j \in \Gamma(u)} x_j + \sum_{j \in \Gamma(u)} x_j^2. \end{aligned}$$

In view of the Rayleigh principle and the fact that $G - u \in \mathcal{P}_{n-1}$, we have

$$\sum_{ij \in E(G-u)} (x_i + x_j)^2 \leq (1 - x^2) q(G - u) \leq (1 - x^2) q(\mathcal{P}_{n-1}).$$

Now, using the equation

$$(q(\mathcal{P}_n) - d(u))x = \sum_{j \in \Gamma(u)} x_j$$

we find that

$$\begin{aligned} d(u)x^2 + 2x \sum_{j \in \Gamma(u)} x_j + \sum_{j \in \Gamma(u)} x_j^2 &\leq d(u)x^2 + 2(q(\mathcal{P}_n) - d(u))x^2 + \sum_{j \in \Gamma(u)} x_j^2 \\ &\leq d(u)x^2 + 2(q(\mathcal{P}_n) - d(u))x^2 + 1 - (n - d(u))x^2 \\ &= 2q(\mathcal{P}_n)x^2 - nx^2 + 1. \end{aligned}$$

We get

$$(1 - x^2)q(\mathcal{P}_{n-1}) + q(\mathcal{P}_n) \geq 2q(\mathcal{P}_n)x^2 - nx^2 + 1,$$

and taking into account that $x^2 \leq 1/n$, we see that

$$q(\mathcal{P}_{n-1}) \geq q(\mathcal{P}_n) \frac{1 - 2x^2}{1 - x^2} + \frac{-nx^2 + 1}{1 - x^2} \geq q(\mathcal{P}_n) \frac{1 - 2/n}{1 - 1/n}.$$

Therefore,

$$\frac{q(\mathcal{P}_{n-1})}{n-2} \geq \frac{q(\mathcal{P}_n)}{n-1}$$

and the limit

$$\lim_{n \rightarrow \infty} \frac{q(\mathcal{P}_n)}{n-1}$$

exists, which implies also that the limit

$$\lim_{n \rightarrow \infty} \frac{q(\mathcal{P}_n)}{n}$$

exists, completing the proof. \square

Proof of Proposition 1.5. Let $K_{r,r}$ be the smallest complete bipartite graph that contains F . Obviously, if a graph G of order n does not contain F , then it does not contain $K_{r,r}$ either. Now the classical result of Kövari, Sós and Turán [14] gives that

$$2e(G) \leq (r-1)^{1/r} (n-r+1) n^{1-1/r} + (r-1)n$$

which implies in turn

$$2e(G) < (n-1)n^{1-c'}$$

for some fixed $c' > 0$ and n sufficiently large. This, together with the upper bound of Das [9]

$$q(G) \leq \frac{2e(G)}{n-1} + n - 2,$$

implies the required inequality for some sufficiently small $c > 0$ and every n . \square

Proof of Theorem 1.6. The theorem follows from the fact that for every graph G and every $k \geq 1$,

$$q(G^{(k)}) = kq(G).$$

Thus, let \mathcal{P} be a hereditary and multiplicative graph property, let $G \in \mathcal{P}_n$ and let $q(\mathcal{P}_n) = q(G_0)$. Then for every $k > 1$, we have

$$\frac{q(G)}{n} = \frac{q(G^{(k)})}{nk} \leq \frac{q(G_0^{(k)})}{kn} \leq \frac{q(\mathcal{P}_{kn})}{kn}$$

and so

$$\frac{q(G)}{n} \leq \nu(\mathcal{P}),$$

completing the proof of the theorem. \square

REFERENCES

- [1] N.M.M de Abreu and V. Nikiforov. Clique number and the signless Laplacian. Submitted for publication.
- [2] N. Alon, J. Balogh, B. Bollobás, and R. Morris. The structure of almost all graphs in a hereditary property. *J. Combin. Theory Ser. B*, 101:85–110, 2011.
- [3] N. Alon, E. Fischer, and I. Newman. Efficient testing of bipartite graphs for forbidden induced subgraphs. *SIAM J. Comput.*, 37:959–976, 2007.
- [4] J. Balogh, B. Bollobás, and D. Weinreich. The speed of hereditary properties of graphs. *J. Combin. Theory Ser. B*, 79:131–156, 2000.
- [5] B. Bollobás and P. Erdős. On the structure of edge graphs. *Bull. London Math. Soc.*, 5:317–321, 1973.
- [6] B. Bollobás and A.G. Thomason, Hereditary and monotone properties of graphs. In: R.L. Graham and J. Nešetřil (editors), *The Mathematics of Paul Erdős, II, Algorithms and Combinatorics*, Springer-Verlag, 14:70–78, 1997.
- [7] G.X. Cai and Y.-Z. Fan. The signless Laplacian spectral radius of graphs with given chromatic number (in Chinese). *Math. Appl. (Wuhan)*, 22(1):161–167, 2009.
- [8] D. Cvetković. Spectral theory of graphs based on the signless Laplacian. Research Report, 2010. Available at http://www.mi.sanu.ac.rs/projects/signless_L_reportApr11.pdf.
- [9] K.C. Das. On conjectures involving second Largest signless Laplacian eigenvalue of graphs. *Linear Algebra Appl.*, 432:3018–3029, 2010.
- [10] P. Erdős and M. Simonovits, A limit theorem in graph theory. *Studia Sci. Math. Hungar.*, 1:51–57, 1966.

- [11] P. Erdős and A.H. Stone. On the structure of linear graphs. *Bull. Amer. Math. Soc.*, 52:1087–1091, 1946.
- [12] P. Hansen and C. Lucas. An inequality for the signless Laplacian index of a graph using the chromatic number. *Graph Theory Notes of New York*, LVII, 39–42, 2009.
- [13] B. He, Y.L. Jin, and X.D. Zhang. Sharp bounds for the signless Laplacian spectral radius in terms of clique number. *Linear Algebra Appl.*, 2011, doi:10.1016/j.laa.2011.10.008.
- [14] T. Kövari, V. Sós, and P. Turán. On a problem of K. Zarankiewicz. *Colloquium Math.*, 3:50–57, 1954.
- [15] V. Nikiforov. Linear combinations of graph eigenvalues. *Electron. J. Linear Algebra*, 15:329–336, 2006.
- [16] V. Nikiforov. The Erdős-Stone theorem for the signless Laplacian, manuscript.
- [17] G. Yu, Y. Wu, and J. Shu. Signless Laplacian spectral radii of graphs with given chromatic number. *Linear Algebra Appl.*, 435:1813–1822, 2011.