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SOME MAJORIZATION INEQUALITIES FOR CONEIGENVALUES*

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Abstract. A new notion of coneigenvalue was introduced by Ikramov in [Kh.D. Ikramov. On pseudo-eigenvalues and singular numbers of a complex square matrix (in Russian). *Zap. Nauchn. Semin. POMI*, 334:111–120, 2006.]. This paper presents some majorization inequalities for coneigenvalues, which extend some classical majorization relations for eigenvalues and singular values, and may serve as a basis for further investigations in this area.

Key words. Majorization, Coneigenvalues, Conjugate normal matrices.

AMS subject classifications. 15A21, 15F42.

1. Preliminaries. The notation $M_n(\mathbb{C})$ means the set of square $n \times n$ complex matrices. For $A \in M_n(\mathbb{C})$, A^T stands for the transpose of A, A^* is the transpose conjugate of A, i.e., $A^* = \overline{A}^T = \overline{A^T}$; the real part (or Hermitian part) of A is denoted by $\mathrm{Re}(A) = \frac{A+A^*}{2}$; A is normal if $A^*A = AA^*$ and is Hermitian if $A = A^*$. Let $\lambda(A)$, $\sigma(A)$ denote the eigenvalue vector, singular value vector of A, respectively, i.e.,

$$\lambda(A) = (\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A)),$$
$$\sigma(A) = (\sigma_1(A), \sigma_2(A), \ldots, \sigma_n(A)),$$

though sometimes $\lambda(A)$ (resp., $\sigma(A)$) is also used to denote the set of eigenvalues (resp., singular values) of A.

We begin with a brief review of the weak majorization and weak log-majorization orders (see [8]). For a real vector $x = (x_1, x_2, \ldots, x_n)$, let $x^{\downarrow}$ be the vector obtained by rearranging the coordinates of x in decreasing order. Thus, $x_1^{\downarrow} \ge x_2^{\downarrow} \ge \cdots \ge x_n^{\downarrow}$.

Definition 1.1. Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be two real vectors. Then we say that x is *weakly majorized by* y, denoted by $x \prec_w y$ (the same as $y \succ_w x$), if $\sum_{j=1}^k x_j^{\downarrow} \le \sum_{j=1}^k y_j^{\downarrow}$ for all k: $1 \le k \le n$. We say that x is *majorized*

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by y, denoted by $x \prec y$ (or $y \succ x$), if further $\sum_{j=1}^n x_j = \sum_{j=1}^n y_j$.

DEFINITION 1.2. Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be two vectors with non-negative entries. Then we say that x is *weakly log-majorized by* y, denoted by $x \prec_{w\log} y$ (the same as $y \succ_{w\log} x$), if $\prod_{j=1}^k x_j^\downarrow \le \prod_{j=1}^k y_j^\downarrow$ for all k: $1 \le k \le n$. We say that x is *log-majorized by* y, denoted by $x \prec_{\log} y$ (or $y \succ_{\log} x$), if further $\prod_{j=1}^n x_j = \prod_{j=1}^n y_j$.

For a complex vector $x = (x_1, x_2, \ldots, x_n)$, its entrywise real part and absolute value are defined by

$$\mathrm{Re}(x) = (\mathrm{Re}(x_1), \mathrm{Re}(x_2), \ldots, \mathrm{Re}(x_n)),$$

$$|x| = (|x_1|, |x_2|, \ldots, |x_n|),$$

respectively. Moreover, if all the entries of x are real and nonnegative, the notation x^r $(r \ge 0)$ means the entrywise rth power of x.

DEFINITION 1.3. A matrix $A \in M_n(\mathbb{C})$ is said to be *conjugate-normal* if

$$AA^* = \overline{A^*A}.$$

In particular, complex symmetric, skew-symmetric, and unitary matrices are special subclasses of conjugate-normal matrices. It seems that the term 'conjugate-normal matrices' was first introduced in [11]. For more properties and characterizations of this kind of matrices, we refer to [3].

2. Introduction.

For $A \in M_n(\mathbb{C})$, define $B = \overline{A}A$. An early result of Djoković [2] says B is similar to R^2, where R is a real matrix. Thus, $\lambda(B) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is symmetric with respect to the real axis and the negative eigenvalues of B (if any) are of even algebraic multiplicity, see also [4].

DEFINITION 2.1. [7] The *coneigenvalues* of $A \in M_n(\mathbb{C})$ are n scalars $\mu_1, \mu_2, \ldots,$ μ_n obtained as follows:

1. If $\lambda_k \in \lambda(B)$ does not lie on the negative real semiaxis, then the corresponding coneigenvalue μ_k is defined as the square root of λ_k with a nonnegative real part. The multiplicity of μ_k is set equal to that of λ_k.
2. With a real negative $\lambda_k \in \lambda(B)$, we associate two conjugate purely imaginary coneigenvalues (i.e., the two square roots of λ_k). The multiplicity of each coneigenvalue is set equal to half the multiplicity of λ_k.

For $A \in M_n(\mathbb{C})$, the vector of its coneigenvalues will be denoted by

$$\mu(A) = (\mu_1(A), \mu_2(A), \ldots, \mu_n(A)).$$

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In the sequel, we will briefly review some known properties related to coneigenvalues.

Define the matrix $\widehat{A} = \begin{bmatrix} 0 & A \\ \overline{A} & 0 \end{bmatrix}$.

Proposition 2.2. [7] *If $\mu_1, \mu_2, \ldots, \mu_n$ are the coneigenvalues of an $n \times n$ matrix A, then*

$$\lambda(\widehat{A}) = (\mu(A), -\mu(A)).$$

Proposition 2.3. [7] *Let A be a conjugate-normal matrix. Then the coneigenvalues of the matrices $\frac{A+A^T}{2}$ and $\frac{A-A^T}{2}$ are the real and imaginary parts, respectively, of the coneigenvalues of A.*

The purpose of this paper is to extend the following classical eigenvalue majorization results to the coneigenvalue case.

Theorem 2.4. (See, e.g., [5]) *Let $A \in M_n(\mathbb{C})$. Then*

$$\lambda\left(\mathrm{Re}(A)\right) \succ \mathrm{Re}(\lambda(A)), \tag{2.1}$$

$$\sigma(A) \succ_{\log} |\lambda(A)|. \tag{2.2}$$

Theorem 2.5. (See, e.g., [5]) *Let $A, B \in M_n(\mathbb{C})$ be Hermitian. Then*

$$\lambda^{\downarrow}(A) + \lambda^{\downarrow}(B) \succ \lambda(A+B), \tag{2.3}$$

$$\lambda(A) \succ \lambda^{\downarrow}(A+B) - \lambda^{\downarrow}(B). \tag{2.4}$$

Applying (2.3) to $\tilde{A} = \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}$ and $\tilde{B} = \begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix}$ gives the analogous majorization for singular values.

Corollary 2.6. (See, e.g., [12]) *Let $A, B \in M_n(\mathbb{C})$. Then*

$$\sigma^{\downarrow}(A) + \sigma^{\downarrow}(B) \succ \sigma(A+B). \tag{2.5}$$

The next proposition shows that (2.3) can be extended to the case of normal matrices.

Proposition 2.7. *Let $A, B \in M_n(\mathbb{C})$ be normal matrices. Then*

$$(\mathrm{Re}(\lambda(A)))^{\downarrow} + (\mathrm{Re}(\lambda(B)))^{\downarrow} \succ \mathrm{Re}(\lambda(A+B)).$$

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Proof.

$$\begin{aligned}\mathrm{Re}(\lambda(A+B)) &\prec \lambda(\mathrm{Re}(A+B))\\ &= \lambda(\mathrm{Re}(A)+\mathrm{Re}(B))\\ &\prec \lambda^{\downarrow}(\mathrm{Re}(A)) + \lambda^{\downarrow}(\mathrm{Re}(B))\\ &= (\mathrm{Re}(\lambda(A)))^{\downarrow} + (\mathrm{Re}(\lambda(B)))^{\downarrow},\end{aligned}$$

where the first majorization is by (2.1) and the second majorization is by (2.3). □

It is natural to ask whether (2.4) also has such an analogue, i.e., if $A, B \in M_n(\mathbb{C})$ are normal matrices, do we have

$$\mathrm{Re}(\lambda(A)) \succ (\mathrm{Re}(\lambda(A+B)))^{\downarrow} - (\mathrm{Re}(\lambda(B)))^{\downarrow}?$$

Unfortunately, the answer is no as the following example shows.

Example 2.8. Taking

$$A = \begin{bmatrix} 0 & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

obviously, A and B are normal. Simple calculation gives

$$\lambda(A) = \{\sqrt{3}i/2, -\sqrt{3}i/2\}, \qquad \lambda(B) = \{1, -1\}, \qquad \lambda(A+B) = \{1/2, -1/2\}.$$

Thus,

$$\mathrm{Re}(\lambda(A)) = (0,0), \qquad (\mathrm{Re}(\lambda(A+B)))^{\downarrow} - (\mathrm{Re}(\lambda(B)))^{\downarrow} = (1/2, -1/2).$$

3. Main results. We start with some observations.

Observation 3.1. The coneigenvalues of a complex symmetric matrix are nonnegative, the coneigenvalues of a complex skew symmetric matrix are purely imaginary.

Proof. If A is complex symmetric, then $\overline{A}A = \overline{A^T}A = A^*A$, thus the coneigenvalues of A coincide with the singular values of A and are thus all nonnegative. The case A being complex skew symmetric can be proved similarly. □

Observation 3.2. Let $A \in M_n(\mathbb{C})$. Then $|\det(A)| = \prod_{k=1}^n \mu_k(A)$. However, we generally do not have $\mathrm{tr}\, A = \sum_{k=1}^n \mu_k(A)$ or $|\mathrm{tr}\, A| = \sum_{k=1}^n \mu_k(A)$.

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Proof. By definition of coneigenvalues, $\prod_{k=1}^n \mu_k^2(A) = \det(A\overline{A}) = |\det(A)|^2$. Moreover, $\mathrm{Re}(\mu_k(A)) \geq 0$ for all k and the multiplicity of $\overline{\mu_k}(A)$ coincides with that of $\mu_k(A)$. Thus, $\prod_{k=1}^n \mu_k(A) \geq 0$. Taking the square root leads to the first claim. For the second claim, we take $A = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$. Then $\operatorname{tr} A = 1 + i$, $|\operatorname{tr} A| = \sqrt{2}$ and $\sum_{k=1}^2 \mu_k(A) = 2$. ◻

LEMMA 3.3. *Let x, y be two nonnegative vectors of the same size. Denote $\widehat{x} = (x, -x)$, $\widehat{y} = (y, -y)$. If $\widehat{x} \prec \widehat{y}$, then*

$$x \prec_w y.$$

Proof. This is trivial by definition of majorization. ◻

LEMMA 3.4. *Let x, y be two nonnegative vectors of the same size. Denote $\widehat{x} = (x, x)$, $\widehat{y} = (y, y)$. If $\widehat{x} \prec_{\log} \widehat{y}$, then*

$$x \prec_{\log} y.$$

Proof. Trivial. ◻

THEOREM 3.5. *Let $A \in M_n(\mathbb{C})$. Then*

$$\mu\left(\frac{A + A^T}{2}\right) \succ_w \mathrm{Re}(\mu(A)). \tag{3.1}$$

Proof. It is clear that the left hand side of (3.1) is a nonnegative vector, since $\frac{A+A^T}{2}$ is complex symmetric.

$$\begin{aligned}
\mathrm{Re}\left(\lambda\left(\begin{bmatrix} 0 & A \\ \overline{A} & 0 \end{bmatrix}\right)\right) &\prec \lambda\left(\mathrm{Re}\left(\begin{bmatrix} 0 & A \\ \overline{A} & 0 \end{bmatrix}\right)\right) \\
&= \lambda\left(\begin{bmatrix} 0 & \frac{A+(\overline{A})^*}{2} \\ \frac{\overline{A}+A^*}{2} & 0 \end{bmatrix}\right) \\
&= \lambda\left(\begin{bmatrix} 0 & \frac{A+A^T}{2} \\ \frac{\overline{A+A^T}}{2} & 0 \end{bmatrix}\right).
\end{aligned}$$

That is,

$$\lambda\left(\begin{bmatrix} 0 & \frac{A+A^T}{2} \\ \frac{\overline{A+A^T}}{2} & 0 \end{bmatrix}\right) \succ \mathrm{Re}\left(\lambda\left(\begin{bmatrix} 0 & A \\ \overline{A} & 0 \end{bmatrix}\right)\right).$$

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By Lemma 3.3, the desired result holds. □

We cannot replace "$\succ_w$" by "$\succ$" in (3.1) as the following example shows.

EXAMPLE 3.6. Let $A = \begin{bmatrix} 1 & 2i \\ 0 & 1 \end{bmatrix}$. Then $\mu(A) = (1,1)$, $\mu\left(\frac{A+A^T}{2}\right) = \sigma\left(\frac{A+A^T}{2}\right) = (\sqrt{2}, \sqrt{2})$. Thus, $\sum_{k=1}^{2} \mu_k\left(\frac{A+A^T}{2}\right) > \sum_{k=1}^{2} \mathrm{Re}(\mu(A))$ in this case.

THEOREM 3.7. *Let $A \in M_n(\mathbb{C})$. Then*

$$\sigma(A) \succ_{\log} |\mu(A)|.$$

Proof. By Proposition 2.2, we have

$$\begin{aligned} (|\mu(A)|, |\mu(A)|) &= |\lambda(\widehat{A})| \\ &\prec_{\log} \sigma(\widehat{A}) \\ &= \lambda^{1/2}\left(\begin{bmatrix} 0 & A \\ \overline{A} & 0 \end{bmatrix}^* \begin{bmatrix} 0 & A \\ \overline{A} & 0 \end{bmatrix}\right) \\ &= \lambda^{1/2}\left(\begin{bmatrix} \overline{A^*A} & 0 \\ 0 & A^*A \end{bmatrix}\right) \\ &= (\sigma(A), \sigma(A)), \end{aligned}$$

where the majorization is by (2.2). Then Lemma 3.4 gives the desired result. □

By the well known fact that log majorization implies weak majorization (see, e.g., [8]), we have the following corollary, which was the first majorization result discovered on coneigenvalues.

COROLLARY 3.8. [7] *Let $A \in M_n(\mathbb{C})$. Then for any $p \geq 0$,*

$$\sigma^p(A) \succ_w |\mu^p(A)|. \tag{3.2}$$

The next corollary is an analogue of the generalized Schur inequality [10] with coneigenvalues involved.

COROLLARY 3.9. *Let $A = [a_{jk}] \in M_n(\mathbb{C})$. Then for any $0 \leq p \leq 2$,*

$$\sum_{j,k=1}^{n} |a_{jk}|^p \geq \sum_{k=1}^{n} \mu_k^p(A). \tag{3.3}$$

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Proof. Note that the right hand side of (3.3) is real. Mond and Pečarić [9] have showed that

$$\sum_{j,k=1}^{n} |a_{jk}|^p \geq \sum_{k=1}^{n} \sigma_k^p(A) \tag{3.4}$$

for $0 \leq p \leq 2$. Thus, (3.3) follows immediately by (3.2). □

Remark 3.10. As pointed out by a referee, though Petri and Ikramov [10] only presented (3.4) for $p \geq 1$ and later a much simpler proof was given in [6], the proofs given there held also for $0 \leq p < 1$.

Theorem 3.11. *Let $A, B \in M_n(\mathbb{C})$ be conjugate normal matrices. Then*

$$(\mathrm{Re}(\mu(A)))^{\downarrow} + (\mathrm{Re}(\mu(B)))^{\downarrow} \succ_w \mathrm{Re}\left(\mu\left(A+B\right)\right).$$

Proof. By Theorem 3.5, we have

$$\begin{aligned}
\mathrm{Re}\left(\mu\left(A+B\right)\right) \prec_w \mu &\left(\frac{A+B+(A+B)^T}{2}\right) \\
= \sigma &\left(\frac{A+B+(A+B)^T}{2}\right) \\
\prec_w \sigma^{\downarrow} &\left(\frac{A+A^T}{2}\right) + \sigma^{\downarrow}\left(\frac{B+B^T}{2}\right) \\
= \mu^{\downarrow} &\left(\frac{A+A^T}{2}\right) + \mu^{\downarrow}\left(\frac{B+B^T}{2}\right) \\
= &\ (\mathrm{Re}(\mu(A)))^{\downarrow} + (\mathrm{Re}(\mu(B)))^{\downarrow},
\end{aligned}$$

where the second weak majorization is by (2.5) and the last equality is by Proposition 2.3. □

Corollary 3.12. *Let $A, B \in M_n(\mathbb{C})$ be symmetric matrices. Then*

$$\mu^{\downarrow}(A) + \mu^{\downarrow}(B) \succ_w \mu\left(A+B\right). \tag{3.5}$$

Remark 3.13. Readers should be able to observe that (3.5) is the same as (2.5).

Theorem 3.14. *Let $A, B \in M_n(\mathbb{C})$ be symmetric matrices. Then*

$$\mu(A) \succ_w |(\mu^{\downarrow}(A+B) - \mu^{\downarrow}(B))|. \tag{3.6}$$

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Proof. Since A, B are symmetric, (3.6) is the same as

$$\sigma(A) \succ_w |(\sigma^{\downarrow}(A+B) - \sigma^{\downarrow}(B))|. \tag{3.7}$$

(3.7) is the singular value counterpart of (2.4) and can be found in, e.g., [1]. $\square$

4. Concluding remarks. For $A \in M_n(\mathbb{C})$, we know that one alternative definition for singular values of A is the nonnegative eigenvalues of the augmented matrix $\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}$. Given the present notion of *coneigenvalue*, the notion of its counterpart, say *consingular value*, seems lacking. What would be a possible definition for *consingular value*? We provide one here, analogous to the definition of singular values in terms of eigenvalues of an augmented matrix.

DEFINITION 4.1. Let $A \in M_n(\mathbb{C})$. The *consingular values* of A are the n scalars $\gamma_1(A), \gamma_2(A), \ldots, \gamma_n(A)$ defined by the coneigenvalues of $\begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}$, with each consingular value taking half the multiplicity of the corresponding coneigenvalue.

We can see that, since $\begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}$ is symmetric,

$$\begin{aligned}
\mu\left(\begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}\right) &= \sigma\left(\begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}\right) \\
&= \lambda^{1/2}\left(\begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}^* \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}\right) \\
&= \lambda^{1/2}\left(\begin{bmatrix} (AA^*)^T & 0 \\ 0 & A^*A \end{bmatrix}\right) \\
&= \sigma\left(\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}\right).
\end{aligned}$$

Thus, with our definition, we have:

The consingular values of a matrix are exactly its singular values.

Theorem 3.7 can thus be rephrased as:

The consingular values of a matrix log majorize its coneigenvalues in absolute value.

Majorization relations for eigenvalues or singular values are currently still an active area of study. It is expected that more results on coneigenvalue majorization will be discovered in the near future.

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REFERENCES

[1] T. Ando. Majorizations and inequalities in matrix theory. *Linear Algebra Appl.*, 199:17–67, 1994.

[2] D.Z. Djoković. On some representations of matrices. *Linear Multilinear Algebra*, 4:33–40, 1976.

[3] H. Faßender and Kh.D. Ikramov. Conjugate-normal matrices: A survey. *Linear Algebra Appl.*, 429:1425–1441, 2008.

[4] R.A. Horn and C.R. Johnson. *Matrix Analysis.* Cambridge University Press, Cambridge, 1990.

[5] R.A. Horn and C.R. Johnson. *Topics in Matrix Analysis.* Cambridge University Press, Cambridge, 1991.

[6] Kh.D. Ikramov. A simple proof of the generalized Schur inequality. *Linear Algebra Appl.*, 199:143–149, 1994.

[7] Kh.D. Ikramov. On pseudo-eigenvalues and singular numbers of a complex square matrix, (in Russian). *Zap. Nauchn. Semin. POMI*, 334:111–120, 2006. Translation in *J. Math. Sci. (N.Y.)*, 141:1639–1642, 2007.

[8] A.W. Marshall, I. Olkin, and B.C. Arnold. *Inequalities: Theory of Majorization and its Applications*, second edition. Springer Series in Statistics, Springer, New York, 2011.

[9] B. Mond and J.E. Pečarić. An extension of the generalised Schur inequality. *Bull. Austral. Math. Soc.*, 52:341–344, 1995.

[10] N.V. Petri and Kh.D. Ikramov. Extremal properties of some matrix norms. *Comput. Math. and Math. Phys.*, 8:219–230, 1968.

[11] M. Vujičić, F. Herbut, and G. Vujičić. Canonical form for matrices under unitary congruence transformations. I: Conjugate-normal matrices. *SIAM J. Appl. Math.*, 23:225–238, 1972.

[12] F. Zhang. *Matrix Theory: Basic Results and Techniques*, second edition. Universitext, Springer-Verlag, New York, 2011.