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EXPLICIT POLAR DECOMPOSITION OF COMPANION MATRICES*

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Abstract. An explicit formula for the polar decomposition of an $n \times n$ nonsingular companion matrix is derived. The proof involves the largest and smallest singular values of the companion matrix.

Key words. companion matrices, polar decomposition, singular values

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1. Introduction.

Let

$$f(z) = z^n - a_{n-1}z^{n-1} - \cdots - a_1 z - a_0, \quad a_0 \neq 0,$$

be a complex polynomial and

$$C = \begin{bmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ a_0 & a_1 & \dots & a_{n-1} \end{bmatrix} \tag{1}$$

be an $n \times n$ nonsingular companion matrix associated with $f(z)$. Let $C = PU$ be the left polar decomposition of C with positive-definite P and unitary U. The singular values of C, i.e., the eigenvalues of P, are well known ([1], [5], [6]). They yield bounds for zeros and for products of zeros of $f(z)$ [6], and they are used for the computation of robustness measures in systems theory [5]. In view of the wide range of applications, both of the polar decomposition and of companion matrices, an explicit formula for $C = PU$ is useful. It is the purpose of this note to derive explicit expressions for the factors P and U in terms of the coefficients a_ν of $f(z)$. As companion matrices have been included in collections of test matrices (see e.g., Table I of [3]) our formula adds yet one more possibility for testing computational algorithms in numerical linear algebra. Our formula also shows that companion matrices belong to the class

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of matrices for which the polar decomposition is finitely computable. Whether all complex square matrices have that property is an open problem, which has been studied in [2].

2. Polar decomposition formula. Our main result, Theorem 2.1, is the explicit formula for P and U in the left polar decomposition of a nonsingular companion matrix C where the coefficients of the polynomial $f(z)$ form the last row.

THEOREM 2.1. *Let the companion matrix C in* (1) *be partitioned as*

$$C = \begin{bmatrix} 0 & I_{n-1} \\ a_0 & d^* \end{bmatrix},$$

with $a_0 \neq 0$ and $d^* = (a_1, \ldots, a_{n-1})$. Define

$$w = \left[(|a_0|+1)^2 + |a_1|^2 + \cdots + |a_{n-1}|^2\right]^{\frac{1}{2}} = \left[(|a_0|+1)^2 + \|d\|^2\right]^{\frac{1}{2}} \tag{2}$$

and

$$v = \frac{a_0}{|a_0| w} \begin{bmatrix} -d \\ 1 + |a_0| \end{bmatrix}. \tag{3}$$

Then

$$P = \frac{1}{w} \begin{bmatrix} wI_{n-1} - (w + |a_0| + 1)^{-1} dd^* & d \\ d^* & w^2 - |a_0| - 1 \end{bmatrix} \tag{4}$$

is positive definite and $P^2 = CC^*$. Assume $P = (p_1, \ldots, p_n)$ and set $U = (v, p_1, \ldots, p_{n-1})$. Then U is unitary and $C = PU$ is the left polar decomposition of (1).

To prove Theorem 2.1 we first consider the singular values of C, i.e. the nonnegative square roots of the eigenvalues of

$$CC^* = \begin{bmatrix} I_{n-1} & d \\ d^* & s \end{bmatrix}, \tag{5}$$

where

$$s = \sum_{i=0}^{n-1} |a_i|^2 = |a_0|^2 + \|d\|^2.$$

Set $a_n = 1$, and define

$$F(z) = z^2 - \left(\sum_{i=0}^{n} |a_i|^2\right) z + |a_0|^2. \tag{6}$$

The following result is known (see, e.g., [1, pp. 224–225], [5], [6]). To make our note self-contained we include a simple proof.

LEMMA 2.2. *Let* $0 < \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_n$ *be the singular values of* C. *Then* $\sigma_2 = \cdots = \sigma_{n-1} = 1$, *and* σ_1^2, σ_n^2 *are the zeros of* $F(z)$ *in* (6).

Proof. From (5) it follows that

$$\begin{aligned} (7)\quad \det(zI_n - CC^*) &= (z-s)\det[(z-1)I_{n-1}] - d^*\mathrm{adj}[(z-1)I_{n-1}]d \\ &= (z-1)^{n-2}\left[z^2 - (s+1)z + s - \|d\|^2\right] \\ &= (z-1)^{n-2}F(z). \end{aligned}$$

Thus CC^* has 1 as eigenvalue of multiplicity at least $(n-2)$. Since the eigenvalues of the principal submatrix I_{n-1} in (5) interlace those of CC^*, it follows that $\sigma_1^2 \leq 1 \leq \sigma_n^2$. □

Note that, as $F(z)$ in (6) is quadratic, the values of σ_1^2, σ_n^2 can be found explicitly in terms of $|a_0|^2$ and $\|d\|^2$, see [5, Th. 3.1]. Also $\sigma_1\sigma_n = |a_0|$ and $\sigma_1^2 + \sigma_n^2 = s+1$. These relations give $\sigma_1 + \sigma_n = w$, and $\|d\|^2 = s - \sigma_1^2\sigma_n^2 = -\left(\sigma_n^2 - 1\right)\left(\sigma_1^2 - 1\right)$. From (2) follows

$$(8)\qquad \|d\|^2 = (w + |a_0| + 1)(w - |a_0| - 1).$$

For the computation of the square root of CC^* only a symmetric 2×2 matrix has to be considered. The following can easily be verified.

LEMMA 2.3. *Let*

$$H = \begin{bmatrix} 1 & \|d\| \\ \|d\| & |a_0|^2 + \|d\|^2 \end{bmatrix}.$$

Then, $\det(zI - H) = F(z) = \left(z - \sigma_1^2\right)\left(z - \sigma_n^2\right)$, *and*

$$H^{\frac{1}{2}} = w^{-1}\begin{bmatrix} 1 + |a_0| & \|d\| \\ \|d\| & w^2 - |a_0| - 1 \end{bmatrix}.$$

Proof of Theorem 2.1. The case with $\sigma_1 = 1$ or $\sigma_n = 1$ is equivalent to $F(1) = 0$, or because of (7), equivalent to $d = 0$. In this case (5) implies

$$P = (CC^*)^{\frac{1}{2}} = \mathrm{diag}(1, \ldots, 1, |a_0|).$$

Furthermore $C = PU$ with P as above and

$$U = \begin{bmatrix} 0 & I_{n-1} \\ \frac{a_0}{|a_0|} & 0 \end{bmatrix},$$

agreeing with (4) and (3).

In the case $\sigma_1 < 1 < \sigma_n$, that is $d \neq 0$, we define vectors

$$v_1 = \frac{1}{\|d\|} \begin{bmatrix} d \\ 0 \end{bmatrix}, \quad v_n = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Then (5), and $CC^* v_1 = v_1 + \|d\| v_n$ and $CC^* v_n = \|d\| v_1 + s v_n$, imply

$$CC^* (v_1, v_n) = (v_1, v_n)\, H.$$

Now consider the eigenvalue 1 of CC^* and let $y_2, \ldots, y_{n-1}$ be an orthonormal set of eigenvectors of CC^* satisfying $CC^* y_i = y_i$, for $i = 2, \ldots, n-1$. Note that for each y_i we have $y_i^* = (x_i^*, 0)$ and $d^* x_i = 0$. Then $V = (y_2, \ldots, y_{n-1}, v_1, v_n)$ is a unitary matrix, and

$$V^* CC^* V = \begin{bmatrix} I_{n-2} & 0 \\ 0 & H \end{bmatrix}.$$

Hence

$$P = (CC^*)^{\frac{1}{2}} = V \begin{bmatrix} I_{n-2} & 0 \\ 0 & H^{\frac{1}{2}} \end{bmatrix} V^*,$$

where $H^{\frac{1}{2}}$ is given in Lemma 2.3. Thus

$$P = I_n + (v_1, v_n)(H^{\frac{1}{2}} - I_2) \begin{bmatrix} v_1^* \\ v_n^* \end{bmatrix}.$$

From (8) it follows that

$$H^{\frac{1}{2}} - I_2 = w^{-1} \begin{bmatrix} -\|d\|^2 (w + |a_0| + 1)^{-1} & \|d\| \\ \|d\| & w^2 - |a_0| - 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

On multiplication, the above expression for P yields (4).

For a nonsingular companion matrix C given by (1) it is well known that

$$C^{-1} = \begin{bmatrix} \frac{-d^*}{a_0} & \frac{1}{a_0} \\ I_{n-1} & 0 \end{bmatrix}$$

For the unitary factor of $C = PU$, we have $U = P(C^{-1})^*$. Hence

$$U = (p_1, \ldots, p_{n-1}, p_n) \begin{bmatrix} \frac{-d}{\bar{a}_0} & I_{n-1} \\ \frac{1}{\bar{a}_0} & 0 \end{bmatrix} = (v, p_1, \ldots, p_{n-1}),$$

with

$$v = \frac{1}{\bar{a}_0} P \begin{bmatrix} -d \\ 1 \end{bmatrix} = \frac{1}{\bar{a}_0 w} \begin{bmatrix} [-w + \|d\|^2 (w + |a_0| + 1)^{-1} + 1] d \\ -\|d\|^2 + w^2 - |a_0| - 1 \end{bmatrix}.$$

Using (8) yields (3) and completes the proof. □

It is well known (see, e.g., [4] or [7]) that for a given nonsingular matrix the unitary factors in the left and in the right polar decomposition are equal. Now define

$$\gamma = 1 - \frac{1}{w + |a_0| + 1}$$

and set

$$Q = \frac{1}{w} \begin{bmatrix} |a_0| + |a_0|^2 & \bar{a}_0 d^* \\ a_0 d & w I_{n-1} + \gamma d d^* \end{bmatrix}.$$

It is not difficult to verify that Q is positive definite and $Q^2 = C^* C$. Hence if C is nonsingular and U is given as in Theorem 2.1, then $C = UQ$ is the right polar decomposition of (1).

Let C_{lr}, C_{lc}, C_{fr}, C_{fc} be the companion matrices where the coefficients of the polynomial $f(z)$ form the last row, last column, first row, first column, respectively. So far in our note we have considered $C = C_{lr}$. Using the $n \times n$ permutation matrix (the reverse unit matrix) $K = (k_{ij})$ where $k_{i,n-i+1} = 1$, and 0 elsewhere, we note that

$$C_{lr} = C^T, \quad C_{fr} = KCK, \quad C_{fc} = KC^T K.$$

Hence the polar decompositions of the preceding three types of companion matrices are products that involve the matrices U, K, and P or Q. For any real nonsingular 2×2 matrix the right polar decomposition in closed form is given in [8].

There is a relation between the singular values σ_1 and σ_n of C and the zeros λ of the polynomial $f(z)$, namely $\sigma_1 \le |\lambda| \le \sigma_n$. Is it possible that the eigenvalues $e^{i\varphi_\nu}$, $\nu = 1, \ldots, n$, of the unitary factor U also provide information on the geometry of the zeros of $f(z)$?

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