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# SINGULAR POINTS OF THE TERNARY POLYNOMIALS ASSOCIATED WITH 4-BY-4 MATRICES*

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**Abstract.** Let $T$ be an $n \times n$ matrix. The numerical range of $T$ is defined as the set

$$W(T) = \{\xi^* T \xi : \xi \in \mathbb{C}^n, \xi^* \xi = 1\}.$$

A homogeneous ternary polynomial associated with $T$ is defined as

$$F(t, x, y) = \det(tI_n + x(T + T^*)/2 + y(T - T^*)/(2i)).$$

The numerical range $W(T)$ is the convex hull of the real affine part of the dual curve of $F(t, x, y, ) = 0$. We classify the numerical ranges of $4 \times 4$ matrices according to the singular points of the curve $F(t, x, y) = 0$.

**Key words.** Numerical range, Homogeneous ternary polynomial, Singular point.

**AMS subject classifications.** 15A60, 14H45.

## 1. Introduction.

The numerical range $W(T)$ of an $n \times n$ complex matrix $T$ was introduced by Toeplitz and defined as the set

$$W(T) = \{\xi^* T \xi : \xi \in \mathbb{C}^n, \xi^* \xi = 1\}.$$

Kippenhahn [11] (see also [20] for an English translation) characterized this range from a viewpoint of the algebraic curve of the homogeneous ternary polynomial associated with $T$:

$$F(t, x, y) = \det(tI_n + x(T + T^*)/2 + y(T - T^*)/(2i)).$$

Let $\Gamma_F$ be the algebraic curve of $F(t, x, y)$, i.e.,

$$\Gamma_F = \{[(t, x, y)] \in \mathbb{CP}^2 : F(t, x, y) = 0\},$$

where $[(t, x, y)]$ is the equivalence class containing $(t, x, y) \in \mathbb{C}^3 - (0, 0, 0)$ under the relation $(t_1, x_1, y_1) \sim (t_2, x_2, y_2)$ if $(t_2, x_2, y_2) = k(t_1, x_1, y_1)$ for some nonzero complex number $k$. The dual curve $\Gamma_F^\wedge$ of $\Gamma_F$ is defined by

$$\Gamma_F^\wedge = \{[(T, X, Y)] \in \mathbb{CP}^2 : T\,t + X\,x + Y\,y = 0 \text{ is a tangent line of } \Gamma_F\}.$$

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Kippenhahn [11] showed that $W(T)$ is the convex hull of $\Gamma_F^\wedge$ in the real affine plane. The real affine part of $\Gamma_F^\wedge$ is called the boundary generating curve of $W(A)$. There have been a number of papers on the boundary generating curves of the numerical range; for example, see [2, 3, 6, 10].

A complete classification of the range $W(T)$ of $3\times 3$ matrices via the factorability of the homogeneous ternary polynomial $F(t,x,y)$ is given in [11] (see also [10]). It shows that the shapes of $W(T)$ fall into four categories, namely,

($i$) a (possibly degenerate) triangle, if $F(t,x,y)$ factors into three real linear factors;
($ii$) a convex hull of a non-degenerate elliptical disc and a point which is possibly contained in the elliptical disc, if $F(t,x,y)$ factors into a real linear factor and an irreducible quadratic factor;
($iii$) a smooth boundary curve with a flat portion, if $F(t,x,y)$ is irreducible and the curve has $\Gamma_F$ has a real node;
($iv$) an ovular, if $F(t,x,y)$ is irreducible and the curve $\Gamma_F$ has no singular point.

Examples of matrices for each category are also given there.

In this paper, we classify the numerical range of $4\times 4$ matrices along Kippenhahn's direction by examining the singular points of the homogeneous ternary polynomial curve $F(t,x,y)=0$. We focus mainly on the case when $F(t,x,y)$ is irreducible.

**2. Singular points.** We outline briefly the classification of singular points of an algebraic curve. For references on the classification, see, for instance, [15] or [18]. Let $G(t,x,y)$ be a complex ternary form of degree $n$. A point $(t_0,x_0,y_0)\neq(0,0,0)$ of $\Gamma_G$ is called a *singular point* if

$$\frac{\partial}{\partial t}G(t_0,x_0,y_0)=\frac{\partial}{\partial x}G(t_0,x_0,y_0)=\frac{\partial}{\partial y}G(t_0,x_0,y_0)=0.$$

If $G(t,x,y)$ is multiplicity free in the polynomial ring $\mathbb{C}[t,x,y]$ (it happens when $G(t,x,y)$ is irreducible), then the number of singular points of $\Gamma_G\subset\mathbb{CP}^2$ is finite. Suppose that $(1,x_0,y_0)$ is a singular point of $\Gamma_G$. We consider the Taylor expansion of $G(1,x,y)$ around $(1,x_0,y_0)$:

$$G(1,x_0+X,y_0+Y)=\prod_{j=1}^{m}(a_jX+b_jY)+\sum_{i+j\geq m+1}c_{i,j}X^iY^j, \tag{2.1}$$

where $(a_j,b_j)\neq(0,0)$, $j=1,2,\ldots,m$, are pairs of complex numbers. The number $m\geq 2$ is called the *multiplicity* of the singular point $(1,x_0,y_0)$. The following three frames are used to provide local expression of the curve $G(1,x,y)=0$ near the point $(x_0,y_0)$. The finest frame is the ring $\mathbb{C}[X]^*$ of fractional formal power series of $X=x-x_0$ of non-negative order. Let

$$G(1,x_0,y)=c_0(y-y_0)^k(y-y_1)^{\ell_1}\cdots(y-y_p)^{\ell_p}$$

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for some $k \geq m$, $c_0 \neq 0$, where $y_1, \dots, y_p$ are roots of the equations of $G(1, x_0, y) = 0$ other than $y_0$. Then there are unique $k$ solutions $y = y_j(x)$ of $G(1, x, y) = 0$ satisfying $y(x_0) = y_0$ expressed in fractional power series

$$y_j(X) = y_0 + a_j X^{m_1/n_1} + b_j X^{m_2/n_2} + \cdots,$$

where $m_1/n_1 < m_2/n_1 < \cdots$ are positive rational numbers [18, Chapter IV]. If we use fractional power series, we can express the curve $G(1, x, y) = 0$ near $(x_0, y_0)$ as the union of $k$ parametrized curves. If we assume sufficient small modulus of $X$, the above series are convergent. The coarse frame is the polynomial ring $\mathbb{C}[x, y]$ itself. If $G(1, x, y)$ is irreducible in this ring, we can do nothing with this frame. The intermediate frame is the ring $\mathbb{C}[[X, Y]]$ of formal power series in $X = x - x_0$ and $Y = y - y_0$. By the analyticity of the function $G(1, x, y)$ near $(x_0, y_0)$, we can replace this ring by a slightly more restrictive one, that is, the ring $A(V)$ of analytic functions in a neighborhood $V$ of $(x_0, y_0)$ in $\mathbb{C}^2$. We assume that $V$ does not contain singular points of the curve $G(1, x, y) = 0$ other than $(x_0, y_0)$. Consider its irreducible decomposition

$$G(1, x, y) = g_0(x, y) g_1(x, y) g_2(x, y) \cdots g_s(x, y)$$

in $A(V)$, where $g_0(x_0, y_0) \neq 0$ and $g_j(x_0, y_0) = 0$ for $j = 1, 2, \dots, s$. Each curve $g_j(x, y) = 0$ $(j = 1, 2, \dots, s)$ is called an *irreducible analytic branch* of $\Gamma_G$ around $(1, x_0, y_0)$. The number $s$ for the singular point is an important invariant of the singular point. To recognize the difference of decompositions in $\mathbb{C}[X]^*$ and in $\mathbb{C}[[X, Y]]$, we provide a simple example. Let $G(t, x, y) = ty^2 - x^3$. Then the point $(t, x, y) = (1, 0, 0)$ is a singular point of multiplicity 2. In the ring $\mathbb{C}[[x, y]]$, the analytic function $y^2 - x^3$ is irreducible. However, the curve $y^2 - x^3 = 0$ is decomposed as the union of the curves $y = x^{2/3}, y = (-1 \pm i\sqrt{3})/2x^{2/3}$.

Now we classify singular points of $\Gamma_G$, we consider two functions

$$g(X, Y) = G(1, x_0 + X, y_0 + Y), \quad g_Y(X, Y) = G_Y(1, x_0 + X, y_0 + Y).$$

The Taylor series of these functions define an ideal $(g, g_Y)$ of the ring $\mathbb{C}[[X, Y]]$ of formal power series in $X, Y$. The dimension of the quotient ring $\mathbb{C}[[X, Y]]/(g, g_Y)$ is finite, and is called the local intersection number of $\Gamma_G, \Gamma_{G_y}$ at $P$. We define

$$\delta(P) = \frac{1}{2}\Big(\dim \mathbb{C}[[X, Y]]/(g, g_Y)) - m + s\Big).$$

This number is always a non-negative integer (cf. [9, 15]). Then *genus* of $\Gamma_G$ is given by

$$g(\Gamma_G) = (1/2)(n-1)(n-2) - \sum_{j=1}^{k} \delta(P_j),$$

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where $P_1, \ldots, P_k$ are singular points of $\Gamma_G$. The dual curve of a plane algebraic curve $\Gamma_G$ has the same genus as the original curve. Table 1 displays some types of singular points (cf. [9, p. 37]).

TABLE 2.1
*Classification of singular points.*

| Types of singular points | multiplicity $m$ | number $s$ | $\delta(P)$ |
|---|---|---|---|
| $O_2$ | 2 | 2 | 1 |
| $O_2'$ | 2 | 2 | 2 |
| $O_2''$ | 2 | 2 | 3 |
| $C_2$ | 2 | 1 | 1 |
| $C_2'$ | 2 | 1 | 2 |
| $C_2''$ | 2 | 1 | 3 |
| $O_3$ | 3 | 3 | 3 |
| $C_3$ | 3 | 1 | 3 |
| $CO$ | 3 | 2 | 3 |

A *node* in the wide sense is a singular point $(x_0, y_0)$ of the curve $\Gamma_G$ for which every irreducible analytic branch around $(x_0, y_0)$ is expressed as

$$x_j(u) = x_0 + b_j u + \sum_{k=2}^{\infty} c_k^{[j]} u^k,$$
$$y_j(u) = y_0 - a_j u + \sum_{k=2} d_k^{[j]} u^k$$

for some $(a_j, b_j) \neq (0,0)$ ( $j = 1, 2, \ldots, m$). Such irreducible analytic branch is called linear. The singular points $O_2, O_2', O_2'', O_3$ belongs to this type. If the coefficients satisfy $a_i b_j \neq a_j b_i$ for $1 \leq i < j \leq m$, then the node $(1, x_0, y_0)$ is called an *ordinary singular point*. A singular point $(1, x_0, y_0)$ of $\Gamma_G$ of multiplicity $m$ is an ordinary $m$-ple point $O_m$ if and only if the coefficients $a_j, b_j$ in (2.1) satisfy $a_j b_k - b_j a_k \neq 0$ for $1 \leq j < k \leq m$. An irreducible analytic branch other than linear type is called a *cusp*. The singular points $C_2, C_2', C_2$", $C_3$ belongs to this class. A tacnode cusp $CO$ composed of a linear-type irreducible analytic branch and a cusp $C_2$. Since we frequently deal with singular points of multiplicity 2, we pay special attention to this type of singular points. We assume that $(1, x_0, y_0)$ is a singular point of one of the type $C_2$, $C_2'$, $C_2''$ or $O_2', O_2''$. By changing coordinates, we may assume that the Taylor expansion of $G(1, x, y)$ around $(1, x_0, y_0)$ satisfies

$$G(1, x_0, y_0) = a\,(y - y_0)^2 + \sum_{i+j \geq 3} c_{i,j} (x - x_0)^i (y - y_0)^j, \tag{2.2}$$

where $a \neq 0$. Under this assumption, if $(1, x_0, y_0)$ is of type $C_2, C_2'$ or $C_2''$, then the

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irreducible analytic branch $g_1(x, y) = 0$ around the singular point is expressed as

$$x = x_0 + u^\ell$$
$$y = y_0 + a_k e^{2\pi ijk/\ell} u^k + a_{k+1} e^{2\pi ij(k+1)/\ell} u^{k+1} + \cdots$$

for some integers $2 \leq \ell < k$ and $j = 0, 1, 2, \dots, \ell - 1$, and $a_k \neq 0$. For the tacnode cusp $CO$, one irreducible analytic branch is expressed in this form. If $(1, x_0, y_0)$ is a singular point of type $O_2'$ or $O_2''$, then the local expression of $\Gamma_G$ in a neighborhood of $(1, x_0, y_0)$ is given by two irreducible analytic branches expressed as

$$y_1 = y_0 + \alpha_2 x^2 + \alpha_3 x^3 + \cdots$$
$$y_2 = y_0 + \beta_2 x^2 + \beta_3 x^3 + \cdots .$$

The node is $O_2'$ if $\alpha_2 \neq \beta_2$, and the node is $O_2''$ if $\alpha_2 = \beta_2, \alpha_3 \neq \beta_3$.

A real homogeneous polynomial $p(x) = p(x_1, x_2, \dots, x_m)$ of degree $n$ is *hyperbolic* with respect to a vector $e = (e_1, e_2, \dots, e_m)$ if $p(e) \neq 0$ and, for all vectors $w \in \mathbf{R}^m$, the univariate polynomial $t \mapsto p(w - te)$ has all real roots (cf. [1]). The following theorem is mentioned in [11, 20] (in the dual form) without a rigorous proof. We give a proof here relying on an affirmative solution to Lax conjecture [8, 12].

THEOREM 2.1. *Let $G(t, x, y)$ be a real homogeneous ternary polynomial of degree $m > 2$. If $G(t, x, y)$ is hyperbolic with respect to $(1, 0, 0)$ then $\Gamma_G$ has no real cusps.*

*Proof.* Suppose $G(t, x, y)$ of degree $m$ is hyperbolic with respect to $(1, 0, 0)$ with $G(1, 0, 0) = 1$. By Theorem 8 in [12], there exists a pair of $m \times m$ real symmetric matrices $S_1, S_2$ satisfying

$$G(t, x, y) = \det(tI_m + xS_1 + yS_2).$$

Then, by Rellich's result [16] on the perturbation of Hermitian matrices, there exist real-valued analytic functions $\lambda_1(\theta), \lambda_2(\theta), \dots, \lambda_n(\theta)$ on the real line with period $2\pi$ such that

$$G(t, -\cos\theta, -\sin\theta) = (t - \lambda_1(\theta))(t - \lambda_2(\theta)) \cdots (t - \lambda_n(\theta)).$$

Hence, every real singular point $(t, x, y)$ is expressed as $(t_0, \cos\theta_0, \sin\theta_0)$ for some real numbers $t_0, \theta_0$. By using a rotation of coordinates, we may assume that $\theta_0 = 0$. A local expression of $G(t, x, y) = 0$ near the point $(t_0, 1, 0)$ is given by

$$(t, y) = (\lambda_j(\theta) \sec\theta, \tan\theta)$$

for indices $j$ satisfying $\lambda_j(0) = t_0$. Thus, the singular point is a node in the wide sense. $\square$

REMARK 2.2. We notice that $G(t, x, y)$ has real coefficients. If $\Gamma_G$ has an imaginary singular point, then its conjugate is also a singular point of the same type.

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**3. Classification.** Let $G(t,x,y)$ be an irreducible quartic ternary form. The complex projective curve $\Gamma_G$ is classified into 21 types according to the number of its singular points and their forms (cf. [15, 17]). Each type usually contains infinitely many projectively inequivalent quartic curves. The following list of 21 types contains the type names, and forms of singular points:

[i] type $I_a$: a ramphoid cusp $C_2''$.
[ii] type $I_b$: a simple cusp $C_3$ of order 3.
[iii] type $II_b$: a tacnode-cusp $CO$.
[iv] type $II\,1/2_b$: a cusp $C_2$ and a double cusp $C_2'$.
[v] type $III_a$: a double cusp $C_2'$ and a node $O_2$.
[vi] type $III_b$: a cusp $C_2$ and a tacnode $O_2'$.
[vii] type $III_d$: three cusps $C_2, C_2, C_2$.
[viii] type $III_f$: a cusp $C_2$ and two nodes $O_2, O_2$.
[ix] type $III_h$: a double cusp $C_2'$.
[x] type $III_k$: a cusp $C_2$ and a node $O_2$.
[xi] type $III_m$: a cusp $C_2$.
[xii] type $II\,1/2_a$: an ordinary triple point $O_3$.
[xiii] type $III_c$: a node $O_2$ and a tacnode $O_2'$.
[xiv] type $III_e$: two cusps $C_2, C_2$ and a node $O_2$.
[xv] type $III_g$: three nodes $O_2, O_2, O_2$.
[xvi] type $II_a$: an osnode $O_2''$.
[xvii] type $III_i$: a tacnode $O_2'$.
[xviii] type $III_j$: two nodes $O_2, O_2$.
[xix] type $III_\ell$: two cusps $C_2, C_2$.
[xx] type $III_n$: a node $O_2$.
[xxi] type $III_o$: no singular points.

Let $T$ be a $4\times 4$ matrix. The boundary generating curve of $W(T)$ can be classified by the factorability of the homogeneous ternary polynomial $F(t,x,y)$ associated with $T$. We obtain the following result.

Theorem 3.1. *The boundary generating curve of the numerical range of a $4\times 4$ matrix falls into one of the following cases:*

*Case 1. The vertices of a (possibly degenerate) quadrilateral.*
*Case 2. A non-degenerate ellipse and two points, one or two of these points may be contained in the elliptical disc.*
*Case 3. Two non-degenerate ellipses, these ellipses may take arbitrary relative position.*
*Case 4. The dual curve of an irreducible cubic curve and a point which may be contained in the convex hull of the dual curve.*
*Case 5. The dual of an irreducible quartic curve.*

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Case 4 of Theorem essentially reduces to the boundary generating curve of an irreducible cubic curve which is analyzed in [11] for $3 \times 3$ matrices. The following result classifies the boundary generating curve of Case 5 for an irreducible quartic homogeneous ternary polynomial.

Theorem 3.2. *Let $T$ be a $4 \times 4$ matrix. If the associated homogeneous ternary polynomial $F(t, x, y)$ is irreducible then $\Gamma_F$ is one of the types [xii], [xiii], ..., [xxi]. Conversely, for each type of [xii], [xiii], ..., [xxi], there exists a $4 \times 4$ matrix so that its associated curve $\Gamma_F$ is of the type required.*

*Proof.* Let $T$ be a $4 \times 4$ matrix, and $F(t, x, y)$ be its associated homogeneous ternary polynomial. It is obvious that $F(t, x, y)$ is hyperbolic with respect to (1,0,0) and $F(1, 0, 0) = 1$. Then, by Theorem 3, $\Gamma_F$ has no real cusp. Since $F$ has real coefficients, if $\Gamma_F$ has an imaginary singular point, then its conjugate is also a singular point of the same type. Hence, $\Gamma_F$ falls into one of the types [xii], [xiii], ..., [xxi]. For the converse part, for each type of [xii], [xiii], ..., [xxi], we give a $4 \times 4$ matrix whose associated curve $\Gamma_F$ is of the type required in Section 4. □

**4. Examples.** We provide 10 examples of matrices to complete the proof of Theorem 3.2. There are two images in each example. The first one is the curve $\Gamma_F$ associated with the matrix $T$, and the second one is its dual curve $\Gamma_F^\wedge$ which is the boundary generating curve of $W(T)$. Example 4.5 in this section fulfills the missing link in [14]. This example disproves the conjecture for the non-existence of type [xvi] mentioned in [14]. The types [xii], [xiii], ..., [xxi] are classified into 4 families via the genus $g$ of $\Gamma_F$. The genus $g$ is 0 for [xii], [xiii],[xiv],[xv], [xvi]. The genus $g$ is 1 for [xvii],[xviii],[xix], the genus $g$ is 2 for [xx], and the genus $g$ is 3 for [xxi].

Example 4.1. **[xii] type** $II\,1/2_a$ :

Let $T$ be a nilpotent $4 \times 4$ matrix given by

$$T = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The form $F(t, x, y)$ is computed that

$$16F(t, x, y) = 16t^4 - 24t^2x^2 - 24t^2y^2 + 16tx^3 + 16tx\,y^2 - 3x^4 - 2x^2y^2 + y^4.$$

The curve $\Gamma_F$ has an ordinary triple point $O_3$ at $(t, x, y) = (1, 2, 0)$. The order of the dual curve of $\Gamma_F$ is 6. The matrix $T$ is a typical example of matrices treated in [2]. The real affine part of $\Gamma_F$ is displayed in Figure 1, and its dual curve is shown in Figure 2.

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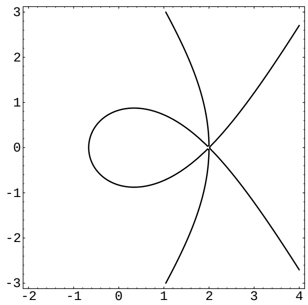


Figure 1. $\Gamma_F$ of Example 4.1.

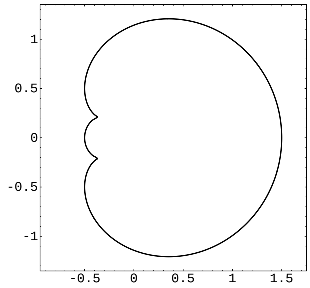


Figure 2. Dual curve of Figure 1.

Example 4.2. **[xiii] type** $III_c$ :

Let $T$ be a $4 \times 4$ real matrix given by

$$T = \begin{pmatrix} 1 & 0 & b & 0 \\ 0 & 1 & 0 & c \\ -b & 0 & -1 & 2 \\ 0 & -c & -2 & -1 \end{pmatrix},$$

where $b = \sqrt{3 + 2\sqrt{2}}$, $c = \sqrt{3 - 2\sqrt{2}}$. Then

$$F(t, x, y) = t^4 - 2t^2x^2 - 10t^2y^2 - 8tx\,y^2 + x^4 + 2x^2y^2 + y^4.$$

The curve $\Gamma_F$ has an ordinary double point $O_2$ at $(t, x, y) = (1, 1, 0)$ and a tacnode $O_2'$ at $(t, x, y) = (1, -1, 0)$. The order of the dual curve of $\Gamma_F$ is 6. The curves $\Gamma_F$ and its dual curve $\Gamma_F^\wedge$ are shown in Figure 3 and Figure 4, respectively.

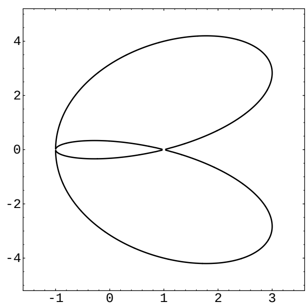


Figure 3. $\Gamma_F$ of Example 4.2.

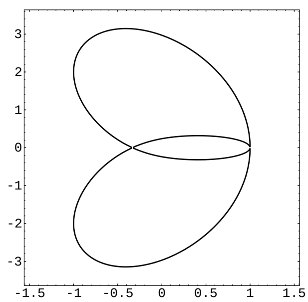


Figure 4. Dual curve of Figure 3.

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EXAMPLE 4.3. **[xiv] type** $III_e$ :

Let $T$ be a $4 \times 4$ real matrix given by

$$T = \begin{pmatrix} -187 & 55\sqrt{3} & 44\sqrt{3} & 0 \\ -55\sqrt{3} & -187 & 22\sqrt{3} & 330 \\ -44\sqrt{3} & -22\sqrt{3} & 11 & 0 \\ 0 & -330 & 0 & 363 \end{pmatrix}.$$

Then the form $F(t,x,y)$ associated with $T$ is given by

$$F(t,x,y) = t^4 - 100914t^2x^2 - 125235t^2y^2 + 11585024tx^3 + 14494590txy^2 \\ +139631217x^4 + 680586885x^2y^2 + 632491200y^4.$$

The curve $\Gamma_F$ has a node $O_2$ at $(t,x,y) = (1, 1/187, 0)$ and two imaginary cusps $C_2$, $C_2$ at $(t,x,y) = (1, 1/37, 10i/407)$ and $(t,x,y) = (1, 1/37, -10i/407)$. The order of the dual curve of $\Gamma_F$ is 4. The dual curve of $\Gamma_F$ is projectively equivalent to $\Gamma_F$. The curve $\Gamma_F$ is known as a limaçon of Pascal (cf. [15]). The real affine part of $\Gamma_F$ and its dual curve are displayed in Figure 5 and Figure 6, respectively.

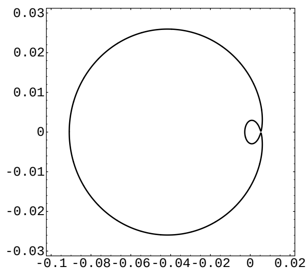


Figure 5. $\Gamma_F$ of Example 4.3.

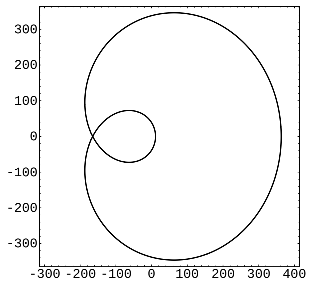


Figure 6. Dual curve of Figure 5.

EXAMPLE 4.4. **[xv] type** $III_g$ :

Let $T$ be a $4 \times 4$ real matrix given by

$$T = \begin{pmatrix} 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 3 \\ 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then we have

$$16F(t,x,y) = 16t^4 - 264t^2(x^2+y^2) + 320tx^3 - 960txy^2 + 225(x^2+y^2)^2.$$

The curve $\Gamma_F$ can be parametrized as

$$x = -\frac{10}{9}(\cos(2s) + \frac{4}{5}\cos s), \quad y = \frac{10}{9}(\sin(2s) - \frac{4}{5}\sin s).$$

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This is a special roulette curve treated in [5]. The curve $\Gamma_F$ has three nodes $O_2, O_2, O_2$ at $(t,x,y) = (1, 2/5, 0)$, $(t,x,y) = (1, -1/5, \sqrt{3}/5)$, $(t,x,y) = (1, -1/5, -\sqrt{3}/5)$. The order of the dual curve of $\Gamma_F$ is 6. The images of the real affine part of $\Gamma_F$ is produced in Figure 7, and its dual curve in Figure 8.

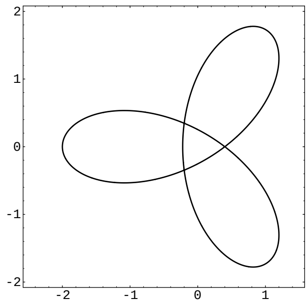


Figure 7. $\Gamma_F$ of Example 4.4.

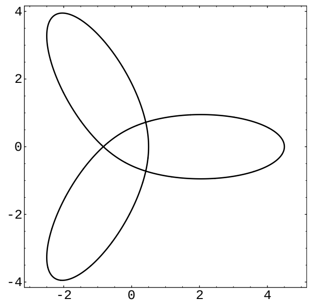


Figure 8. Dual curve of Figure 7.

Example 4.5. **[xvi] type** $II_a$ :

Let $T$ be a $4 \times 4$ real matrix given by

$$T = \begin{pmatrix} -1 & 0 & 0 & 2/\sqrt{7} \\ 0 & -1 & 2/\sqrt{5} & 0 \\ 0 & -2/\sqrt{5} & 3/5 & 2/\sqrt{35} \\ -2/\sqrt{7} & 0 & -2/\sqrt{35} & 1/7 \end{pmatrix}.$$

Then the form $F(t,x,y)$ associated with $T$ is given by

$$35F(t,x,y) = 35t^4 - 44t^3x - 14t^2x^2 - 52t^2y^2 + 20tx^3 + 40txy^2 + 3x^4 + 12x^2y^2 + 16y^4.$$

The curve $\Gamma_F$ has an osnode $O_2''$ at $(t,x,y) = (1,1,0)$. The order of the dual curve of $\Gamma_F$ is 6 The real affine part of $\Gamma_F$ is shown in Figure 9, and the real affine part of the dual curve of $\Gamma_F$ is displayed in Figure 10. The form $F(t,x,y)$ can also be obtained by a deformation of a ternary quartic form $G(x,y,z)$ provided in [17]:

$$G(x,y,z) = x^2z^2 - 2xy^2z + y^4 + y^2z^2 - z^4.$$

The curve $\Gamma_G$ has an osnode at $(x,y,z) = (1,0,0)$. The matrix $T$ is constructed from the ternary form $F(t,x,y)$ by solving some algebraic equations.

Example 4.6. **[xvii] type** $III_i$ :

Let $T$ be a $4 \times 4$ real matrix given by

$$T = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ 0 & -1 & -1 & 2 \end{pmatrix}.$$

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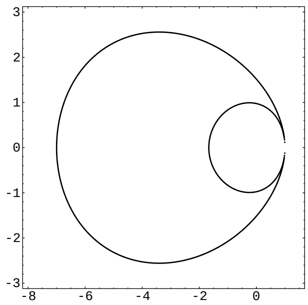

Figure 9. $\Gamma_F$ of Example 4.5.

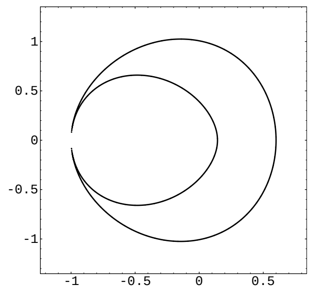

Figure 10. Dual curve of Figure 9.

Then

$$F(t,x,y) = t^4 - 3t^2x^2 - 4t^2y^2 + 2tx^3 + txy^2 + 3x^2y^2 + y^4.$$

The curve $\Gamma_F$ has a tacnode $O_2'$ at $(t,x,y) = (1,1,0)$. The order of the dual curve of $\Gamma_F$ is 8. The images $\Gamma_F$ and its dual curve $\Gamma_F^\wedge$ are displayed in Figure 11 and Figure 12, respectively.

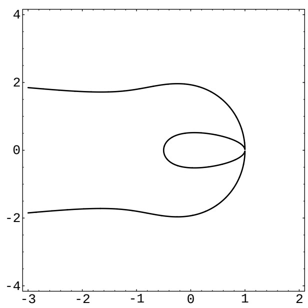

Figure 11. $\Gamma_F$ of Example 4.6.

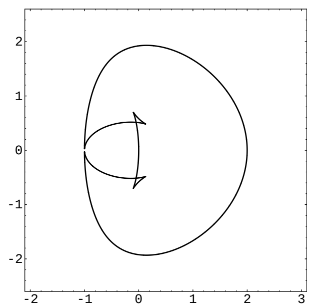

Figure 12. Dual curve of Figure 11.

Example 4.7. **[xviii] type** $III_j$ :

Let $T$ be a $4 \times 4$ real matrix given by

$$T = \begin{pmatrix} 0 & 2e/(1-e^2) & 0 & 2/(1-e^2) \\ 0 & 0 & 2/(1-e^2) & 0 \\ 0 & 0 & 0 & 2e/(1-e^2) \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $\epsilon$ is an arbitrary constant satisfying $0 < \epsilon < 1$. Then the form $F(t,x,y)$ associated with $T$ is given by

$$F(t,x,y) = t^4 - \frac{2(1+\epsilon^2)}{(1-\epsilon^2)^2}(t^2x^2 + t^2y^2) + \frac{1}{(1-\epsilon^2)^2}x^4 + \frac{2(1+\epsilon^4)}{(1-\epsilon^2)^4}x^2y^2 + \frac{(1+\epsilon^2)^2}{(1-\epsilon^2)^4}y^4.$$

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The curve $\Gamma_F$ has two nodes $O_2$, $O_2$ at $(t,x,y) = (1,0,(1-\epsilon^2)/\sqrt{1+\epsilon^2})$, $(t,x,y) = (1,0,-(1-\epsilon^2)/\sqrt{1+\epsilon^2})$. The real affine curve

$$\{(x,y) \in \mathbf{R}^2 : F(1,x,y) = 0\}$$

is symmetric with respect to the $y$-axis and this curve consists of two analytic branches. The right branch is expressed as

$$x = \cos\theta + \epsilon\sqrt{1-\epsilon^2\sin^2\theta},$$
$$y = (1-\epsilon^2)\sin\theta,$$

$0 \le \theta \le 2\pi$. This curve has a historical background, it was treated by Fladt in [7] as one of Kepler's models of planetary orbits. We provide the image of the real affine part of $\Gamma_F$ for $e = 1/5$ in Figure 13, and its dual curve in Figure 14.

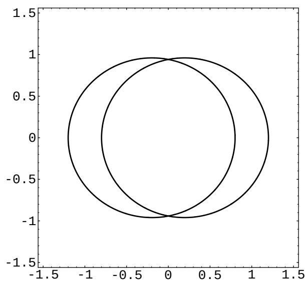


Figure 13. $\Gamma_F$ of Example 4.7.

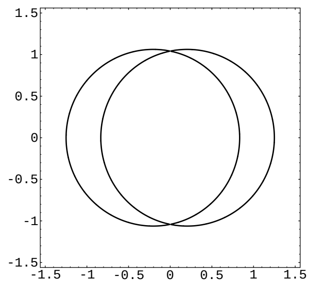


Figure 14. Dual curve of Figure 13.

EXAMPLE 4.8. **[xix] type** $III_\ell$ :

Let $T$ be a $4 \times 4$ real matrix given by

$$T = \begin{pmatrix} 2 & a(k) & 0 & 0 \\ -a(k) & 2(3-k)/(3+3k) & 0 & b(k) \\ 0 & 0 & -2 & c(k) \\ 0 & -b(k) & -c(k) & 2(1-3k)/(3+3k) \end{pmatrix},$$

where $0 < k < 1$, and entries $a(k), b(k), c(k)$ are given by

$$a(k) = \frac{4k\sqrt{3+k}}{\sqrt{3}\sqrt{1+5k+7k^2+3k^3}},$$
$$b(k) = \frac{16\sqrt{k}}{3\sqrt{3+10k+3k^2}},$$
$$c(k) = \frac{4\sqrt{1+5k+7k^2+3k^3}}{\sqrt{3}(1+k)^2\sqrt{3+k}}.$$

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We compute that the associated form $F(t,x,y)$ which is given by

$$9(1+k)^4F(t,x,y) = 9(1+k)^4t^4 + 24(1-k)(k+1)^3t^3x$$
$$-8(k+1)^2(3k^2+14k+3)t^2x^2 - 16(k+1)^2(k^2+8k+1)t^2y^2$$
$$-64(1-k^2)(k^2+4k+1)txy^2 - 16(k+1)^2(3-k)(1-3k)x^4$$
$$-64(k+1)^2(k^2-4k+1)x^2y^2 + 256k^2y^4$$

(cf. [14]). The associated curve $\Gamma_F$ has two imaginary cusps $C_2$, $C_2$ at

$$(t,x,y) = (1, -\frac{1+k}{2(1-k)}, i\frac{1+k}{2(1-k)}), \quad (t,x,y) = (1, -\frac{1+k}{2(1-k)}, -i\frac{1+k}{2(1-k)}).$$

The images of the real affine part of $\Gamma_F$ for $k = 2/3$ is shown in Figure 15, and its real affine part of dual curve $\Gamma_F^\wedge$ in Figure 16.

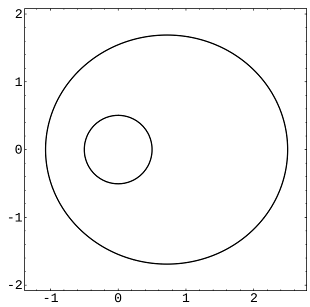

Figure 15. $\Gamma_F$ of Example 4.8.

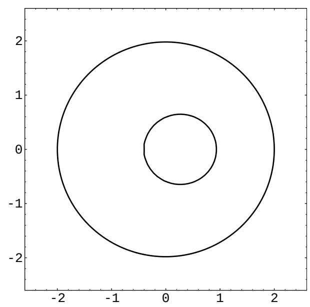

Figure 16. Dual curve of Figure 15.

EXAMPLE 4.9. **[xx] type** $III_n$ :

Let $T$ be a $4\times 4$ tridiagonal matrix given by

$$T = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & -1 & 2 & 0 \\ 0 & -2 & 2 & 1 \\ 0 & 0 & -1 & 3 \end{pmatrix}.$$

Then

$$F(t,x,y) = t^4 + 3t^3x - 3t^2x^2 - 6t^2y^2 - 7tx^3 - 11txy^2 + 6x^4 + 5x^2y^2 + y^4.$$

The curve $\Gamma_F$ has a node $O_2$ at $(t,x,y) = (1,1,0)$. The order of the dual curve of $\Gamma_F$ is 10. The real affine parts of $\Gamma_F$ and $\Gamma_F^\wedge$ are displayed in Figure 17 and Figure 18, respectively.

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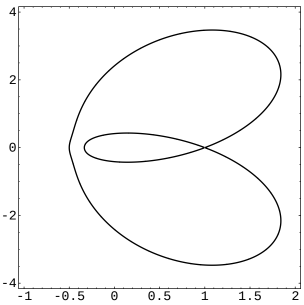

Figure 17. $\Gamma_F$ of Example 4.9.

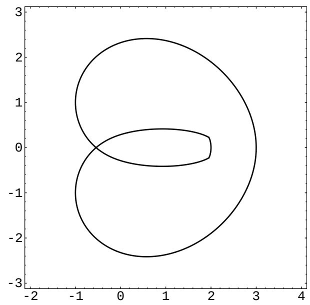

Figure 18. Dual curve of Figure 17.

EXAMPLE 4.10. **[xxi] type** $III_o$ :

Let $T$ be a $4 \times 4$ upper triangular matrix given by

$$T = \begin{pmatrix} 4/5 & 12/25 & -36/125 & -81/625 \\ 0 & 3/5 & 16/25 & 36/125 \\ 0 & 0 & -3/5 & 12/25 \\ 0 & 0 & 0 & -4/5 \end{pmatrix}.$$

Then

$$5^8 \times 2^2 F(t,x,y) = 1562500t^4 - 1973861t^2x^2 - 411361t^2y^2 + 485809x^4 \\ +130993x^2y^2 + 5184y^4.$$

The matrix $T$ is a typical example of matrices treated in [4]. The boundary generating curve of $W(T)$ for this matrix satisfies a Poncelet property with the unit circle(see [4, 13, 19] for Poncelet property). The curve $\Gamma_F$ has no singular points. The order of the dual curve of $\Gamma_F$ is 12. The real affine parts of $\Gamma_F$ and $\Gamma_F^\wedge$ are displayed in Figure 19 and Figure 20, respectively.

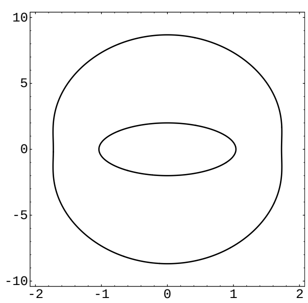

Figure 19. $\Gamma_F$ of Example 4.10.

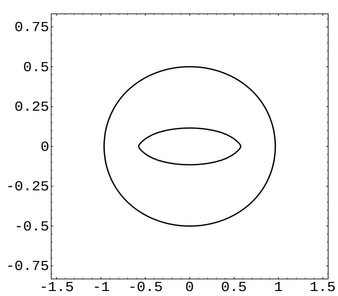

Figure 20. Dual curve of Figure 19.

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