SINGULAR POINTS OF THE TERNARY POLYNOMIALS ASSOCIATED WITH 4-BY-4 MATRICES

MAO-TING CHIEN† AND HIROSHI NAKAZATO‡

Abstract. Let \( T \) be an \( n \times n \) matrix. The numerical range of \( T \) is defined as the set
\[
W(T) = \{ \xi^* T \xi : \xi \in \mathbb{C}^n, \xi^* \xi = 1 \}.
\]
A homogeneous ternary polynomial associated with \( T \) is defined as
\[
F(t, x, y) = \det(tI_n + x(T + T^*)/2 + y(T - T^*)/(2i)).
\]
The numerical range \( W(T) \) is the convex hull of the real affine part of the dual curve of \( F(t, x, y) = 0 \).
We classify the numerical ranges of 4 \times 4 \) matrices according to the singular points of the curve \( F(t, x, y) = 0 \).

Key words. Numerical range, Homogeneous ternary polynomial, Singular point.

AMS subject classifications. 15A60, 14H45.

1. Introduction. The numerical range \( W(T) \) of an \( n \times n \) complex matrix \( T \) was introduced by Toeplitz and defined as the set
\[
W(T) = \{ \xi^* T \xi : \xi \in \mathbb{C}^n, \xi^* \xi = 1 \}.
\]
Kippenhahn [11] (see also [20] for an English translation) characterized this range from a viewpoint of the algebraic curve of the homogeneous ternary polynomial associated with \( T \):
\[
F(t, x, y) = \det(tI_n + x(T + T^*)/2 + y(T - T^*)/(2i)).
\]
Let \( \Gamma_F \) be the algebraic curve of \( F(t, x, y) \), i.e.,
\[
\Gamma_F = \{ [(t, x, y)] \in \mathbb{CP}^2 : F(t, x, y) = 0 \},
\]
where \( [(t, x, y)] \) is the equivalence class containing \( (t, x, y) \in \mathbb{C}^3 \) \( - (0, 0, 0) \) under the relation \( (t_1, x_1, y_1) \sim (t_2, x_2, y_2) \) if \( (t_2, x_2, y_2) = k(t_1, x_1, y_1) \) for some nonzero complex number \( k \). The dual curve \( \Gamma_F^\ast \) of \( \Gamma_F \) is defined by
\[
\Gamma_F^\ast = \{ [(T, X, Y)] \in \mathbb{CP}^2 : T t + X x + Y y = 0 \text{ is a tangent line of } \Gamma_F \}.\]
Kippenhahn [11] showed that \( W(T) \) is the convex hull of \( \Gamma_F \) in the real affine plane. The real affine part of \( \Gamma_F \) is called the boundary generating curve of \( W(A) \). There have been a number of papers on the boundary generating curves of the numerical range; for example, see [2, 3, 6, 10].

A complete classification of the range \( W(T) \) of \( 3 \times 3 \) matrices via the factorability of the homogeneous ternary polynomial \( F(t, x, y) \) is given in [11] (see also [10]). It shows that the shapes of \( W(T) \) fall into four categories, namely,

(i) a (possibly degenerate) triangle, if \( F(t, x, y) \) factors into three real linear factors;
(ii) a convex hull of a non-degenerate elliptical disc and a point which is possibly contained in the elliptical disc, if \( F(t, x, y) \) factors into a real linear factor and an irreducible quadratic factor;
(iii) a smooth boundary curve with a flat portion, if \( F(t, x, y) \) is irreducible and the curve \( \Gamma_F \) has a real node;
(iv) an ovular, if \( F(t, x, y) \) is irreducible and the curve \( \Gamma_F \) has no singular point.

Examples of matrices for each category are also given there.

In this paper, we classify the numerical range of \( 4 \times 4 \) matrices along Kippenhahn’s direction by examining the singular points of the homogeneous ternary polynomial curve \( F(t, x, y) = 0 \). We focus mainly on the case when \( F(t, x, y) \) is irreducible.

2. Singular points. We outline briefly the classification of singular points of an algebraic curve. For references on the classification, see, for instance, [15] or [18].

Let \( G(t, x, y) \) be a complex ternary form of degree \( n \). A point \((t_0, x_0, y_0) \neq (0, 0, 0)\) of \( \Gamma_G \) is called a singular point if

\[
\frac{\partial}{\partial t} G(t_0, x_0, y_0) = \frac{\partial}{\partial x} G(t_0, x_0, y_0) = \frac{\partial}{\partial y} G(t_0, x_0, y_0) = 0.
\]

If \( G(t, x, y) \) is multiplicity free in the polynomial ring \( \mathbb{C}[t, x, y] \) (it happens when \( G(t, x, y) \) is irreducible), then the number of singular points of \( \Gamma_G \subset \mathbb{C}P^2 \) is finite.

Suppose that \((1, x_0, y_0)\) is a singular point of \( \Gamma_G \). We consider the Taylor expansion of \( G(1, x, y) \) around \((1, x_0, y_0)\):

\[
G(1, x_0 + X, y_0 + Y) = \prod_{j=1}^{m} (a_j X + b_j Y) + \sum_{i+j \geq m+1} c_{i,j} X^i Y^j,
\]

(2.1)

where \((a_j, b_j) \neq (0, 0), j = 1, 2, \ldots, m\), are pairs of complex numbers. The number \( m \geq 2 \) is called the multiplicity of the singular point \((1, x_0, y_0)\). The following three frames are used to provide local expression of the curve \( G(1, x, y) = 0 \) near the point \((x_0, y_0)\). The finest frame is the ring \( \mathbb{C}[X]^* \) of fractional formal power series of \( X = x - x_0 \) of non-negative order. Let

\[
G(1, x_0, y) = c_0(y - y_0)^k (y - y_1)^{\ell_1} \cdots (y - y_p)^{\ell_p}
\]
Ternary Polynomials Associated With 4-by-4 Matrices

for some \( k \geq m, c_0 \neq 0 \), where \( y_1, \ldots, y_p \) are roots of the equations of \( G(1, x_0, y) = 0 \) other than \( y_0 \). Then there are unique \( k \) solutions \( y = y_j(x) \) of \( G(1, x, y) = 0 \) satisfying \( y(x_0) = y_0 \) expressed in fractional power series

\[
y_j(X) = y_0 + a_jX^{m_1/n_1} + b_jX^{m_2/n_2} + \cdots,
\]

where \( m_1/n_1 < m_2/n_1 < \cdots \) are positive rational numbers [18 Chapter IV]. If we use fractional power series, we can express the curve \( G(1, x, y) = 0 \) near \( (x_0, y_0) \) as the union of \( k \) parametrized curves. If we assume sufficient small modulus of \( X, Y \), there are unique \( \Gamma \) solutions \( (a, b, c, d) \). The number \( \delta(P) = \frac{1}{2} \left( \dim \mathbb{C}[X, Y]/\langle g, gv \rangle - m + s \right) \).

This number is always a non-negative integer (cf. [9, 13]). Then genus of \( \Gamma \) is given by

\[
g(\Gamma) = (1/2)(n - 1)(n - 2) - \sum_{j=1}^{k} \delta(P_j),
\]
where \( P_1, \ldots, P_k \) are singular points of \( \Gamma_G \). The dual curve of a plane algebraic curve \( \Gamma_G \) has the same genus as the original curve. Table 1 displays some types of singular points (cf. [9, p. 37]).

<table>
<thead>
<tr>
<th>Types of singular points</th>
<th>multiplicity ( m )</th>
<th>number ( s )</th>
<th>( \delta(P) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( O_2 )</td>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>( O'_2 )</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>( O''_2 )</td>
<td>2</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>( C_2 )</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( C'_2 )</td>
<td>2</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>( C''_2 )</td>
<td>2</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>( O_3 )</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>( C_3 )</td>
<td>3</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>( CO )</td>
<td>3</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

A node in the wide sense is a singular point \((x_0, y_0)\) of the curve \( \Gamma_G \) for which every irreducible analytic branch around \((x_0, y_0)\) is expressed as

\[
x_j(u) = x_0 + b_j u + \sum_{k=2}^{\infty} c_{j,k}^i u^k,
\]

\[
y_j(u) = y_0 - a_j u + \sum_{k=2}^{\infty} d_{j,k}^i u^k
\]

for some \((a_j, b_j) \neq (0,0)\) \((j = 1, 2, \ldots, m)\). Such irreducible analytic branch is called linear. The singular points \( O_2, O'_2, O''_2, O_3 \) belongs to this type. If the coefficients satisfy \( a_i b_j \neq a_j b_i \) for \( 1 \leq i < j \leq m \), then the node \((1, x_0, y_0)\) is called an ordinary singular point. A singular point \((1, x_0, y_0)\) of \( \Gamma_G \) of multiplicity \( m \) is an ordinary \( m \)-ple point \( O_m \) if and only if the coefficients \( a_j, b_j \) in (2.1) satisfy \( a_j b_k - b_j a_k \neq 0 \) for \( 1 \leq j < k \leq m \). An irreducible analytic branch other than linear type is called a cusp. The singular points \( C_2, C'_2, C''_2, C_3 \) belongs to this class. A tacnode cusp \( CO \) composed of a linear-type irreducible analytic branch and a cusp \( C_2 \). Since we frequently deal with singular points of multiplicity 2, we pay special attention to this type of singular points. We assume that \((1, x_0, y_0)\) is a singular point of one of the type \( C_2, C'_2, C''_2 \) or \( O'_2, O''_2 \). By changing coordinates, we may assume that the Taylor expansion of \( G(1, x, y) \) around \((1, x_0, y_0)\) satisfies

\[
G(1, x_0, y_0) = a (y - y_0)^2 + \sum_{i+j \geq 3} c_{i,j} (x - x_0)^i (y - y_0)^j, \tag{2.2}
\]

where \( a \neq 0 \). Under this assumption, if \((1, x_0, y_0)\) is of type \( C_2, C'_2 \) or \( C''_2 \), then the
irreducible analytic branch \( g_1(x, y) = 0 \) around the singular point is expressed as
\[
\begin{align*}
x &= x_0 + u^\ell \\
y &= y_0 + a_k e^{2\pi ij k/\ell} u^k + a_{k+1} e^{2\pi ij (k+1)/\ell} u^{k+1} + \ldots
\end{align*}
\]
for some integers \( 2 \leq \ell < k \) and \( j = 0, 1, 2, \ldots, \ell - 1 \), and \( a_k \neq 0 \). For the tacnode cusp \( CO \), one irreducible analytic branch is expressed in this form. If \( (1, x_0, y_0) \) is a singular point of type \( O_2' \) or \( O_3' \), then the local expression of \( \Gamma_G \) in a neighborhood of \( (1, x_0, y_0) \) is given by two irreducible analytic branches expressed as
\[
\begin{align*}
y_1 &= y_0 + \alpha_2 x^2 + \alpha_3 x^3 + \cdots \\
y_2 &= y_0 + \beta_2 x^2 + \beta_3 x^3 + \cdots.
\end{align*}
\]
The node is \( O_2' \) if \( \alpha_2 \neq \beta_2 \), and the node is \( O_3' \) if \( \alpha_2 = \beta_2, \alpha_3 \neq \beta_3 \).

A real homogeneous polynomial \( p(x) = p(x_1, x_2, \ldots, x_m) \) of degree \( n \) is hyperbolic with respect to a vector \( e = (e_1, e_2, \ldots, e_m) \) if \( p(e) \neq 0 \) and, for all vectors \( w \in \mathbb{R}^m \), the univariate polynomial \( t \mapsto p(w - te) \) has all real roots (cf. [11]). The following theorem is mentioned in [11, 20] (in the dual form) without a rigorous proof. We give a proof here relying on an affirmative solution to Lax conjecture [8, 12].

**Theorem 2.1.** Let \( G(t, x, y) \) be a real homogeneous ternary polynomial of degree \( m > 2 \). If \( G(t, x, y) \) is hyperbolic with respect to \( (1, 0, 0) \) then \( \Delta_G \) has no real cusps.

**Proof.** Suppose \( G(t, x, y) \) of degree \( m \) is hyperbolic with respect to \( (1, 0, 0) \) with \( G(1, 0, 0) = 1 \). By Theorem 8 in [12], there exists a pair of \( m \times m \) real symmetric matrices \( S_1, S_2 \) satisfying
\[
G(t, x, y) = \det(tI_m + xS_1 + yS_2).
\]
Then, by Rellich’s result [10] on the perturbation of Hermitian matrices, there exist real-valued analytic functions \( \lambda_1(\theta), \lambda_2(\theta), \ldots, \lambda_m(\theta) \) on the real line with period \( 2\pi \) such that
\[
G(t, -\cos \theta, -\sin \theta) = (t - \lambda_1(\theta))(t - \lambda_2(\theta)) \cdots (t - \lambda_m(\theta)).
\]
Hence, every real singular point \( (t, x, y) \) is expressed as \( (t_0, \cos \theta_0, \sin \theta_0) \) for some real numbers \( t_0, \theta_0 \). By using a rotation of coordinates, we may assume that \( \theta_0 = 0 \). A local expression of \( G(t, x, y) = 0 \) near the point \( (t_0, 1, 0) \) is given by
\[
(t, y) = (\lambda_j(\theta) \sec \theta, \tan \theta)
\]
for indices \( j \) satisfying \( \lambda_j(0) = t_0 \). Thus, the singular point is a node in the wide sense.

**Remark 2.2.** We notice that \( G(t, x, y) \) has real coefficients. If \( \Gamma_G \) has an imaginary singular point, then its conjugate is also a singular point of the same type.
3. Classification. Let \( G(t,x,y) \) be an irreducible quartic ternary form. The complex projective curve \( \Gamma_G \) is classified into 21 types according to the number of its singular points and their forms (cf. [15, 17]). Each type usually contains infinitely many projectively inequivalent quartic curves. The following list of 21 types contains the type names, and forms of singular points:

- Type \( I_0 \): a ramphoid cusp \( C''_2 \).
- Type \( I_1 \): a simple cusp \( C_3 \) of order 3.
- Type \( I_{II} \): a tacnode-cusp \( CO \).
- Type \( II_{1/2} \): a cusp \( C_2 \) and a double cusp \( C'_2 \).
- Type \( II_3 \): a double cusp \( C'_2 \) and a node \( O_2 \).
- Type \( III_6 \): a cusp \( C_2 \) and a tacnode \( O'_2 \).
- Type \( III_7 \): three cusps \( C_2, C_2, C_2 \).
- Type \( III_8 \): a cusp \( C_2 \) and two nodes \( O_2, O_2 \).
- Type \( III_9 \): a double cusp \( C''_2 \).
- Type \( III_{10} \): a cusp \( C_2 \) and a node \( O_2 \).
- Type \( III_{11} \): an ordinary triple point \( O_3 \).
- Type \( III_{12} \): a node \( O_2 \) and a tacnode \( O'_2 \).
- Type \( III_{13} \): two cusps \( C_2, C_2 \) and a node \( O_2 \).
- Type \( III_{14} \): three nodes \( O_2, O_2, O_2 \).
- Type \( III_{15} \): an osnode \( O''_2 \).
- Type \( III_{16} \): a tacnode \( O'_2 \).
- Type \( III_{17} \): two nodes \( O_2, O_2 \).
- Type \( III_{18} \): two cusps \( C_2, C_2 \).
- Type \( III_{19} \): a node \( O_2 \).
- Type \( III_{20} \): no singular points.

Let \( T \) be a \( 4 \times 4 \) matrix. The boundary generating curve of \( W(T) \) can be classified by the factorability of the homogeneous ternary polynomial \( F(t,x,y) \) associated with \( T \). We obtain the following result.

**Theorem 3.1.** The boundary generating curve of the numerical range of a \( 4 \times 4 \) matrix falls into one of the following cases:

- **Case 1.** The vertices of a (possibly degenerate) quadrilateral.
- **Case 2.** A non-degenerate ellipse and two points, one or two of these points may be contained in the elliptical disc.
- **Case 3.** Two non-degenerate ellipses, these ellipses may take arbitrary relative position.
- **Case 4.** The dual curve of an irreducible cubic curve and a point which may be contained in the convex hull of the dual curve.
- **Case 5.** The dual of an irreducible quartic curve.
Case 4 of Theorem essentially reduces to the boundary generating curve of an irreducible cubic curve which is analyzed in [11] for $3 \times 3$ matrices. The following result classifies the boundary generating curve of Case 5 for an irreducible quartic homogeneous ternary polynomial.

**Theorem 3.2.** Let $T$ be a $4 \times 4$ matrix. If the associated homogeneous ternary polynomial $F(t, x, y)$ is irreducible then $\Gamma_F$ is one of the types [xii], [xiii], ..., [xxi]. Conversely, for each type of [xii], [xiii], ..., [xxi], there exists a $4 \times 4$ matrix so that its associated curve $\Gamma_F$ is of the type required.

**Proof.** Let $T$ be a $4 \times 4$ matrix, and $F(t, x, y)$ be its associated homogeneous ternary polynomial. It is obvious that $F(t, x, y)$ is hyperbolic with respect to $(1,0,0)$ and $F(1,0,0) = 1$. Then, by Theorem [3] $\Gamma_F$ has no real cusp. Since $F$ has real coefficients, if $\Gamma_F$ has an imaginary singular point, then its conjugate is also a singular point of the same type. Hence, $\Gamma_F$ falls into one of the types [xii], [xiii], ..., [xxi]. For the converse part, for each type of [xii], [xiii], ..., [xxi], we give a $4 \times 4$ matrix whose associated curve $\Gamma_F$ is of the type required in Section 4.

### 4. Examples

We provide 10 examples of matrices to complete the proof of Theorem 3.2. There are two images in each example. The first one is the curve $\Gamma_F$ associated with the matrix $T$, and the second one is its dual curve $\Gamma_F^\perp$ which is the boundary generating curve of $W(T)$. Example 4.5 in this section fulfills the missing link in [14]. This example disproves the conjecture for the non-existence of type [xvi] mentioned in [14]. The genus $g$ is 0 for [xii], [xiii], ..., [xxi] are classified into 4 families via the genus $g$ of $\Gamma_F$. The genus $g$ is 1 for [xii], [xiii], ..., [xxi], the genus $g$ is 2 for [xx], and the genus $g$ is 3 for [xxi].

**Example 4.1.** [xii] type II $1/2_a$:

Let $T$ be a nilpotent $4 \times 4$ matrix given by

$$T = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$ 

The form $F(t, x, y)$ is computed that

$$16F(t, x, y) = 16t^4 - 24t^2x^2 - 24t^2y^2 + 16txy^2 + 16tx^2y^2 - 3x^4 - 2x^2y^2 + y^4.$$ 

The curve $\Gamma_F$ has an ordinary triple point $O_3$ at $(t, x, y) = (1, 2, 0)$. The order of the dual curve of $\Gamma_F$ is 6. The matrix $T$ is a typical example of matrices treated in [2]. The real affine part of $\Gamma_F$ is displayed in Figure 1, and its dual curve is shown in Figure 2.
Example 4.2. \[\text{type III}_c:\]

Let \( T \) be a \( 4 \times 4 \) real matrix given by

\[
T = \begin{pmatrix}
1 & 0 & b & 0 \\
0 & 1 & 0 & c \\
-b & 0 & -1 & 2 \\
0 & -c & -2 & -1
\end{pmatrix},
\]

where \( b = \sqrt{3 + 2\sqrt{2}}, \) \( c = \sqrt{3 - 2\sqrt{2}}. \) Then

\[
F(t, x, y) = t^4 - 2t^2x^2 - 10t^2y^2 - 8txy^2 + x^4 + 2x^2y^2 + y^4.
\]

The curve \( \Gamma_F \) has an ordinary double point \( O_2 \) at \( (t, x, y) = (1, 1, 0) \) and a tacnode \( O'_2 \) at \( (t, x, y) = (1, -1, 0). \) The order of the dual curve of \( \Gamma_F \) is 6. The curves \( \Gamma_F \) and its dual curve \( \Gamma_F^\wedge \) are shown in Figure 3 and Figure 4, respectively.
Example 4.3. [xiv] type $III_e$:

Let $T$ be a $4 \times 4$ real matrix given by

$$T = \begin{pmatrix} -187 & 55\sqrt{3} & 44\sqrt{3} & 0 \\ -55\sqrt{3} & -187 & 22\sqrt{3} & 330 \\ -44\sqrt{3} & -22\sqrt{3} & 11 & 0 \\ 0 & -330 & 0 & 363 \end{pmatrix}.$$  

Then the form $F(t, x, y)$ associated with $T$ is given by

$$F(t, x, y) = t^4 - 100914t^2x^2 - 125235t^2y^2 + 11585024tx^3 + 14494590txy^2 + 139631217x^4 + 680586885x^2y^2 + 632491200y^4.$$  

The curve $\Gamma_F$ has a node $O_2$ at $(t, x, y) = (1, 1/187, 0)$ and two imaginary cusps $C_2$ at $(t, x, y) = (1, 1/37, 10i/407)$ and $(t, x, y) = (1, 1/37, -10i/407)$. The order of the dual curve of $\Gamma_F$ is 4. The dual curve of $\Gamma_F$ is projectively equivalent to $\Gamma_F$. The curve $\Gamma_F$ is known as a limaçon of Pascal (cf. [15]). The real affine part of $\Gamma_F$ and its dual curve are displayed in Figure 5 and Figure 6, respectively.

![Figure 5](image1.png)  
![Figure 6](image2.png)

Figure 5. $\Gamma_F$ of Example 4.3.  
Figure 6. Dual curve of Figure 5.

Example 4.4. [xv] type $III_g$:

Let $T$ be a $4 \times 4$ real matrix given by

$$T = \begin{pmatrix} 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 3 \\ 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$  

Then we have

$$16F(t, x, y) = 16t^4 - 264t^2(x^2 + y^2) + 320tx^3 - 960txy^2 + 225(x^2 + y^2)^2.$$  

The curve $\Gamma_F$ can be parametrized as

$$x = -\frac{10}{9}(\cos(2s) + \frac{4}{5}\cos s), \quad y = \frac{10}{9}(\sin(2s) - \frac{4}{5}\sin s).$$
This is a special roulette curve treated in [5]. The curve $\Gamma_F$ has three nodes $O_2, O_2, O_2$ at $(t, x, y) = (1, 2/5, 0)$, $(t, x, y) = (1, -1/5, \sqrt{3}/5)$, $(t, x, y) = (1, -1/5, -\sqrt{3}/5)$. The order of the dual curve of $\Gamma_F$ is 6. The images of the real affine part of $\Gamma_F$ is produced in Figure 7, and its dual curve in Figure 8.

Example 4.5. \textbf{[xvi] type II\textsubscript{a}}:

Let $T$ be a $4 \times 4$ real matrix given by

$$
T = \begin{pmatrix}
-1 & 0 & 0 & 2/\sqrt{7} \\
0 & -1 & 2/\sqrt{5} & 0 \\
-2/\sqrt{7} & 0 & -2/\sqrt{35} & 1/7 \\
-2/\sqrt{5} & 3/5 & 2/\sqrt{35} & 0
\end{pmatrix}.
$$

Then the form $F(t, x, y)$ associated with $T$ is given by

$$
35F(t, x, y) = 35t^4 - 44t^3x - 14t^2x^2 - 52tx^3 + 20txy^2 + 40tx^2y + 3x^4 + 12x^2y^2 + 16y^4.
$$

The curve $\Gamma_F$ has an osnode $O_2'$ at $(t, x, y) = (1, 1, 0)$. The order of the dual curve of $\Gamma_F$ is 6 The real affine part of $\Gamma_F$ is shown in Figure 9, and the real affine part of the dual curve of $\Gamma_F$ is displayed in Figure 10. The form $F(t, x, y)$ can also be obtained by a deformation of a ternary quartic form $G(x, y, z)$ provided in [17]:

$$
G(x, y, z) = x^2z^2 - 2xy^2z + y^4 + y^2z^2 - z^4.
$$

The curve $\Gamma_G$ has an osnode at $(x, y, z) = (1, 0, 0)$. The matrix $T$ is constructed from the ternary form $F(t, x, y)$ by solving some algebraic equations.

Example 4.6. \textbf{[xvii] type III\textsubscript{i}}:

Let $T$ be a $4 \times 4$ real matrix given by

$$
T = \begin{pmatrix}
-1 & 0 & 1 & 0 \\
0 & -1 & 1 & 1 \\
-1 & -1 & 0 & 1 \\
0 & -1 & -1 & 2
\end{pmatrix}.
$$

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Then

$$F(t, x, y) = t^4 - 3t^2x^2 - 4t^2y^2 + 2tx^3 + txy^2 + 3x^2y^2 + y^4.$$  

The curve $\Gamma_F$ has a tacnode $O_2'$ at $(t, x, y) = (1, 1, 0)$. The order of the dual curve of $\Gamma_F$ is 8. The images $\Gamma_F$ and its dual curve $\Gamma_F^\wedge$ are displayed in Figure 11 and Figure 12, respectively.

**Example 4.7.** \[\text{xviii}] \text{type III}_j:

Let $T$ be a $4 \times 4$ real matrix given by

$$T = \begin{pmatrix} 0 & 2e/(1 - \epsilon^2) & 0 & 2/(1 - \epsilon^2) \\ 0 & 0 & 2/(1 - \epsilon^2) & 0 \\ 0 & 0 & 0 & 2e/(1 - \epsilon^2) \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $\epsilon$ is an arbitrary constant satisfying $0 < \epsilon < 1$. Then the form $F(t, x, y)$ associated with $T$ is given by

$$F(t, x, y) = t^4 - \frac{2(1 + \epsilon^2)}{(1 - \epsilon^2)^2}(t^2x^2 + t^2y^2) + \frac{1}{(1 - \epsilon^2)^2}x^4 + \frac{2(1 + \epsilon^4)}{(1 - \epsilon^2)^4}x^2y^2 + \frac{(1 + \epsilon^2)^2}{(1 - \epsilon^2)^4}y^4.$$
The curve $\Gamma_F$ has two nodes $O_2, O_2$ at $(t, x, y) = (1, 0, (1 - \epsilon^2)/\sqrt{1 + \epsilon^2})$, $(t, x, y) = (1, 0, - (1 - \epsilon^2)/\sqrt{1 + \epsilon^2})$. The real affine curve

$$\{(x, y) \in \mathbb{R}^2 : F(1, x, y) = 0\}$$

is symmetric with respect to the $y$-axis and this curve consists of two analytic branches. The right branch is expressed as

$$x = \cos \theta + \epsilon \sqrt{1 - \epsilon^2} \sin^2 \theta,$$

$$y = (1 - \epsilon^2) \sin \theta,$$

$0 \leq \theta \leq 2\pi$. This curve has a historical background, it was treated by Fladt in [7] as one of Kepler’s models of planetary orbits. We provide the image of the real affine part of $\Gamma_F$ for $\epsilon = 1/5$ in Figure 13, and its dual curve in Figure 14.

**Example 4.8.** [xix] type III$_L$:

Let $T$ be a $4 \times 4$ real matrix given by

$$T = \begin{pmatrix}
2 & a(k) & 0 & 0 \\
-a(k) & 2(3 - k)/(3 + 3k) & 0 & b(k) \\
0 & 0 & -2 & c(k) \\
0 & -b(k) & -c(k) & 2(1 - 3k)/(3 + 3k)
\end{pmatrix},$$

where $0 < k < 1$, and entries $a(k), b(k), c(k)$ are given by

$$a(k) = \frac{4k\sqrt{3 + k}}{3\sqrt{1 + 5k + 7k^2 + 3k^3}},$$

$$b(k) = \frac{16\sqrt{k}}{3\sqrt{3 + 10k + 3k^2}},$$

$$c(k) = \frac{4\sqrt{1 + 5k + 7k^2 + 3k^3}}{\sqrt{3}(1 + k)^2\sqrt{3 + k}}.$$
We compute that the associated form $F(t,x,y)$ which is given by

$$9(1 + k)^4 F(t, x, y) = 9(1 + k)^4 t^4 + 24(1 - k)(k + 1)^3 t^3 x$$
$$- 8(k + 1)^2(3k^2 + 14k + 3)t^2 x^2 - 16(k + 1)^2(k^2 + 8k + 1)t^2 y^2$$
$$- 64(1 - k^2)(k^2 + 4k + 1)txy - 16(k + 1)^2(3 - k)(1 - 3k)x^4$$
$$- 64(k + 1)^2(k^2 - 4k + 1)x^2 y^2 + 256k^2 y^4$$

(cf. [14]). The associated curve $\Gamma_F$ has two imaginary cusps $C_2, C_2$ at

$$(t, x, y) = (1, -\frac{1 + k}{2(1 - k)}, i\frac{1 + k}{2(1 - k)}), \quad (t, x, y) = (1, -i\frac{1 + k}{2(1 - k)}, -\frac{1 + k}{2(1 - k)}).$$

The images of the real affine part of $\Gamma_F$ for $k = 2/3$ is shown in Figure 15, and its real affine part of dual curve $\Gamma_F^\wedge$ in Figure 16.

**Example 4.9.** [xx] type $III_n$:

Let $T$ be a $4 \times 4$ tridiagonal matrix given by

$$T = \begin{pmatrix}
-1 & 1 & 0 & 0 \\
-1 & -1 & 2 & 0 \\
0 & -2 & 2 & 1 \\
0 & 0 & -1 & 3
\end{pmatrix}.$$

Then

$$F(t, x, y) = t^4 + 3t^3 x - 3t^2 x^2 - 6t^2 y^2 - 7tx^3 - 11txy^2 + 6x^4 + 5x^2 y^2 + y^4.$$

The curve $\Gamma_F$ has a node $O_2$ at $(t, x, y) = (1, 1, 0)$. The order of the dual curve of $\Gamma_F$ is 10. The real affine parts of $\Gamma_F$ and $\Gamma_F^\wedge$ are displayed in Figure 17 and Figure 18, respectively.
Example 4.10. [xxi] type $III_a$:

Let $T$ be a $4 \times 4$ upper triangular matrix given by

$$T = \begin{pmatrix}
\frac{4}{5} & \frac{12}{25} & -\frac{36}{125} & -\frac{81}{625} \\
0 & \frac{3}{5} & \frac{16}{25} & \frac{36}{125} \\
0 & 0 & -\frac{3}{5} & \frac{12}{25} \\
0 & 0 & 0 & -\frac{4}{5}
\end{pmatrix}.$$  

Then

$$5^8 \times 2^2 F(t, x, y) = 1562500t^4 - 1973861t^2x^2 - 411361t^2y^2 + 485809x^4$$

$$+ 130993x^2y^2 + 5184y^4.$$  

The matrix $T$ is a typical example of matrices treated in [3]. The boundary generating curve of $W(T)$ for this matrix satisfies a Poncelet property with the unit circle (see [3] [13] [19] for Poncelet property). The curve $\Gamma_F$ has no singular points. The order of the dual curve of $\Gamma_F$ is 12. The real affine parts of $\Gamma_F$ and $\Gamma_F^\wedge$ are displayed in Figure 19 and Figure 20, respectively.
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