Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 731-754, August 2012

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ALLOW PROBLEMS CONCERNING SPECTRAL PROPERTIES OF PATTERNS*

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Abstract. Let $S \subseteq \{0, +, -, +_0, -_0, *, \#\}$ be a set of symbols, where $+$ (resp., $-$, $+_0$ and $-_0$) denotes a positive (resp., negative, nonnegative and nonpositive) real number, and $*$ (resp., $\#$) denotes a nonzero (resp., arbitrary) real number. An S-pattern is a matrix with entries in S. In particular, a $\{0, +, -\}$-pattern is a sign pattern and a $\{0, *\}$-pattern is a zero-nonzero pattern. This paper extends the following problems concerning spectral properties of sign patterns and zero-nonzero patterns to S-patterns: spectrally arbitrary patterns; inertially arbitrary patterns; refined inertially arbitrary patterns; potentially nilpotent patterns; potentially stable patterns; and potentially purely imaginary patterns. Relationships between these classes of S-patterns are given and techniques that appear in the literature are extended. Some interesting examples and properties of patterns when $\#$ belongs to the symbol set are highlighted. For example, it is shown that there is a $\{0, +, \#\}$-pattern of order n that is spectrally arbitrary with exactly $2n-1$ nonzero entries. Finally, a modified version of the nilpotent-Jacobian method is presented that can be used to show a pattern is inertially arbitrary.

Key words. Generalized sign patterns, Inertially arbitrary, Potentially nilpotent, Potentially purely imaginary, Potentially stable, Refined inertially arbitrary, Sign patterns, Spectrally arbitrary, Zero-nonzero patterns.

AMS subject classifications. 15A18, 15A29, 05C50.

1. Introduction: Definitions, background and motivation.

1.1. Definitions. Throughout this paper, we assume all matrices are square and use the notation $\mathbb{S}$ to denote the set of symbols $\mathbb{S} = \{0, +, -, +_0, -_0, *, \#\}$, where $+$ (resp., $-$) represents a positive (resp., negative) real number, $+_0$ (resp., $-_0$) represents a nonnegative (resp., nonpositive) real number, and $*$ (resp., $\#$) represents a nonzero (resp., arbitrary) real number. For a symbol set $S \subseteq \mathbb{S}$, an *S-pattern* is a matrix with entries in S. In particular, a $\{0, +, -\}$-pattern is a *sign pattern*, a $\{0, *\}$-pattern is a *zero-nonzero pattern*, a $\{+, -\}$-pattern is a *full sign pattern*, a $\{0, +\}$-pattern is a *nonnegative sign pattern*, a $\{+\}$-pattern is a *positive sign pattern*, and a $\{0, +, -, \#\}$-pattern is a *generalized sign pattern*. We use the term *pattern* when statements hold for all S-patterns with $S \subseteq \mathbb{S}$.

*Received by the editors on January 23, 2012. Accepted for publication on August 18, 2012. Handling Editor: Leslie Hogben.

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Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 731-754, August 2012

There are many well known spectral properties of nonnegative and positive sign patterns, for example, the Perron-Frobenius theory. Generalized sign patterns were defined in [22] but were also studied in earlier papers. The symbol # was introduced in [16] to denote the ambiguous sign and provides a way to deal with powers of sign patterns. We define the multiplication and addition of symbols using the following tables:

$\cdot$	0	$+$	$-$	$+_0$	$-_0$	$*$	$\#$
0	0	0	0	0	0	0	0
$+$	0	$+$	$-$	$+_0$	$-_0$	$*$	$\#$
$-$	0	$-$	$+$	$-_0$	$+_0$	$*$	$\#$
$+_0$	0	$+_0$	$-_0$	$+_0$	$-_0$	$\#$	$\#$
$-_0$	0	$-_0$	$+_0$	$-_0$	$+_0$	$\#$	$\#$
$*$	0	$*$	$*$	$\#$	$\#$	$*$	$\#$
$\#$	0	$\#$	$\#$	$\#$	$\#$	$\#$	$\#$

$+$	0	$+$	$-$	$+_0$	$-_0$	$*$	$\#$
0	0	$+$	$-$	$+_0$	$-_0$	$*$	$\#$
$+$	$+$	$+$	$\#$	$+$	$\#$	$\#$	$\#$
$-$	$-$	$\#$	$-$	$\#$	$-$	$\#$	$\#$
$+_0$	$+_0$	$+$	$\#$	$+_0$	$\#$	$\#$	$\#$
$-_0$	$-_0$	$\#$	$-$	$\#$	$-_0$	$\#$	$\#$
$*$	$*$	$\#$	$\#$	$\#$	$\#$	$\#$	$\#$
$\#$	$\#$	$\#$	$\#$	$\#$	$\#$	$\#$	$\#$

Both multiplication and addition defined in this way are commutative and associative, and multiplication is distributive with respect to addition. The addition and multiplication of patterns are defined in the usual way, so that the sum and product of two patterns are $\mathbb{S}$-patterns. The *negative* of a pattern $\mathcal{A}$, denoted by $-\mathcal{A}$, is obtained by multiplying each entry of $\mathcal{A}$ by $-$.

The *sign* of a real number a, denoted by $sgn(a)$, is defined as

$$sgn(a) = \begin{cases} + & \text{if } a > 0, \\ - & \text{if } a < 0, \text{ and} \\ 0 & \text{if } a = 0. \end{cases}$$

The *sign pattern of a real matrix* A, denoted by $sgn(A)$, is the $\{0, +, -\}$-pattern obtained from A by replacing each entry by its sign. The *qualitative class* of a pattern $\mathcal{A} = [\alpha_{ij}]$, denoted by $Q(\mathcal{A})$, is the set of all real matrices $A = [a_{ij}]$ such that:

(i) if $\alpha_{ij} = *$, then $a_{ij} \neq 0$,
(ii) if $\alpha_{ij} = +_0$, then $a_{ij} \geq 0$,
(iii) if $\alpha_{ij} = -_0$, then $a_{ij} \leq 0$,
(iv) if $\alpha_{ij} = +$, then $a_{ij} > 0$,
(v) if $\alpha_{ij} = -$, then $a_{ij} < 0$, and
(vi) if $\alpha_{ij} = 0$, then $a_{ij} = 0$.

If $\alpha_{ij} = \#$, then a_{ij} is regarded as being completely free. If $A \in Q(\mathcal{A})$, then A is a *realization* of $\mathcal{A}$. Furthermore, if $A \in Q(\mathcal{A})$ is a nilpotent matrix, that is, A has all of its eigenvalues equal to 0, then A is said to be a *nilpotent realization* of $\mathcal{A}$.

The *inertia* of a matrix A, denoted by $i(A)$, is the ordered triple (n_+, n_-, n_0) of nonnegative integers where n_+ (resp., n_- and n_0) is the number of eigenvalues of A

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with positive (resp., negative and zero) real part. The *inertia* of a pattern $\mathcal{A}$ is the set $i(\mathcal{A}) = \{i(A) : A \in Q(\mathcal{A})\}$. A pattern $\mathcal{A}$ of order n is an *inertially arbitrary pattern*, denoted by IAP, if $i(\mathcal{A})$ contains every ordered triple (n_+, n_-, n_0) of nonnegative integers with $n_+ + n_- + n_0 = n$.

The *refined inertia* of a matrix A, denoted by $ri(A)$, is the ordered 4-tuple $(n_+, n_-, n_z, 2n_p)$ of nonnegative integers where n_+ (resp., n_-) is the number of eigenvalues of A with positive (resp., negative) real part, and n_z (resp., $2n_p$) is the number of zero (resp., nonzero pure imaginary) eigenvalues of A. In this context, the inertia of a matrix is then $i(A) = (n_+, n_-, n_z + 2n_p)$. The *refined inertia* of a pattern $\mathcal{A}$ is the set $ri(\mathcal{A}) = \{ri(A) : A \in Q(\mathcal{A})\}$. A pattern $\mathcal{A}$ of order n is a *refined inertially arbitrary pattern*, denoted by rIAP, if $ri(\mathcal{A})$ contains every ordered 4-tuple $(n_+, n_-, n_z, 2n_p)$ of nonnegative integers with $n_+ + n_- + n_z + 2n_p = n$.

The *spectrum* of a matrix A, denoted by $\sigma(A)$, is the multiset of eigenvalues of A. The *spectrum* of a pattern $\mathcal{A}$ is the set $\sigma(\mathcal{A}) = \{\sigma(A) : A \in Q(\mathcal{A})\}$. A pattern $\mathcal{A}$ of order n is a *spectrally arbitrary pattern*, denoted by SAP, if every multiset of n complex numbers, closed under complex conjugation, is in the spectrum of $\mathcal{A}$.

In the literature, for example, see [3], properties that matrix patterns may allow or require have been defined in the following manner: a pattern $\mathcal{A}$ *allows* (resp., *requires*) a property $\mathbf{P}$ if some (resp., every) matrix $A \in Q(\mathcal{A})$ has property $\mathbf{P}$. In this paper, we consider the problems of a pattern being IAP (resp., rIAP or SAP) to be properties a pattern may allow in the sense that the pattern must allow all possible inertias (resp., refined inertias or spectra closed under complex conjugation). Note that these three spectral properties do not directly satisfy the above definition of allows. To formalize this, we call a property $\mathbf{P}$ *simple* when $\mathcal{A}$ allows property $\mathbf{P}$ if and only if there is a matrix $A \in Q(\mathcal{A})$ that has property $\mathbf{P}$. Suppose an arbitrary property $\mathbf{P}$ can be written as a union of simple properties, that is, $\mathbf{P} = \cup_i \mathbf{P}_i$, where each $\mathbf{P}_i$ is a simple property. Then we say $\mathcal{A}$ *allows* property $\mathbf{P}$, if for every simple property $\mathbf{P}_i$, $\mathcal{A}$ allows $\mathbf{P}_i$. With this definition of allows, the property of $\mathcal{A}$ being IAP (resp., rIAP) means that $\mathcal{A}$ allows every simple property $\mathbf{P}_i$, where each $\mathbf{P}_i$ represents a distinct inertia (resp., refined inertia). Moreover, if $\mathcal{A}$ is SAP, then for each property $\mathbf{P}_i$ representing a distinct spectrum closed under complex conjugation, $\mathcal{A}$ allows $\mathbf{P}_i$, that is, $\mathcal{A}$ allows each spectrum closed under complex conjugation.

In addition to IAP, rIAP and SAP, we give names to three more spectral problems that are simple properties a pattern may allow. In particular, a pattern $\mathcal{A}$ of order n is a *potentially stable pattern*, denoted by PS, if $\mathcal{A}$ allows the inertia $(0, n, 0)$. A pattern $\mathcal{A}$ of order n is a *potentially purely imaginary pattern*, denoted by PPI, if $\mathcal{A}$ allows the inertia $(0, 0, n)$. Note that we follow the convention that 0 is a purely imaginary number. A pattern $\mathcal{A}$ is a *potentially nilpotent pattern*, denoted by PN, if $\mathcal{A}$ allows a nilpotent realization. Note that a pattern $\mathcal{A}$ is PN if and only if $(0, 0, n, 0) \in ri(\mathcal{A})$.

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Associated with a pattern $\mathcal{A} = [\alpha_{ij}]$ of order n is a *labelled directed graph*, denoted by $\mathcal{G}(\mathcal{A})$, with vertices $1, 2, \ldots, n$, and an arc from i to j with label α_{ij} if and only if $\alpha_{ij} \neq 0$. When $\alpha_{ij} \neq 0$, we say the *sign* of the arc from i to j is α_{ij}. For $q \geq 2$, if $i_1, i_2, \ldots, i_q$ are distinct integers from $\{1, 2, \ldots, n\}$ and the product $\alpha_{i_1,i_2}\alpha_{i_2,i_3} \cdots \alpha_{i_{q-1},i_q}\alpha_{i_q,i_1} \neq 0$, then we say $\mathcal{G}(\mathcal{A})$ has a *q-cycle* on vertices $i_1, i_2, \ldots, i_q$. The *sign of a q-cycle* is the product of the signs of its arcs. For $q \geq 2$, we say $\mathcal{G}(\mathcal{A})$ *allows a positive (*resp.*, negative) q-cycle* if $\mathcal{G}(\mathcal{A})$ has a q-cycle whose sign belongs to the set $\{+, +_0, *, \#\}$ (resp., $\{-, -_0, *, \#\}$). If $\alpha_{ii} \neq 0$, then $\mathcal{G}(\mathcal{A})$ has a 1-*cycle* (or *loop*) at vertex i with *sign* α_{ii}. We say $\mathcal{G}(\mathcal{A})$ *allows a positive (*resp.*, negative) loop* if $\mathcal{G}(\mathcal{A})$ has a loop whose sign belongs to the set $\{+, +_0, *, \#\}$ (resp., $\{-, -_0, *, \#\}$). Further, we say $\mathcal{G}(\mathcal{A})$ *allows two oppositely signed loops*, if there is a pair of loops in $\mathcal{G}(\mathcal{A})$, one of which has sign belonging to $\{+, +_0, *, \#\}$ and the other with sign belonging to $\{-, -_0, *, \#\}$.

A pattern $\mathcal{A}$ is said to be PN^+ if $\mathcal{A}$ is potentially nilpotent and $\mathcal{G}(\mathcal{A})$ allows a positive loop, allows a negative loop, and allows a negative 2-cycle. Similarly, a pattern $\mathcal{A}$ is said to be PPI^+ if $\mathcal{A}$ is potentially purely imaginary and $\mathcal{G}(\mathcal{A})$ allows a positive loop, allows a negative loop, and allows a negative 2-cycle. The property PN^+ was defined in [14] for sign patterns but explicitly required that $\mathcal{G}(\mathcal{A})$ has at least two loops. In our definition, it is possible that $\mathcal{A}$ can be PN^+ (or PPI^+) with the property that $\mathcal{G}(\mathcal{A})$ has exactly one loop with sign $\#$. The definition of PN^+ from [14] is motivated in part by Theorem 2.4 below, which is a generalization of [5, Lemma 5.1]. The definition of PN^+ in this paper is motivated by Corollary 2.8 and is equivalent to the definition in [14] for $\{0, +, -, *\}$-patterns.

Let $\mathcal{A}$ and $\mathcal{B}$ be two patterns of order n. We say $\mathcal{A}$ is a *subpattern* of $\mathcal{B}$, if $\mathcal{A}$ can be obtained from $\mathcal{B}$ by replacing some (or possibly none) of the nonzero symbols in $\mathcal{B}$ with 0. If $\mathcal{A}$ is a subpattern of $\mathcal{B}$, then we also say $\mathcal{B}$ is a *superpattern* of $\mathcal{A}$. If $\mathcal{A}$ is a subpattern (resp., superpattern) of $\mathcal{B}$ and $\mathcal{A} \neq \mathcal{B}$, then we say $\mathcal{A}$ is a *proper* subpattern (resp., superpattern) of $\mathcal{B}$.

A *permutation pattern* $\mathcal{P}$ is a $\{0, +\}$-pattern, where the symbol $+$ occurs precisely once in each row and once in each column. A *permutation similarity* of a pattern $\mathcal{A}$ is a product of the form $\mathcal{P}^T\mathcal{A}\mathcal{P}$, where $\mathcal{P}$ is a permutation pattern. A *signature pattern* $\mathcal{S}$ is a $\{0, +, -\}$-pattern, each of whose diagonal entries belong to the set $\{+, -\}$ and every off-diagonal entry is 0. A *signature similarity* of a pattern $\mathcal{A}$ is a product of the form $\mathcal{S}\mathcal{A}\mathcal{S}$, where $\mathcal{S}$ is a signature pattern.

For any matrix A, we say A is *reducible* if there is a permutation matrix P such that

$$P^TAP = \begin{bmatrix} A_1 & A_2 \\ \mathbf{0} & A_3 \end{bmatrix},$$

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where A_1 and A_3 are square matrices of order at least one and $\mathbf{0}$ is the zero matrix. If A is not reducible, then we call A *irreducible*. We can treat reducibility and irreducibility as properties that a pattern may allow or require. In particular, we say a pattern $\mathcal{A}$ *allows reducibility* (resp., *allows irreducibility*), if there is a matrix $A \in Q(\mathcal{A})$ such that A is reducible (resp., irreducible). Additionally, we say $\mathcal{A}$ *requires reducibility* (resp., *requires irreducibility*), if every matrix $A \in Q(\mathcal{A})$ is reducible (resp., irreducible). Note that it is possible for a pattern to allow both reducibility and irreducibility.

The spectrum of a pattern is preserved under transposition, permutation similarity and signature similarity. If $\mathcal{A}$ allows property $\mathbf{P}$ for one of SAP, rIAP, IAP, PPI$^+$, PPI, PN$^+$ or PN, then $-\mathcal{A}$ allows property $\mathbf{P}$. With the exception of potential stability, when dealing with spectral properties of patterns described in this paper, we say two patterns $\mathcal{A}$ and $\mathcal{B}$ are *equivalent* if one can be obtained from the other by any combination of negation, transposition, permutation similarity and signature similarity. For the *equivalence* of potentially stable patterns, the operation of negation is not permitted, but any combination of transposition, permutation similarity and signature similarity is allowed.

We now consider another operation on patterns. Let $\mathcal{A} = [\alpha_{ij}]$ and $\mathcal{B} = [\beta_{ij}]$ be two patterns of order n. We say $\mathcal{B}$ is a *relaxation* of $\mathcal{A}$ if the following hold:

(i) if $\alpha_{ij} = \#$, then $\beta_{ij} = \#$,
(ii) if $\alpha_{ij} = *$, then $\beta_{ij} \in \{*, \#\}$,
(iii) if $\alpha_{ij} = +_0$, then $\beta_{ij} \in \{+_0, \#\}$,
(iv) if $\alpha_{ij} = -_0$, then $\beta_{ij} \in \{-_0, \#\}$,
(v) if $\alpha_{ij} = +$, then $\beta_{ij} \in \{+, +_0, *, \#\}$,
(vi) if $\alpha_{ij} = -$, then $\beta_{ij} \in \{-, -_0, *, \#\}$, and
(vii) if $\alpha_{ij} = 0$, then $\beta_{ij} \in \{0, +_0, -_0, \#\}$.

If $\mathcal{B}$ is a relaxation of $\mathcal{A}$, then we also say $\mathcal{A}$ is a *signing* of $\mathcal{B}$. Equivalently, $\mathcal{B}$ is a relaxation (resp., signing) of $\mathcal{A}$ if and only if $Q(\mathcal{A}) \subseteq Q(\mathcal{B})$ (resp., $Q(\mathcal{B}) \subseteq Q(\mathcal{A})$). If $\mathcal{A}$ is a sign pattern that is a signing of $\mathcal{B}$, then we say $\mathcal{A}$ is a *complete signing* of $\mathcal{B}$. If $\mathcal{A}$ is a signing (resp., relaxation) of $\mathcal{B}$ and $\mathcal{A} \neq \mathcal{B}$, then we say $\mathcal{A}$ is a *proper* signing (resp., relaxation) of $\mathcal{B}$.

Given a fixed $S \subseteq \mathbb{S}$, an *S-relaxation* (resp., *S-signing*, *S-superpattern*, *S-subpattern*) of an $\mathbb{S}$-pattern $\mathcal{A}$, is a relaxation (resp., signing, superpattern, subpattern) $\mathcal{B}$ of $\mathcal{A}$ such that $\mathcal{B}$ is an S-pattern. Note that no such $\mathcal{B}$ may exist for certain symbol sets S and $\mathbb{S}$-patterns $\mathcal{A}$, in which case the set of S-relaxations (resp., S-signings, S-superpatterns, S-subpatterns) of $\mathcal{A}$ is the empty set. For example, given any $\{0, \#\}$-pattern $\mathcal{A}$, there are no $\{+, -\}$-relaxations of $\mathcal{A}$ and only one $\{\#\}$-relaxation of $\mathcal{A}$. The set of all S-patterns is closed under taking S-relaxations, S-signings, S-superpatterns and S-subpatterns.

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A publication of the International Linear Algebra Society
Volume 23, pp. 731-754, August 2012

There are two notions of minimality when dealing with patterns and spectral properties. The notion that appears in the literature is minimality with respect to the number of 0 entries. To distinguish the two notions of minimality, for $S \subseteq \mathbb{S}$, we say a pattern $\mathcal{A}$ is *minimal with respect to subpatterns over* S *for property* $\mathbf{P}$, if $\mathcal{A}$ is an S-pattern with property $\mathbf{P}$ and no proper S-subpattern of $\mathcal{A}$ has property $\mathbf{P}$. On the other hand, we say $\mathcal{A}$ is *minimal with respect to signings over* S *for property* $\mathbf{P}$, if $\mathcal{A}$ is an S-pattern with property $\mathbf{P}$ and no proper S-signing of $\mathcal{A}$ has property $\mathbf{P}$.

1.2. Brief background and motivation. In the literature, allow problems for S-patterns have been well studied when $S = \{0, +, -\}$ and $S = \{0, *\}$. Below, we briefly describe some techniques and results for these problems.

In 1965, the stability of sign patterns was investigated in [20] and in 1969 some results appeared in [18]. In 1988, the identification of potentially nilpotent sign patterns was listed as an open problem in [12]. In 2000, spectrally and inertially arbitrary sign patterns were introduced in [11]. Currently, the nilpotent-Jacobian method (see [2, 4, 5, 11, 17]) is often used to prove that a pattern is spectrally (and hence inertially) arbitrary. For a survey discussing sign patterns that are SAP, IAP, PN or PS, see [3]. More recently, constructions of $\{0, +, -\}$-patterns that are PS are given in [13] and constructions of $\{0, *\}$-patterns that are PN are given in [1]. In 2009, refined inertially arbitrary sign patterns were defined in [14] and later studied in [9].

The goal of this paper is to extend techniques that appear in the literature for sign patterns and zero-nonzero patterns to $\mathbb{S}$-patterns. One underlying objective for studying $\mathbb{S}$-patterns is to bridge the gap between sign patterns and zero-nonzero patterns and to gain a better understanding of spectral properties for such patterns. In particular, we provide an inertially arbitrary $\{0, +, -, *\}$-pattern of order 4 that has no complete signing or proper subpattern that is inertially arbitrary (see Example 3.4). Such an example demonstrates that when classifying $\{0, +, -, *\}$-patterns that are IAP, it is not sufficient to characterize the sign patterns that are IAP and then take relaxations. This is contrary to the properties PS, PN and PPI, as demonstrated in Lemma 2.3.

Further motivation to study $\mathbb{S}$-patterns is that often, the location of the zero entries in a pattern prevents it from allowing a spectral property $\mathbf{P}$ (in the sense that any other pattern with zeros in the exact same locations would also not allow the property $\mathbf{P}$). The study of $\{0, \#\}$-patterns indicates the importance the location of the 0's have when dealing with certain spectral properties. In particular, if a $\{0, \#\}$-pattern $\mathcal{A}$ does not allow property $\mathbf{P}$, for one of the eight spectral properties discussed in this paper, then any signing of $\mathcal{A}$ also does not allow property $\mathbf{P}$ (see Corollary 2.2). Such insight may help with problems like the $2n$*-conjecture* that asks if every $\{0, +, -, *\}$-pattern of order n that is SAP must have at least $2n$ nonzero entries.

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1.3. Outline and summary. In Section 2, we first investigate general results regarding allow properties and then state implications amongst the spectral properties defined in this paper. We also generalize known results to $\mathbb{S}$-patterns and discuss techniques in the context of $\mathbb{S}$-patterns, e.g., the nilpotent-Jacobian method. Additionally, we state a modification the nilpotent-Jacobian method that can be used to show a pattern and its superpatterns are inertially arbitrary. In Section 3, for every $S \subseteq \mathbb{S}$, we characterize the S-patterns of order 2 that are either SAP, rIAP, IAP, PS, PPI$^+$, PPI, PN$^+$ or PN. In Section 4, we discuss the $2n$-conjecture for $\{0, +, -, *\}$-patterns and the effect # belonging to the symbol set has on this conjecture.

2. General results on allow problems concerning spectral properties. In this section, we first discuss some observations regarding allow and require properties. Next, we investigate necessary conditions on $\mathcal{G}(\mathcal{A})$ in order for a pattern $\mathcal{A}$ to have a specified property. We observe implications amongst spectral properties of patterns and the effect # belonging to the symbol set can have these properties. We also discuss the nilpotent-Jacobian method in the context of $\mathbb{S}$-patterns and provide a modification that can be used to show a non-SAP pattern is IAP.

2.1. Remarks on allow and require properties. In the next few results, we investigate operations on patterns that perserve allow and require properties.

LEMMA 2.1. *Let $\mathcal{A}$ be a pattern and $\mathbf{P}$ be a property.*

(1) If $\mathcal{A}$ allows property $\mathbf{P}$, then every relaxation of $\mathcal{A}$ allows property $\mathbf{P}$.
(2) If $\mathcal{A}$ requires property $\mathbf{P}$, then every signing of $\mathcal{A}$ requires property $\mathbf{P}$.

Proof. (1) Let $\mathcal{B}$ be a relaxation of $\mathcal{A}$. Then $Q(\mathcal{A}) \subseteq Q(\mathcal{B})$. If $\mathcal{A}$ allows property $\mathbf{P} = \cup_i \mathbf{P}_i$, where each property $\mathbf{P}_i$ is simple, then $\mathcal{A}$ allows each $\mathbf{P}_i$. Hence, for each i, there is an $A_i \in Q(\mathcal{A}) \subseteq Q(\mathcal{B})$ such that A_i has property $\mathbf{P}_i$. Thus, for each i, $\mathcal{B}$ allows property $\mathbf{P}_i$, and hence, $\mathcal{B}$ allows property $\mathbf{P}$.
(2) Let $\mathcal{B}$ be a signing of $\mathcal{A}$. Then $Q(\mathcal{B}) \subseteq Q(\mathcal{A})$. If $\mathcal{A}$ requires property $\mathbf{P}$, then for every $A \in Q(\mathcal{A})$, A has property $\mathbf{P}$. Thus, for every $B \in Q(\mathcal{B}) \subseteq Q(\mathcal{A})$, B has property $\mathbf{P}$, and hence, $\mathcal{B}$ requires property $\mathbf{P}$. $\square$

A consequence to Lemma 2.1 is the following.

COROLLARY 2.2. *Let $\mathcal{A}$ be a pattern and $\mathbf{P}$ be a property. If $\mathcal{A}$ does not allow property $\mathbf{P}$, then every signing of $\mathcal{A}$ does not allow property $\mathbf{P}$. Furthermore, if $\mathcal{A}$ does not require property $\mathbf{P}$, then every relaxation of $\mathcal{A}$ does not require property $\mathbf{P}$.*

When the property $\mathbf{P}$ is a simple property, sign patterns play a crucial role as demonstrated in the next result.

LEMMA 2.3. *Let $\mathcal{A}$ be a pattern and $\mathbf{P}$ be a simple property. Then $\mathcal{A}$ allows*

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A publication of the International Linear Algebra Society
Volume 23, pp. 731-754, August 2012

property $\mathbf{P}$ *if and only if* $\mathcal{A}$ *is a relaxation of a sign pattern that allows property* $\mathbf{P}$.

Proof. If $\mathcal{A}$ is a relaxation of a sign pattern that allows property $\mathbf{P}$, then $\mathcal{A}$ allows property $\mathbf{P}$ by Lemma 2.1. Now, suppose $\mathcal{A}$ allows a simple property $\mathbf{P}$. Then there is an $A \in Q(\mathcal{A})$ that has property $\mathbf{P}$. As $A \in Q(sgn(A))$, the sign pattern $sgn(A)$ allows property $\mathbf{P}$. But $\mathcal{A}$ is a relaxation of $sgn(A)$, therefore, $\mathcal{A}$ is a relaxation of a sign pattern that allows property $\mathbf{P}$. ◻

Lemma 2.3 only holds for simple properties, e.g., PPI, PN and PS. In general, only one direction holds for properties that are the union of simple properties, e.g., IAP, rIAP, SAP, PPI$^+$ and PN$^+$. In particular, Lemma 2.3 implies that in order to characterize $\mathbb{S}$-patterns that allow a simple property $\mathbf{P}$, it is sufficient to characterize all sign patterns that allow $\mathbf{P}$ and then take all possible relaxations. It is usually difficult to characterize sign patterns that allow a simple property, however, important information can sometimes be obtained about sign patterns through the study of $\mathbb{S}$-patterns. For example, Corollary 2.2 implies that if a $\{0, \#\}$-pattern $\mathcal{A}$ does not allow property $\mathbf{P}$, then any sign pattern with 0's in the same entries as in $\mathcal{A}$ also does not allow property $\mathbf{P}$.

2.2. Necessary conditions and implications amongst spectral properties. We next look at how allowing the inertia $(0, 0, n)$ can affect the labelled directed graph of a pattern. Note that the proof of [5, Lemma 5.1] does not depend on the symbol set and rather relies on the existence of a realization $A \in Q(\mathcal{A})$ with $i(A) = (0, 0, n)$ that has at least one nonzero diagonal element.

THEOREM 2.4. *Let* $S \subseteq \mathbb{S}$ *and* $\mathcal{A}$ *be an* S*-pattern of order* n*. If there is a realization* $A \in Q(\mathcal{A})$ *with* $i(A) = (0, 0, n)$ *and at least one nonzero entry on the main diagonal of* A*, then* $\mathcal{G}(\mathcal{A})$ *allows two oppositely signed loops and allows a negative* 2*-cycle.*

We emphasize that in Theorem 2.4 we require the realization $A \in Q(\mathcal{A})$ with $i(A) = (0, 0, n)$ to have a nonzero entry on its main diagonal (and hence, A has at least two nonzero entries of opposite sign on its main diagonal). For $\{0, +, -, *\}$-patterns, if $\mathcal{G}(\mathcal{A})$ has a loop, then every $A \in Q(\mathcal{A})$ has at least one nonzero entry on its main diagonal.

COROLLARY 2.5. *Let* $S \subseteq \{0, +, -, *\}$ *and* $\mathcal{A}$ *be an* S*-pattern of order* n*. If* $\mathcal{G}(\mathcal{A})$ *has a loop and* $\mathcal{A}$ *allows the inertia* $(0, 0, n)$*, then* $\mathcal{G}(\mathcal{A})$ *allows two oppositely signed loops and allows a negative* 2*-cycle.*

Analogues of Corollary 2.5 have appeared in the literature for sign patterns with the stronger assumption that $\mathcal{A}$ is IAP [5, Lemma 5.1], and also with the assumption that $\mathcal{G}(\mathcal{A})$ contains a loop and $\mathcal{A}$ is PN [10, Lemma 3.2]. It has also appeared for zero-nonzero patterns with the assumption that $\mathcal{A}$ is SAP [8, Lemma 3.3].

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A publication of the International Linear Algebra Society
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For $S \subseteq \{0, +, -, *\}$, Figure 2.1 illustrates the known relationships among the different properties for S-patterns of order $n \geq 2$. The arrows in Figure 2.1 indicate implication, for example, the top arrow represents the statement 'if $\mathcal{A}$ is SAP, then $\mathcal{A}$ is rIAP'. The only nontrivial implications are rIAP $\rightarrow$ PN$^+$ and IAP $\rightarrow$ PPI$^+$, which

$$\begin{tikzcd}
 & & \text{SAP} \arrow[d] & & \\
 & & \text{rIAP} \arrow[dl] \arrow[dr] & & \\
 & \text{PN}^+ \arrow[dl] \arrow[dr] & & \text{IAP} \arrow[dl] \arrow[dr] & \\
\text{PN} \arrow[dr] & & \text{PPI}^+ \arrow[dl] & & \text{PS} \\
 & \text{PPI} & & &
\end{tikzcd}$$

FIG. 2.1. *Implications amongst spectral properties for $\{0, +, -, *\}$-patterns of order $n \geq 2$.*

are true by Corollary 2.5. Thus, if $\mathcal{A}$ is SAP, rIAP, IAP, PN$^+$ or PPI$^+$, then $\mathcal{G}(\mathcal{A})$ must allow two oppositely signed loops and allow a negative 2-cycle. In general, this is not true for $\mathbb{S}$-patterns as demonstrated in the following example. First note that if a pattern $\mathcal{A} = [\alpha_{ij}]$ of order n allows the inertia $(0, 0, n)$ and $\mathcal{G}(\mathcal{A})$ does not allow a negative 2-cycle, then we must have $\alpha_{ii} \in \{0, +_0, -_0, \#\}$ for each i (this follows from the proof of [5, Lemma 5.1] as otherwise $A \in Q(\mathcal{A})$ with $i(A) = (0, 0, n)$ would have a nonzero on the diagonal forcing the coefficient of x^{n-2} in the characteristic polynomial of A to be negative, a contradiction).

EXAMPLE 2.6. Consider the following diagonal pattern $\mathcal{D}_n$ of order n,

$$\mathcal{D}_n = \begin{bmatrix} \# & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \# \end{bmatrix}.$$

For $n \geq 2$, note that $\mathcal{D}_n$ is IAP, PN, PS and PPI, however, $\mathcal{D}_n$ is not PPI$^+$, PN$^+$, rIAP or SAP. For $n = 1$, $\mathcal{D}_1$ is SAP, rIAP, IAP, PN, PS and PPI, however, $\mathcal{D}_1$ is not PPI$^+$ or PN$^+$.

As demonstrated in Example 2.6, for symbol sets S with $\# \in S$, the labelled directed graph of an S-pattern that is IAP does not need to allow a negative 2-cycle. We next provide an example that illustrates if $\mathcal{A}$ is SAP, then $\mathcal{G}(\mathcal{A})$ can have a single loop which is contrary to that of sign patterns and zero-nonzero patterns.

EXAMPLE 2.7. Consider the companion pattern $\mathcal{C}_n$ of order n that arises from

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A publication of the International Linear Algebra Society
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the companion matrix, that is,

$$\mathcal{C}_n = \begin{bmatrix} 0 & + & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & + \\ \# & \# & \cdots & \cdots & \# \end{bmatrix}.$$

For $n \geq 2$, $\mathcal{C}_n$ is SAP, rIAP, IAP, PS, PPI$^+$, PPI, PN$^+$ and PN. Observe that $\mathcal{G}(\mathcal{C}_n)$ allows a positive loop, allows a negative loop, and allows a negative 2-cycle, but $\mathcal{G}(\mathcal{C}_n)$ does not allow two oppositely signed loops. Also, note that even though $\mathcal{C}_n$ is an $\mathbb{S}$-pattern that allows the inertia $(0, 0, n)$, the conditions of Theorem 2.4 are not satisfied since every realization $A \in Q(\mathcal{C}_n)$ with $i(A) = (0, 0, n)$ has zeros on the main diagonal.

If $\mathcal{A}$ is an $\mathbb{S}$-pattern of order $n \geq 2$ that is rIAP or SAP, then $\mathcal{A}$ must allow a positive loop and allow a negative loop. Furthermore, by using the refined inertia $(0, 0, n-2, 2)$ in the proof of [5, Lemma 5.1] in place of the inertia $(0, 0, n)$, it can be shown that $\mathcal{G}(\mathcal{A})$ must allow a negative 2-cycle.

Corollary 2.8. *Let $S \subseteq \mathbb{S}$ and $\mathcal{A}$ be an S-pattern of order $n \geq 2$. If $\mathcal{A}$ is rIAP, then $\mathcal{G}(\mathcal{A})$ allows a positive loop, allows a negative loop, and allows a negative 2-cycle. Furthermore, if $\mathcal{G}(\mathcal{A})$ has exactly one loop, then the loop has sign $\#$.*

For $S \subseteq \mathbb{S}$, Figure 2.2 illustrates the known relationships among the different properties for S-patterns of order $n \geq 2$. The implication rIAP $\rightarrow$ PN$^+$ is true by

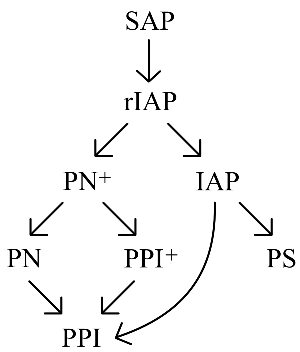

Fig. 2.2. *Implications amongst spectral properties for $\mathbb{S}$-patterns of order $n \geq 2$.*

Corollary 2.8. As illustrated with Example 2.6, if $\mathcal{A}$ is IAP then $\mathcal{A}$ does not need to be PPI$^+$.

Although, in general, the converse implications in Figures 2.1 and 2.2 do not hold,

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A publication of the International Linear Algebra Society
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for certain classes of patterns some of the converse implications can be shown to hold for all patterns in that class. In [19], it is shown that every $\{+,-\}$-pattern that is PN is also SAP. This proof can be modified for $\{+,-,*,\#\}$-patterns by noting that if $\mathcal{A}$ is a $\{+,-,*,\#\}$-pattern of order n, then for every $A \in Q(\mathcal{A})$ and $B \in M_n(\mathbb{R})$, we have $A + \epsilon B \in Q(\mathcal{A})$, for every sufficiently small $\epsilon > 0$.

Lemma 2.9. *Let $\mathcal{A}$ be a $\{+,-,*,\#\}$-pattern. The following are equivalent:*

(i) $\mathcal{A}$ is SAP.
(ii) $\mathcal{A}$ is rIAP.
(iii) $\mathcal{A}$ is PN$^+$.
(iv) $\mathcal{A}$ is PN.

Note that if $+_0$ (resp., 0 and $-_0$) belongs to the symbol set, then Lemma 2.9 need not hold in general. Observe that the pattern where every entry is $+_0$ (resp., 0 and $-_0$) is PN but not SAP.

We also observe that the technique used in [19] can be modified to prove the following result.

Lemma 2.10. *If $\mathcal{A}$ is a $\{+,-,*,\#\}$-pattern of order n and $\mathcal{A}$ allows the refined inertia $(0,0,r,n-r)$, where $r \geq 2$, then $\mathcal{A}$ is IAP.*

Proof. Take a matrix A with refined inertia $(0,0,r,n-r)$, where $r \geq 2$. Decompose A into real Jordan canonical form, i.e., $A = SJS^{-1}$. If $\epsilon > 0$ is sufficiently small, then by replacing appropriate entries on the main diagonal of J by ϵ or $-\epsilon$, we can construct a matrix $B \in Q(\mathcal{A})$ with arbitrary inertia (n_+, n_-, n_0) where $n_+ + n_- + n_0 = n$. As $r \geq 2$, such a matrix B can always be constructed. □

It is an open question whether or not a $\{+,-,*,\#\}$-pattern that is IAP is also PN (and hence SAP by Lemma 2.9).

2.3. Extending the nilpotent-Jacobian method. Currently, the nilpotent-Jacobian method (see [2, 4, 5, 11, 17]) is the most widely used tool in showing a $\{0,+,-,*\}$-pattern is SAP. In the context of $\{0,+,-,*,\#\}$-patterns, the nilpotent-Jacobian method may still be used to show a pattern is SAP and the proof is identical to that of [21, Theorem 4].

Theorem 2.11. *Let $\mathcal{A} = [\alpha_{ij}]$ be a $\{0,+,-,*,\#\}$-pattern of order n and suppose there exists some nilpotent matrix $A \in Q(\mathcal{A})$. Let $a_{i_1j_1}, \ldots, a_{i_nj_n}$ be n entries of A such that the corresponding symbols in $\mathcal{A}$ are nonzero (that is, $\alpha_{i_kj_k} \neq 0$ for each k). Let X be the real matrix obtained by replacing these entries in A by variables $x_1, \ldots, x_n$, and let the characteristic polynomial of X be given by*

$$p_X(x) = x^n + c_1x^{n-1} + c_2x^{n-2} + \cdots + c_{n-1}x + c_n,$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 731-754, August 2012

where $c_i = c_i(x_1, \ldots, x_n)$ is differentiable in each x_j. If the Jacobian matrix of order n with (i, j) entry equal to $\frac{\partial c_i}{\partial x_j}$ is nonsingular at $(x_1, \ldots, x_n) = (a_{i_1 j_1}, \ldots, a_{i_n j_n})$, then every superpattern of $\mathcal{A}$ (including $\mathcal{A}$ itself) is SAP.

In Theorem 2.11 we omit the symbols $+_0$ and $-_0$ from our symbol set. This is because it is possible for a nilpotent realization $A \in Q(\mathcal{A})$ to have a zero entry corresponding to a $+_0$ or $-_0$ symbol in $\mathcal{A}$. If this zero entry of A is chosen as a variable x_k, then sufficiently small perturbations of this entry may give matrix realizations not belonging to $Q(\mathcal{A})$. However, this problem does not arise when a zero entry of A corresponding to a # symbol in $\mathcal{A}$ is chosen as a variable.

When # is in the symbol set, the nilpotent-Jacobian method can be used on patterns with $2n - 1$ nonzero entries as demonstrated in the next example. This is contrary to that of sign patterns and zero-nonzero patterns (see [21, Theorem 6]).

Example 2.12. Consider the companion pattern $\mathcal{C}_n$ from Example 2.7. Let

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ x_n & \cdots & \cdots & \cdots & x_1 \end{bmatrix},$$

so that A is a nilpotent realization of $\mathcal{C}_n$. The characteristic polynomial of X is

$$p_X(x) = x^n + c_1 x^{n-1} + c_2 x^{n-2} + \cdots + c_{n-1} x + c_n,$$

where $c_i = -x_i$ for each i. Note that

$$\frac{\partial c_i}{\partial x_j} = \begin{cases} -1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the Jacobian matrix evaluated at $(x_1, \ldots, x_n) = (0, \ldots, 0)$ is the negative of the identity matrix of order n, and hence, is nonsingular. Therefore, by Theorem 2.11, every superpattern of $\mathcal{C}_n$ is SAP.

We next present a variation on the nilpotent-Jacobian method that is useful in showing a pattern is IAP.

Theorem 2.13. *Let $\mathcal{A} = [\alpha_{ij}]$ be a $\{0, +, -, *, \#\}$-pattern of order n and suppose there exists a matrix $A \in Q(\mathcal{A})$ with refined inertia $ri(A) = (0, 0, r, n - r)$, for some $r \geq 2$. Let $a_{i_1 j_1}, \ldots, a_{i_n j_n}$ be n entries of A such that the corresponding symbols in $\mathcal{A}$ are nonzero (that is, $\alpha_{i_k j_k} \neq 0$ for each k). Let X be the real matrix obtained by replacing these entries in A by variables $x_1, \ldots, x_n$, and let the characteristic polynomial*

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A publication of the International Linear Algebra Society
Volume 23, pp. 731-754, August 2012

of X be given by

$$p_X(x) = x^n + c_1x^{n-1} + c_2x^{n-2} + \cdots + c_{n-1}x + c_n,$$

where $c_i = c_i(x_1,\ldots,x_n)$ is differentiable in each x_j. If the Jacobian matrix of order n with (i,j) entry equal to $\frac{\partial c_i}{\partial x_j}$ is nonsingular at $(x_1,\ldots,x_n) = (a_{i_1j_1},\ldots,a_{i_nj_n})$, then every superpattern of $\mathcal{A}$ (including $\mathcal{A}$ itself) is IAP. Furthermore, for every $s \le r$ with $n-s$ even, $\mathcal{A}$ (and its superpatterns) allow the refined inertia $(0,0,s,n-s)$.

Proof. Suppose that the Jacobian matrix of order n with (i,j) entry equal to $\frac{\partial c_i}{\partial x_j}$ is nonsingular at $(x_1,\ldots,x_n) = (a_{i_1j_1},\ldots,a_{i_nj_n})$. Define $f:\mathbb{R}^n \to \mathbb{R}^n$ by

$$f(x_1,\ldots,x_n) = (c_1(x_1,\ldots,x_n),\ldots,c_n(x_1,\ldots,x_n)).$$

Then the Implicit Function Theorem asserts that there exist open neighbourhoods M of $(a_{i_1j_1},\ldots,a_{i_nj_n})$ and N of $f(a_{i_1j_1},\ldots,a_{i_nj_n})$ such that f maps M bijectively to N. But

$$f(a_{i_1j_1},\ldots,a_{i_nj_n}) = (k_1,k_2,\ldots,k_n),$$

where k_i is the coefficient of x^{n-i} in the characteristic polynomial of A, that is,

$$p_A(x) \quad = \quad x^n + k_1x^{n-1} + k_2x^{n-2} + \cdots + k_{n-1}x + k_n.$$

It follows that there is an $\epsilon > 0$, such that for any $b_1,b_2,\ldots,b_n \in \mathbb{R}$ with each $|b_i| < \epsilon$, there is a matrix $B \in Q(\mathcal{A})$ with characteristic polynomial

$$p_B(x) = x^n + (b_1+k_1)x^{n-1} + \cdots + (b_{n-1}+k_{n-1})x + (b_n+k_n).$$

As A has refined inertia $(0,0,r,n-r)$, we can factor the characteristic polynomial of A as

$$p_A(x) \quad = \quad x^r\left(x^2+\beta_1\right)\left(x^2+\beta_2\right)\cdots\left(x^2+\beta_{\frac{n-r}{2}}\right),$$

for some $\beta_i \ge 0$, $i = 1,2,\ldots,\frac{n-r}{2}$. Note that for any $\beta > 0$ and sufficiently small $\delta \in (-1,1)$, the polynomial $x^2+\delta x+\beta$ has two roots with positive (resp., negative or zero) real part whenever $\delta < 0$ (resp., $\delta > 0$ or $\delta = 0$).

Let (n_+,n_-,n_0) be a nonnegative integer triple with $n_+ + n_- + n_0 = n$. Now, choose sufficiently small $\zeta_1,\zeta_2,\ldots,\zeta_r,\delta_1,\delta_2,\ldots,\delta_{\frac{n-r}{2}} \in (-1,1)$, so that the polynomial

$$p(x) = [(x+\zeta_1)\cdots(x+\zeta_r)]\cdot\left[(x^2+\delta_1x+\beta_1)\cdots\left(x^2+\delta_{\frac{n-r}{2}}x+\beta_{\frac{n-r}{2}}\right)\right]$$

has n_+ (resp., n_- and n_0) roots with positive (resp., negative and zero) real part, with the added property that $p(x)$ can be written as

$$p(x) = x^n + (b_1+k_1)x^{n-1} + \cdots + (b_{n-1}+k_{n-1})x + (b_n+k_n),$$

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A publication of the International Linear Algebra Society
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for some $b_1, b_2, \ldots, b_n \in \mathbb{R}$ with each $|b_i| < \epsilon$, $i = 1, 2, \ldots, n$. Since $r \geq 2$, such a polynomial $p(x)$ can always be constructed. Thus, there is a $B \in Q(\mathcal{A})$ with characteristic polynomial $p_B(x) = p(x)$ and inertia $i(B) = (n_+, n_-, n_0)$. As (n_+, n_-, n_0) was an arbitrary nonnegative integer triple with $n_+ + n_- + n_0 = n$, we have that $\mathcal{A}$ is IAP. Furthermore, an argument similar to that in [11] can be used to show that every superpattern of $\mathcal{A}$ is also IAP.

Finally, for every $s \leq r$ with $n - s$ even, the above argument can be modified to show that $\mathcal{A}$ (and its superpatterns) allows the refined inertia $(0, 0, s, n - s)$. In particular, if $\beta > 0$ is sufficiently small, then there is a $B \in Q(\mathcal{A})$ with characteristic polynomial

$$p_B(x) = x^s (x^2 + \beta)^{\frac{r-s}{2}} \left(x^2 + \beta_1\right) \cdots \left(x^2 + \beta_{\frac{n-r}{2}}\right).$$

Such a B has refined inertia $(0, 0, s, n - s)$. □

In the next example, we show how Theorem 2.13 can be used to prove a pattern is inertially arbitrary.

Example 2.14. Consider the following signing of the companion pattern:

$$\hat{\mathcal{C}}_n = \begin{bmatrix} 0 & + & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & + \\ s_n & \cdots & \cdots & s_2 & s_1 \end{bmatrix},$$

where

$$s_i = \begin{cases} - & \text{if } i \text{ is even and } i \leq n-2, \\ \# & \text{otherwise.} \end{cases}$$

Let X be defined as in Example 2.12. Note that $\hat{\mathcal{C}}_n$ allows the refined inertia $(0, 0, 2, n - 2)$ if n is even or $(0, 0, 3, n - 3)$ if n is odd. Let $A \in Q(\hat{\mathcal{C}}_n)$ be such a realization of the form X. Suppose k_i is the coefficient of x^{n-i} in the characteristic polynomial $p_A(x)$ of A. As in Example 2.12, the Jacobian evaluated at $(x_1, \ldots, x_n) = (k_1, \ldots, k_n)$ is the negative of the identity matrix of order n, and hence, is nonsingular. Therefore, by Theorem 2.13, every superpattern of $\hat{\mathcal{C}}_n$ is IAP. In fact, $\hat{\mathcal{C}}_n$ is a minimal IAP with respect to both signings and subpatterns ([15, Lemma 20] implies any signing of $\hat{\mathcal{C}}_n$ is not IAP).

Unlike spectrally arbitrary patterns, Example 2.14 shows that it is *not* necessary for a pattern that is IAP to allow a positive and negative principal minor of order k for all $k = 1, \ldots, n$. We remark that in [11] it is stated that a sign pattern that is

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A publication of the International Linear Algebra Society
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IAP must allow a positive and a negative principal minor of every order k, however, no proof of this result or counter-example (for sign patterns) is known. Necessary conditions on the signs of principal minors for an IAP can be found in [7, Lemma 1] and [15, Lemma 20].

We next show how Theorem 2.13 can be used on sign patterns that have appeared in the literature. Note that the next example also demonstrates that we can use Theorem 2.13 on (sign) patterns of order n with $2n-1$ nonzeros.

EXAMPLE 2.15. For $n \geq 6$, the pattern

$$\mathcal{W}_n = \begin{bmatrix} + & - & 0 & 0 & 0 & \cdots & 0 \\ + & 0 & - & 0 & 0 & \cdots & \vdots \\ 0 & 0 & 0 & - & 0 & \ddots & \vdots \\ 0 & + & 0 & 0 & - & \ddots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \ddots & 0 \\ 0 & + & 0 & \cdots & \cdots & 0 & - \\ 0 & + & 0 & \cdots & \cdots & 0 & - \end{bmatrix}$$

of order n appeared in [7] and was shown to be an inertially arbitrary sign pattern that is minimal with respect to subpatterns. Let

$$X = \begin{bmatrix} x_1 & -1 & 0 & 0 & 0 & \cdots & 0 \\ x_3 & 0 & -1 & 0 & 0 & \cdots & \vdots \\ 0 & 0 & 0 & -1 & 0 & \ddots & \vdots \\ 0 & x_4 & 0 & 0 & -1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \ddots & 0 \\ 0 & x_{n-1} & 0 & \cdots & \cdots & 0 & -1 \\ 0 & x_n & 0 & \cdots & \cdots & 0 & -x_2 \end{bmatrix},$$

and let $A \in Q(\mathcal{W}_n)$ be the matrix where $x_1 = x_2 = 1$ and $x_i = 6$ for $i = 3, 4, \ldots, n$. By [7, Lemma 5] and relabelling, the characteristic polynomial of X is

$$\begin{aligned} p_X(x) = \quad & x^n + (x_2 - x_1)x^{n-1} + (x_3 - x_1x_2)x^{n-2} \\ & +(x_2x_3 - x_4)x^{n-3} + (x_5 - x_4(x_2 - x_1))x^{n-4} \\ & + \textstyle\sum_{k=2}^{n-4} (-1)^k (x_{k+3}(x_2 - x_1) + x_1x_2x_{k+2} - x_{k+4})x^{n-3-k} \\ & +(-1)^n x_1(x_n - x_2x_{n-1}). \end{aligned}$$

Note that the characteristic polynomial of A is $p_A(x) = x^{n-4}(x^4 + 5x^2 + 6)$, hence, A has refined inertia $(0, 0, n-4, 4)$. Let c_i denote the coefficient of x^{n-i} in $p_X(x)$ so that the Jacobian matrix has (i, j) entry equal to $\frac{\partial c_i}{\partial x_j}$. Although we do not explicitly

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A publication of the International Linear Algebra Society
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display the Jacobian matrix, a similar argument to that in [7, Lemma 5] shows that the Jacobian matrix is nonsingular. Thus, by Theorem 2.13, every superpattern of $\mathcal{W}_n$ (including $\mathcal{W}_n$ itself) is IAP.

The next example demonstrates that Theorem 2.13 cannot always be used to show a pattern is IAP.

EXAMPLE 2.16. Consider the following $\{0, \#\}$-pattern of order 4 which is a relaxation of $\mathcal{N}$ in [5]:

$$\mathcal{N}^{\#} = \begin{bmatrix} \# & \# & 0 & 0 \\ 0 & 0 & \# & \# \\ \# & \# & 0 & 0 \\ 0 & 0 & \# & \# \end{bmatrix}.$$

By Lemma 2.1, $\mathcal{N}^{\#}$ is IAP since it is a relaxation of an IAP. We next show that $\mathcal{N}^{\#}$ allows every refined inertia except $(0,0,0,4)$. Each realization of $\mathcal{N}$ provided in [5] having inertia (n_+, n_-, n_0) with $n_0 \le 1$ may be extended to a realization of $\mathcal{N}^{\#}$ with refined inertia $(n_+, n_-, n_0, 0)$. The following eight matrices in $Q(\mathcal{N}^{\#})$ have respective refined inertias $(0,0,4,0)$, $(0,0,2,2)$, $(1,0,3,0)$, $(1,0,1,2)$, $(1,1,0,2)$, $(1,1,2,0)$, $(2,0,2,0)$ and $(2,0,0,2)$:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -2 & -2 & 0 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ -2 & -1 & 0 & 0 \\ 0 & 0 & -2 & -1 \end{bmatrix},$$

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ -3 & -2 & 0 & 0 \\ 0 & 0 & -2 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -2 & -1 & 0 & 0 \\ 0 & 0 & -2 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -2 & -2 \end{bmatrix},$$

$$\begin{bmatrix} 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ -2 & -2 & 0 & 0 \\ 0 & 0 & -2 & -1 \end{bmatrix} \text{ and } \begin{bmatrix} 4 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ -5 & -3 & 0 & 0 \\ 0 & 0 & -5 & -1 \end{bmatrix}.$$

As $A \in Q(\mathcal{N}^{\#})$ if and only if $-A \in Q(\mathcal{N}^{\#})$, it follows that $\mathcal{N}^{\#}$ allows every refined inertia except possibly $(0,0,0,4)$. To see that $\mathcal{N}^{\#}$ does not allow the refined inertia $(0,0,0,4)$, suppose that

$$A = \begin{bmatrix} a & b & 0 & 0 \\ 0 & 0 & c & d \\ e & f & 0 & 0 \\ 0 & 0 & g & h \end{bmatrix}$$

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A publication of the International Linear Algebra Society
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has refined inertia $ri(A) = (0, 0, 0, 4)$, where $a, b, c, d, e, f, g, h \in \mathbb{R}$. The characteristic polynomial of A is

$$x^4 - (a+h)x^3 + (ah - cf)x^2 + [(af - be)c + f(ch - dg)]x - (ch - dg)(af - be).$$

As $ri(A) = (0, 0, 0, 4)$, we require:

$$\begin{aligned}
&(2.1) & a + h &= 0,\\
&(2.2) & ah - cf &> 0,\\
&(2.3) & (af - be)c + f(ch - dg) &= 0,\\
&(2.4) & -(ch - dg)(af - be) &> 0.
\end{aligned}$$

By (2.1) and (2.2) we have $cf < 0$, that is, c and f are nonzero and of opposite sign. As c is nonzero, by (2.3) and (2.4) we have $\frac{f}{c}(ch - dg)^2 > 0$, thus, c and f are of the same sign, a contradiction. Therefore, $\mathcal{N}^{\#}$ allows every refined inertia except $(0, 0, 0, 4)$, and hence $\mathcal{N}^{\#}$ is not rIAP. Thus, Theorem 2.13 cannot be used to show $\mathcal{N}^{\#}$ is IAP, otherwise it would also imply that $\mathcal{N}^{\#}$ can attain the refined inertia $(0, 0, 0, 4)$.

3. Patterns of small order that allow certain spectral properties. In this section, for $S \subseteq \mathbb{S}$, we characterize the S-patterns of order $n \leq 2$ that are either SAP, rIAP, IAP, PS, PPI$^+$, PPI, PN$^+$ or PN. In the results that follow, we often use the statement '$\mathcal{A}$ is an S-relaxation' of a certain collection of patterns. If no such S-relaxation exists the statement should be interpreted as 'there are no such S-patterns with the stated property'.

In the literature, patterns of order 1 are often excluded, however, they are important in the general case as $[\#]$ is SAP, which can be used to form reducible spectrally arbitrary patterns of higher orders.

THEOREM 3.1. *Let $S \subseteq \mathbb{S}$ and $\mathcal{A}$ be an S-pattern of order* 1.

(1) The following statements are equivalent:
- *(i) $\mathcal{A}$ is SAP.*
- *(ii) $\mathcal{A}$ is rIAP.*
- *(iii) $\mathcal{A}$ is IAP.*
- *(iv) $\mathcal{A}$ is an S-relaxation of* $[\#]$.

(2) The following statements are equivalent:
- *(i) $\mathcal{A}$ is PPI.*
- *(ii) $\mathcal{A}$ is PN.*
- *(iii) $\mathcal{A}$ is an S-relaxation of* $[0]$.

(3) The following statements are equivalent:
- *(i) $\mathcal{A}$ is PS.*

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(ii) $\mathcal{A}$ is an S-relaxation of $[-]$.
(4) There is no such $\mathcal{A}$ that is PN$^+$ or PPI$^+$.

The proof of Theorem 3.1 is easy as there are only seven possible $\mathbb{S}$-patterns of order 1.

Sign patterns that are PN or PS have been well studied in the literature (see, for example, the survey paper [3]). Since sign patterns of order 2 that are PPI (resp., PN and PS) are known or easy to classify, by Lemma 2.3, taking relaxations then generates all such $\mathbb{S}$-patterns of order 2 that are PPI (resp., PN and PS).

THEOREM 3.2. *Let $S \subseteq \mathbb{S}$ and $\mathcal{A}$ be an S-pattern of order* 2.

(1) The following statements are equivalent:
(i) $\mathcal{A}$ is PPI.
(ii) $\mathcal{A}$ is equivalent to an S-relaxation of

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & + \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} + & + \\ - & - \end{bmatrix} \textit{ or } \begin{bmatrix} 0 & + \\ - & 0 \end{bmatrix}.$$

(2) The following statements are equivalent:
(i) $\mathcal{A}$ is PN.
(ii) $\mathcal{A}$ is equivalent to an S-relaxation of

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & + \\ 0 & 0 \end{bmatrix} \textit{ or } \begin{bmatrix} + & + \\ - & - \end{bmatrix}.$$

(3) The following statements are equivalent:
(i) $\mathcal{A}$ is PS.
(ii) $\mathcal{A}$ is equivalent (excluding negation) to an S-relaxation of a superpattern of

$$\begin{bmatrix} - & 0 \\ 0 & - \end{bmatrix} \textit{ or } \begin{bmatrix} 0 & + \\ - & - \end{bmatrix}.$$

(4) The following statements are equivalent:
(i) $\mathcal{A}$ is PPI$^+$.
(ii) $\mathcal{A}$ is equivalent to an S-relaxation of

$$\begin{bmatrix} 0 & + \\ - & \# \end{bmatrix} \textit{ or } \begin{bmatrix} + & + \\ - & - \end{bmatrix}.$$

(5) The following statements are equivalent:
(i) $\mathcal{A}$ is PN$^+$.
(ii) $\mathcal{A}$ is equivalent to an S-relaxation of

$$\begin{bmatrix} 0 & + \\ -_0 & \# \end{bmatrix} \textit{ or } \begin{bmatrix} + & + \\ - & - \end{bmatrix}.$$

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(6) The following statements are equivalent:
(i) $\mathcal{A}$ is SAP.
(ii) $\mathcal{A}$ is rIAP.
(iii) $\mathcal{A}$ is equivalent to an S-relaxation of

$$\begin{bmatrix} 0 & + \\ \# & \# \end{bmatrix} \text{ or } \begin{bmatrix} + & + \\ - & - \end{bmatrix}.$$

(7) The following statements are equivalent:
(i) $\mathcal{A}$ is IAP.
(ii) $\mathcal{A}$ is equivalent to an S-relaxation of

$$\begin{bmatrix} 0 & + \\ \# & \# \end{bmatrix}, \begin{bmatrix} + & + \\ - & - \end{bmatrix}, \begin{bmatrix} \# & 0 \\ 0 & \# \end{bmatrix} \text{ or } \begin{bmatrix} \# & + \\ 0 & \# \end{bmatrix}.$$

Proof. Let $S \subseteq \mathbb{S}$. It suffices to prove the theorem for $\mathbb{S}$-patterns, since then a characterization for S-patterns is obtained by taking S-relaxations. Thus, suppose that $S = \mathbb{S}$.

Cases $(1) - (3)$**:** By Lemma 2.3, it is sufficient to characterize the sign patterns of order 2 that are PPI, PN or PS. This is an easy task that is left to the reader.

Case (4)**:** Let $\mathcal{A}$ be a pattern of order 2 that is PPI$^+$. Then $\mathcal{A}$ is PPI and $\mathcal{G}(\mathcal{A})$ allows a positive loop, allows a negative loop, and allows a negative 2-cycle. Hence, by Case (1), $\mathcal{A}$ is equivalent to a relaxation of

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & + \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} + & + \\ - & - \end{bmatrix} \text{ or } \begin{bmatrix} 0 & + \\ - & 0 \end{bmatrix},$$

with the property that $\mathcal{G}(\mathcal{A})$ allows a positive loop, allows a negative loop, and allows a negative 2-cycle. It is easy to check that over all such relaxations of these four sign patterns, the ones whose corresponding labelled directed graphs allow a positive loop, allow a negative loop, and allow a negative 2-cycle are equivalent to relaxations of

$$\begin{bmatrix} 0 & + \\ - & \# \end{bmatrix} \text{ or } \begin{bmatrix} + & + \\ - & - \end{bmatrix}.$$

Case (5)**:** Let $\mathcal{A}$ be a pattern of order 2 that is PN$^+$. Thus, $\mathcal{A}$ is PPI$^+$ and $\mathcal{A}$ allows nilpotence. Hence, by Case (4), $\mathcal{A}$ is equivalent to relaxations of

$$\begin{bmatrix} 0 & + \\ - & \# \end{bmatrix} \text{ or } \begin{bmatrix} + & + \\ - & - \end{bmatrix}$$

that allow nilpotence. The second pattern allows nilpotence, however, the first pattern does not allow nilpotence. Thus, looking at all proper relaxations of the first pattern

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that allow nilpotence, gives that, $\mathcal{A}$ is equivalent to a relaxation of

$$\begin{bmatrix} 0 & + \\ -_0 & \# \end{bmatrix} \text{ or } \begin{bmatrix} + & + \\ - & - \end{bmatrix}.$$

Case (6)**:** Let $\mathcal{A}$ be a pattern of order 2 that is SAP. Then $\mathcal{A}$ is rIAP, and hence, PN^+. Hence, by Case (5), $\mathcal{A}$ is equivalent to relaxations of

$$\begin{bmatrix} 0 & + \\ -_0 & \# \end{bmatrix} \text{ or } \begin{bmatrix} + & + \\ - & - \end{bmatrix}.$$

The second pattern is SAP, however, the first pattern does not allow the refined inertia $(1,1,0,0)$. Thus, looking at all proper relaxations of the first pattern that allow the refined inertia $(1,1,0,0)$, gives that, $\mathcal{A}$ is equivalent to a relaxation of

$$\begin{bmatrix} 0 & + \\ \# & \# \end{bmatrix} \text{ or } \begin{bmatrix} + & + \\ - & - \end{bmatrix}.$$

Each of these patterns is SAP.

Case (7)**:** Let $\mathcal{A}$ be a pattern of order 2 that is IAP. Then $\mathcal{A}$ is PPI, hence, $\mathcal{A}$ is equivalent to a relaxation of

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & + \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} + & + \\ - & - \end{bmatrix} \text{ or } \begin{bmatrix} 0 & + \\ - & 0 \end{bmatrix}.$$

Every relaxation of the third pattern is IAP. The only relaxations of the first and second patterns that are IAP are relaxations of

$$\begin{bmatrix} \# & 0 \\ 0 & \# \end{bmatrix}, \begin{bmatrix} \# & + \\ 0 & \# \end{bmatrix} \text{ and } \begin{bmatrix} 0 & + \\ \# & \# \end{bmatrix}.$$

Note that

$$\begin{bmatrix} 0 & * \\ * & \# \end{bmatrix} \text{ and } \begin{bmatrix} 0 & + \\ -_0 & \# \end{bmatrix}$$

have nonzero determinant and nonnegative determinant, respectively. Thus, any signing of these two patterns is not IAP as a signing will not allow either the inertia $(1,0,1)$ or $(1,1,0)$. Therefore, if $\mathcal{A}$ is IAP then $\mathcal{A}$ is equivalent to a relaxation of

$$\begin{bmatrix} 0 & + \\ \# & \# \end{bmatrix}, \begin{bmatrix} + & + \\ - & - \end{bmatrix}, \begin{bmatrix} \# & 0 \\ 0 & \# \end{bmatrix} \text{ or } \begin{bmatrix} \# & + \\ 0 & \# \end{bmatrix}.$$

Each of these patterns is IAP. $\square$

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In the case of S-patterns where $S \subseteq \{0,+,-,*\}$, we have the following consequence which is well known for sign patterns and zero-nonzero patterns.

Corollary 3.3. *Let $S \subseteq \{0,+,-,*\}$ and $\mathcal{A}$ be an S-pattern of order 2. The following statements are equivalent:*

(i) $\mathcal{A}$ is SAP,
(ii) $\mathcal{A}$ is rIAP,
(iii) A is IAP,
(iv) $\mathcal{A}$ is PN$^+$,
(v) $\mathcal{A}$ is PPI$^+$,
(vi) $\mathcal{A}$ is equivalent to an S-relaxation of

$$\begin{bmatrix} + & + \\ - & - \end{bmatrix}.$$

We leave it as an open problem to characterize $\mathbb{S}$-patterns of order 3 that are either SAP, rIAP, IAP, PS, PPI$^+$, PPI, PN$^+$ or PN. We end this section with a pattern of order 4 that is a minimal IAP with respect to both subpatterns and signings over $\{0,+,-,*\}$.

Example 3.4. Consider the following $\{0,+,-,*\}$-pattern of order 4 which is a signing of $\mathcal{N}_3^*$ in [6]:

$$\mathcal{A}_4 = \begin{bmatrix} + & + & + & 0 \\ - & - & - & 0 \\ + & 0 & 0 & + \\ * & 0 & 0 & 0 \end{bmatrix}.$$

As shown in [6, Proposition 3.5], there is no complete signing of $\mathcal{A}_4$ that is IAP. Furthermore, no proper subpattern of $\mathcal{A}_4$ is IAP (otherwise the $\{0,*\}$ relaxation of such a subpattern would be IAP and this would contradict [6, Theorem 2.5]). Note that the following matrices in $Q(\mathcal{A}_4)$ have respective inertias $(4,0,0)$, $(0,4,0)$, $(0,0,4)$, $(3,1,0)$, $(0,1,3)$, $(0,3,1)$, $(1,3,0)$, $(3,0,1)$, $(1,0,3)$, $(2,1,1)$, $(1,1,2)$, $(1,2,1)$, $(2,2,0)$, $(2,0,2)$ and $(0,2,2)$:

$$\begin{bmatrix} 3 & 4 & 3 & 0 \\ -4 & -2 & -1 & 0 \\ 2 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 3 & 1 & 0 \\ -4 & -4 & -1 & 0 \\ 1 & 0 & 0 & 2 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 3 & 2 & 0 \\ -4 & -2 & -1 & 0 \\ 2 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ -1 & -1 & -2 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 0 \\ -4 & -2 & -2 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 1 & 0 \\ -3 & -2 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix},$$

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$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ -3 & -2 & -2 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 2 & 2 & 0 \\ -4 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 1 & 0 \\ -4 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ -1 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 0 \\ -4 & -3 & -4 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 0 \\ -1 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ -1 & -2 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 2 & 1 & 0 \\ -3 & -1 & -1 & 0 \\ 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 & 1 & 0 \\ -4 & -3 & -1 & 0 \\ 1 & 0 & 0 & 3 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, $\mathcal{A}_4$ is IAP. Furthermore, $\mathcal{A}_4$ is a minimal IAP with respect to both subpatterns and signings over $\{0,+,-,*\}$. Note that $\mathcal{A}_4$ is also PPI, PS and PPI$^+$. However, $\mathcal{A}_4$ is not PN, PN$^+$, rIAP or SAP, as any $A \in Q(\mathcal{A}_4)$ with $\det(A) = 0$ is forced to have a nonzero coefficient of x in its characteristic polynomial. This also implies that $\mathcal{A}_4$ does not allow the refined inertias $(0,0,2,2)$ or $(0,0,4,0)$ and hence, Theorem 2.13 cannot be used to show $\mathcal{A}_4$ is IAP. This example demonstrates that when classifying $\{0,+,-,*\}$-patterns that are IAP, it is not sufficient to characterize the sign patterns that are IAP and then take relaxations.

4. Discussion on the $2n$-conjecture. If $\mathcal{A}$ is an irreducible sign pattern or zero-nonzero pattern that is SAP, then it can be shown that the number of nonzero symbols in $\mathcal{A}$ is at least $2n-1$ (see [2]). A $\{0,+,-,*\}$-pattern that is SAP with exactly $2n-1$ nonzeros is not known, however, many classes of patterns that are SAP with $2n$ nonzero symbols have appeared in the literature. The question of the existence of a $\{0,+,-,*\}$-pattern that is SAP with $2n-1$ nonzeros is known as the $2n$-conjecture and is usually stated to include reducible patterns:

Conjecture 4.1. *Let $S \subseteq \{0,+,-,*\}$ and $\mathcal{A}$ be an S-pattern of order n that is SAP. Then $\mathcal{A}$ has at least $2n$ nonzero entries.*

The proof in [2] that an irreducible spectrally arbitrary sign pattern requires $2n-1$ nonzeros can be extended to include $\mathbb{S}$-patterns by noting that $n-1$ off-diagonal elements can be scaled to lie in the set $\{-1,0,1\}$. In particular, for any labelled digraph D, let $G(D)$ denote the underlying multigraph of D, that is, the graph obtained from D by ignoring the direction of each arc. Observe that if T is a subdigraph of $\mathcal{G}(A)$ such that the underlying multigraph of T is a forest, then $\mathcal{A}$ has a realization that is positive diagonally similar to A such that each entry corresponding

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to an arc of T lies in the set $\{-1, 0, 1\}$. Furthermore, if the sign of each arc in T belongs to the set $\{+, -, *\}$, then $\mathcal{A}$ has a realization that is positive diagonally similar to A such that each entry corresponding to an arc of T lies in the set $\{-1, 1\}$.

LEMMA 4.2. *Let $\mathcal{A}$ be a pattern of order n and $A \in Q(\mathcal{A})$.*

(1) If $\mathcal{A}$ requires irreducibility, then $\mathcal{A}$ must have a realization with at least $n-1$ off-diagonal entries in $\{-1, 1\}$ that is positive diagonally similar to A.
(2) If $\mathcal{A}$ allows irreducibility, then $\mathcal{A}$ must have a realization with at least $n-1$ off-diagonal entries in $\{-1, 0, 1\}$ that is positive diagonally similar to A. Furthermore, if $A \in Q(\mathcal{A})$ is an irreducible matrix, then $\mathcal{A}$ must have a realization with at least $n-1$ off-diagonal entries in $\{-1, 1\}$ that is positive diagonally similar to A.

Lemma 4.2 along with the proof of [2, Theorem 6.2] gives the following consequence.

THEOREM 4.3. *Let $S \subseteq \mathbb{S}$ and $\mathcal{A}$ be an S-pattern of order n that is SAP and allows irreducibility. Then $\mathcal{A}$ has at least $2n-1$ nonzero entries. Furthermore, there exist $\mathbb{S}$-patterns that allow irreducibility and are SAP with exactly $2n-1$ nonzero entries.*

An $\mathbb{S}$-pattern that is SAP and allows irreducibility with exactly $2n-1$ nonzero entries is the companion pattern in Example 2.7. However, the companion pattern also has only one nonzero symbol on the diagonal and does not require irreducibility. This motivates the following questions.

QUESTION 4.4. Does there exist an $\mathbb{S}$-pattern $\mathcal{A}$ of order n that allows irreducibility and with exactly $2n-1$ nonzero entries such that $\mathcal{A}$ is SAP and $\mathcal{A}$ has at least two nonzero entries on the main diagonal?

QUESTION 4.5. Does there exist an $\mathbb{S}$-pattern $\mathcal{A}$ of order n that requires irreducibility and is SAP with exactly $2n-1$ nonzero entries?

Acknowledgment. The authors thank the referee for a careful reading and helpful suggestions.

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