

SOME TOPOLOGICAL PROPERTIES OF THE SET OF LINEAR PRESERVERS OF MAJORIZATION*

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Abstract. It is shown that the set of all bounded linear preservers of majorization on $\ell^p(I)$, for $p \geq 1$ and an infinite set I , is closed under the norm topology. Therefore, if $(T_n)_{n \in \mathbb{N}}$ is a sequence of linear preservers of majorization on $\ell^p(I)$ which converges to some bounded linear map $T : \ell^p(I) \rightarrow \ell^p(I)$, then T is also a preserver of majorization.

Key words. Majorization, Bounded linear preservers, Closedness.

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1. Introduction. The theory of majorization which was introduced in the beginning of the 20th century as a kind of comparison between two vectors x and y of $\mathbb{R}^n$, arises in different topics of mathematics such as matrix theory [1], graph theory [5, 7], operator theory [6], frame theory [2, 11], and combinatorics [12]. We also refer to the excellent text in this subject by Marshall and Olkin [9] for more applications.

For two vectors $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ in $\mathbb{R}^n$, x is majorized by y , denoted $x \prec y$, if $\sum_{i=1}^k x_i^\downarrow \leq \sum_{i=1}^k y_i^\downarrow$ for all $k = 1, \dots, n$ and $\sum_{i=1}^n x_i^\downarrow = \sum_{i=1}^n y_i^\downarrow$. Here, $(x_1^\downarrow, \dots, x_n^\downarrow)$ represents the non-increasing rearrangement of a vector $(x_1, \dots, x_n)$. There are several equivalent statements for the relation $x \prec y$ among which the expression according to the doubly stochastic matrices is of special importance for us. A $n \times n$ matrix D of non-negative entries is called doubly stochastic if all its row sums and column sums are equal to 1. Let $\mathcal{DS}(\mathbb{R}^n)$ denote the set of all $n \times n$ doubly stochastic matrices. It is known that $x \prec y$ for two vectors $x, y \in \mathbb{R}^n$, if and only if $x = Dy$ for some $D \in \mathcal{DS}(\mathbb{R}^n)$. We refer the reader to [9] for its proof and also other equivalent statements of majorization. This equivalent condition for the majorization relation makes it possible to extend this relation to other spaces such as the space of all $m \times n$ real matrices, or the space of $\ell^p(I) = \{f : I \rightarrow \mathbb{R} \mid \sum_{i \in I} |f(i)|^p < +\infty\}$ for a non-empty set I and $p \geq 1$. See, for example, [3] and [4, 10].

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In this paper, we consider the majorization relation on $\ell^p(I)$ and investigate some topological properties of the set of all bounded linear preservers of this relation. Let us first consider the definition of majorization on the real Banach space $\ell^p(I)$, according to [3].

For a non-empty set I and a real $p \geq 1$, a bounded linear operator $D : \ell^p(I) \rightarrow \ell^p(I)$ is *doubly stochastic* if it is positive, in the sense that $Df \geq 0$ for all $f \geq 0$, and

$$\forall i \in I, \quad \sum_{j \in I} De_j(i) = 1 \quad \text{and} \quad \forall j \in I, \quad \sum_{i \in I} De_j(i) = 1.$$

Here, for each $i \in I$, $e_i \in \ell^p(I)$ represents the function $e_i(k) = 0$ if $k \neq i$ and $e_i(i) = 1$. The set of all doubly stochastic operators on $\ell^p(I)$ is denoted by $\mathcal{DS}(\ell^p(I))$. A bounded linear map $P : \ell^p(I) \rightarrow \ell^p(I)$ is a *permutation* if there exists a bijection $\theta : I \rightarrow I$ such that

$$\forall f \in \ell^p(I), \quad \forall i \in I, \quad (Pf)(i) = f(\theta(i)).$$

Clearly, every permutation is a doubly stochastic operator.

For $f, g \in \ell^p(I)$, f is *majorized* by g , denoted $f \prec g$, if $f = Dg$ for some $D \in \mathcal{DS}(\ell^p(I))$. Hence, for example, if $f = Pg$ for some permutation $P : \ell^p(I) \rightarrow \ell^p(I)$, then $f \prec g$. Note that in this special case, since a permutation is invertible and its inverse is also a permutation, we have also the reverse relation, i.e., $g \prec f$. For f and g in $\ell^p(I)$, we use the notation $f \sim g$ whenever $f \prec g$ and $g \prec f$. It is known that $f \sim g$ if and only if there exists a permutation P for which $f = Pg$ [3, Theorem 3.5] or, equivalently, if there is a bijection $\theta : I \rightarrow I$ for which $f(i) = g(\theta(i))$ for all $i \in I$.

A bounded linear map $T : \ell^p(I) \rightarrow \ell^p(I)$ is called a *preserver of majorization* if whenever $f \prec g$, for $f, g \in \ell^p(I)$, then $Tf \prec Tg$. The set of all bounded linear preservers of majorization on $\ell^p(I)$ is denoted by $\mathcal{M}_{pr}(\ell^p(I))$. Our aim is to prove that this set is a closed subset of $\mathcal{B}(\ell^p(I))$, the set of all bounded linear operators on $\ell^p(I)$, under the norm topology.

In the case where I is a finite set of, for example, n elements there is a natural isomorphism $L : (\ell^p(I), \|\cdot\|_p) \rightarrow (\mathbb{R}^n, \|\cdot\|_1)$, where $\|\cdot\|_1 : \mathbb{R}^n \rightarrow \mathbb{R}$ is defined for each $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ by $\|x\|_1 = \sum_{i=1}^n |x_i|$. It is easily seen that this isomorphism preserves majorization, i.e.,

$$\forall f, g \in \ell^p(I), \quad f \prec g \iff L(f) \prec L(g).$$

Thus, one may define a one-to-one homeomorphism between the two sets $\mathcal{M}_{pr}(\ell^p(I))$ and $\mathcal{M}_{pr}(\mathbb{R}^n)$ given by $T \mapsto LTL^{-1}$, for each $T \in \mathcal{M}_{pr}(\ell^p(I))$. Hence, in order to prove that $\mathcal{M}_{pr}(\ell^p(I))$ is closed, it suffices to consider the same problem for $\mathcal{M}_{pr}(\mathbb{R}^n)$.

It is easily seen that if $D : (\mathbb{R}^n, \|\cdot\|_1) \rightarrow (\mathbb{R}^n, \|\cdot\|_1)$ is a doubly stochastic operator then $\|D\| = 1$, and that the set $\mathcal{DS}(\mathbb{R}^n)$ is closed in $\mathcal{B}(\mathbb{R}^n)$. Since the unit

ball of $\mathcal{B}(\mathbb{R}^n)$ is compact, the set $\mathcal{DS}(\mathbb{R}^n)$ is also compact. Now suppose $(T_k)_{k \in \mathbb{N}}$ is a sequence in $\mathcal{M}_{P_r}(\mathbb{R}^n)$ which converges to a linear map $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$. For x and y in $\mathbb{R}^n$ with $x \prec y$, $T_k x \prec T_k y$ for each $k \in \mathbb{N}$. Hence, there exists $D_k \in \mathcal{DS}(\mathbb{R}^n)$ such that $T_k x = D_k T_k y$. According to the previous argument, the sequence $(D_k)_{k \in \mathbb{N}}$ contains a convergent subsequence in $\mathcal{DS}(\mathbb{R}^n)$. Therefore, there exists $D \in \mathcal{DS}(\mathbb{R}^n)$ and a subsequence $(D_{k_m})_{m \in \mathbb{N}}$ such that $D_{k_m} \rightarrow D$. Using the fact that

$$\forall m \in \mathbb{N}, \quad T_{k_m} x = D_{k_m} T_{k_m} y,$$

and passing to the limit, we obtain the relation $Tx = DTy$ which shows that $Tx \prec Ty$, i.e., $T \in \mathcal{M}_{P_r}(\mathbb{R}^n)$. This proves our claim in the case where I is finite.

Clearly, in this argument the compactness of the set $\mathcal{DS}(\mathbb{R}^n)$ plays a crucial role. As the following example shows, in the case where I is an infinite set, the set $\mathcal{DS}(\ell^p(I))$ is not closed.

EXAMPLE 1.1. Let I be an infinite set and $J = \{i_k \mid k \in \mathbb{N}\} \subset I$ be an infinite countable subset. For each $i, j \in I$ and $n \in \mathbb{N}$, let the real number $d_{ij,n}$ be defined as follows.

$$d_{ij,n} = \begin{cases} \frac{1}{n} & \text{if } i = i_1 \text{ and } j \in \{i_1, \dots, i_n\}, \\ 0 & \text{if } i = i_1 \text{ and } j \in I \setminus \{i_1, \dots, i_n\}, \\ 1 - \frac{1}{n} & \text{if } i = i_k, \text{ for some } k \geq 2, \text{ and } j = i_{k-1}, \\ \frac{1}{n} & \text{if } i = i_k, \text{ for some } k \geq 2, \text{ and } j = i_{n+k-1}, \\ 0 & \text{if } i = i_k, \text{ for some } k \geq 2, \text{ and } j \in I \setminus \{i_{k-1}, i_{n+k-1}\}, \\ 1 & \text{if } i \notin J \text{ and } j = i, \\ 0 & \text{if } i \notin J \text{ and } j \neq i. \end{cases}$$

Then a simple calculation shows that for each $n \in \mathbb{N}$,

$$\forall i \in I, \sum_{j \in I} d_{ij,n} = 1 \quad \text{and} \quad \forall j \in I, \sum_{i \in I} d_{ij,n} = 1.$$

By Proposition 2.6 of [3], for each $n \in \mathbb{N}$ there exists a doubly stochastic operator $D_n : \ell^p(I) \rightarrow \ell^p(I)$ such that $D_n e_j(i) = d_{ij,n}$, for every $i, j \in I$. For each $f \in \ell^p(I)$, we have

$$D_n f = \left(\frac{1}{n} \sum_{k=1}^n f(i_k) \right) e_{i_1} + \sum_{k=1}^{\infty} \left(\left(1 - \frac{1}{n} \right) f(i_k) + \frac{1}{n} f(i_{n+k}) \right) e_{i_{k+1}} + \sum_{i \in I \setminus J} f(i) e_i.$$

If the bounded linear operator $D : \ell^p(I) \rightarrow \ell^p(I)$ is defined by

$$\forall f \in \ell^p(I), \quad Df = \sum_{k=1}^{\infty} f(i_k) e_{i_{k+1}} + \sum_{i \in I \setminus J} f(i) e_i,$$

then for each f ,

$$\begin{aligned}\|D_n f - Df\|_p^p &= \left| \frac{1}{n} \sum_{k=1}^n f(i_k) \right|^p + \frac{1}{n^p} \sum_{k=1}^{\infty} \left| f(i_{n+k}) - f(i_k) \right|^p \\ &\leq \frac{\|f\|_p^p}{n} + \frac{2^p \|f\|_p^p}{n^p} \leq \frac{3^p \|f\|_p^p}{n}.\end{aligned}$$

Therefore, $\|D_n - D\| \leq \frac{3}{\sqrt[n]{n}}$, which shows that $D_n \rightarrow D$. However, since $\sum_{j \in I} De_j(i_1) = 0$, D is not a doubly stochastic operator.

As the previous example indicates, the above mentioned proof for the closedness of $\mathcal{M}_{pr}(\ell^p(I))$ cannot be used in the case where I is an infinite set. In the next section, we give a proof of this property based on the characterization of linear preservers of majorization obtained in [3].

2. Closedness of $\mathcal{M}_{pr}(\ell^p(I))$. Throughout this section, we assume that I is an infinite set. For $f \in \ell^p(I)$, the support of f , i.e., the set $\{i \in I \mid f(i) \neq 0\}$, and its range are denoted, respectively, by $\text{supp}(f)$ and $\text{Im}(f)$. The following lemma plays a crucial role in this paper.

LEMMA 2.1. *Let $p \geq 1$ and suppose the two sequences $(f_n)_{n \in \mathbb{N}}$ and $(g_n)_{n \in \mathbb{N}}$ in $\ell^p(I)$ converge, respectively, to f and g . If for each $n \in \mathbb{N}$, $f_n \sim g_n$ then there exists a bijection $\theta : \text{supp}(f) \rightarrow \text{supp}(g)$ such that*

$$\forall i \in \text{supp}(f), \quad f(i) = g(\theta(i)).$$

Proof. Since $\text{Im}(|f|) \cup \text{Im}(|g|)$ is a countable subset of $\mathbb{R}$ with 0 as the only possible limit point, we can choose a sequence $(\lambda_n)_{n \in \mathbb{N}}$ of real numbers which is strictly decreasing with $\lambda_n \rightarrow 0$ and such that

$$\forall n \in \mathbb{N}, \quad \lambda_n \notin \text{Im}(|f|) \cup \text{Im}(|g|).$$

For a function $h \in \ell^p(I)$, let $I_k(h) := \{i \in I \mid \lambda_k \leq |h(i)| < \lambda_{k-1}\}$ for each $k \in \mathbb{N}$, with the convention that $\lambda_0 = +\infty$. Then for each $h \in \ell^p(I)$, $\{I_k(h) \mid k \in \mathbb{N}\}$ form a partition of $\text{supp}(h)$ with each $I_k(h)$ at most a finite set.

Let $k \in \mathbb{N}$ be fixed and suppose $i \in I_k(f)$. Then $\lambda_k \leq |f(i)| < \lambda_{k-1}$. But $\lambda_k \notin \text{Im}(|f|)$. Therefore, there exists $r > 0$ such that $|f(i)| \in (\lambda_k + r, \lambda_{k-1} - r)$. Using the convergence $f_n \rightarrow f$, there is $N_1 \in \mathbb{N}$ with $\|f_n - f\|_p < r$ for all $n \geq N_1$. Hence, $|f_n(i) - f(i)| < r$ from which it follows that

$$|f_n(i)| \in (|f(i)| - r, |f(i)| + r) \subset (\lambda_k, \lambda_{k-1}).$$

Thus, $i \in I_k(f_n)$. Therefore, $I_k(f) \subset I_k(f_n)$ for each $n \geq N_1$. On the other hand, using the fact that $\lambda_k, \lambda_{k-1} \notin \text{Im}(|f|)$, there exists $\epsilon > 0$ with $(\lambda_k - \epsilon, \lambda_{k-1} + \epsilon) \cap \{|f(i)| \mid i \in I \setminus I_k\} = \emptyset$. Again by the convergence $f_n \rightarrow f$, there exists $N_2 \in \mathbb{N}$ such that $\|f_n - f\|_p < \frac{\epsilon}{2}$, and therefore,

$$|f(i)| - \frac{\epsilon}{2} < |f_n(i)| < |f(i)| + \frac{\epsilon}{2}$$

for all $n \geq N_2$. Hence, if $i \in I \setminus I_k(f)$ then $|f_n(i)| \notin (\lambda_k - \frac{\epsilon}{2}, \lambda_{k-1} + \frac{\epsilon}{2})$, i.e., $i \notin I_k(f_n)$. Therefore, $I_k(f) = I_k(f_n)$ for each $n \geq \max\{N_1, N_2\}$. By a similar argument, there exists $N \in \mathbb{N}$ such that

$$I_k(f) = I_k(f_n) \quad \text{and} \quad I_k(g) = I_k(g_n),$$

for all $n \geq N$. Let $\delta := \min\{|f(i) - g(j)| \mid i \in I_k(f), j \in I_k(g) \text{ and } f(i) \neq g(j)\}$. Then $\delta > 0$ and there exists $n_0 \geq N$ with

$$(2.1) \quad |f_{n_0}(i) - f(i)| < \frac{\delta}{2} \quad \text{and} \quad |g_{n_0}(i) - g(i)| < \frac{\delta}{2},$$

for each $i \in I$. Since $f_{n_0} \sim g_{n_0}$ there is a bijection $\theta_k : I \rightarrow I$ for which

$$(2.2) \quad \forall i \in I, \quad f_{n_0}(i) = g_{n_0}(\theta_k(i)).$$

Hence, for each $i \in I_k(f) = I_k(f_{n_0})$,

$$g_{n_0}(\theta_k(i)) = f_{n_0}(i) \in [\lambda_k, \lambda_{k-1}).$$

Therefore, $\theta_k(i) \in I_k(g_{n_0}) = I_k(g)$. A similar argument shows that $\theta_k : I_k(f) \rightarrow I_k(g)$ is onto. Thus, $\theta_k : I_k(f) \rightarrow I_k(g)$ is a bijection for every $k \in \mathbb{N}$. Note that, by (2.1) and (2.2), for each $i \in I_k(f)$,

$$\begin{aligned} |f(i) - g(\theta_k(i))| &\leq |f(i) - f_{n_0}(i)| + |f_{n_0}(i) - g(\theta_k(i))| \\ &= |f(i) - f_{n_0}(i)| + |g_{n_0}(\theta_k(i)) - g(\theta_k(i))| \\ &< \delta, \end{aligned}$$

which, by the definition of δ , implies that $f(i) = g(\theta_k(i))$.

In short, we have shown that for each $k \in \mathbb{N}$, there exists a bijection $\theta_k : I_k(f) \rightarrow I_k(g)$ for which $f(i) = g(\theta_k(i))$ for all $i \in I_k(f)$. Since for each $h \in \ell^p(I)$, $\text{supp}(h) = \cup_{k \in \mathbb{N}} I_k(h)$, and the family $\{I_k(h) \mid k \in \mathbb{N}\}$ is pair-wise disjoint, one can define a bijection $\theta : \text{supp}(f) \rightarrow \text{supp}(g)$ such that $f(i) = g(\theta(i))$, for each $i \in \text{supp}(f)$. $\square$

It should be noted that under the conditions of the Lemma 2.1, it does not necessarily follow that $f \sim g$. As an example, let $(f_n)_{n \in \mathbb{N}}$ and $(g_n)_{n \in \mathbb{N}}$ be defined as follows.

$$\forall n \in \mathbb{N}, \quad f_n = \sum_{k=1}^{\infty} \frac{1}{2^k} e_{k+1} \quad \text{and} \quad g_n = \sum_{k=1}^n \frac{1}{2^k} e_k + \sum_{k=n+1}^{\infty} \frac{1}{2^k} e_{k+1}.$$

Then, for each $n \in \mathbb{N}$, there is a permutation P_n for which $f_n = P_n g_n$, and therefore $f_n \sim g_n$. We have

$$f = \lim f_n = \sum_{k=1}^{\infty} \frac{1}{2^k} e_{k+1} \quad \text{and} \quad g = \lim g_n = \sum_{k=1}^{\infty} \frac{1}{2^k} e_k.$$

Now, if $f = Dg$ for some doubly stochastic operator $D \in \mathcal{DS}(\ell^p(I))$, then

$$0 = f(1) = \sum_{k=1}^{\infty} g(k) D e_k(1) = \sum_{k=1}^{\infty} \frac{D e_k(1)}{2^k}.$$

Therefore, $D e_k(1) = 0$ for each $k \in \mathbb{N}$, which contradicts the fact that D is a doubly stochastic operator. Hence, $f \not\sim g$.

In the next lemma, we add a condition to Lemma 2.1 under which the relation $f \sim g$ is guaranteed.

LEMMA 2.2. *If the sequences $(f_n)_{n \in \mathbb{N}}$ and $(g_n)_{n \in \mathbb{N}}$ satisfy the assumptions of Lemma 2.1 and*

$$\forall n \in \mathbb{N}, \quad \forall i \in I, \quad f_n(i) g_n(i) = 0,$$

then $f \sim g$.

Proof. According to the previous lemma, there exists a bijection $\theta_1 : \text{supp}(f) \rightarrow \text{supp}(g)$ which satisfies $f(i) = g(\theta_1(i))$ for all $i \in \text{supp}(f)$.

If $\text{supp}(f) \cap \text{supp}(g) \neq \emptyset$ then there exists $i_0 \in I$ with $f(i_0) g(i_0) \neq 0$. But then, using the facts that $f_n \rightarrow f$ and $g_n \rightarrow g$, one finds $n \in \mathbb{N}$ for which both $f_n(i_0)$ and $g_n(i_0)$ are non-zero, and this contradicts our assumption. Hence, $\text{supp}(f) \cap \text{supp}(g) = \emptyset$. Therefore, there exists a bijection between the two sets $I_f^0 := I \setminus \text{supp}(f)$ and $I_g^0 := I \setminus \text{supp}(g)$. Let $\theta_0 : I_f^0 \rightarrow I_g^0$ be a bijection and define $\theta : I \rightarrow I$ by

$$\forall i \in I, \quad \theta(i) = \begin{cases} \theta_1(i) & \text{if } i \in \text{supp}(f), \\ \theta_0(i) & \text{if } i \in I_f^0. \end{cases}$$

It is now easily seen that $f(i) = g(\theta(i))$ for each $i \in I$, which completes the proof. $\square$

The following theorem collects the necessary characterizations of bounded linear preservers of majorization on $\ell^p(I)$ and will be used to prove our claim. For the proof of different parts of it, we refer the reader to Theorem 4.9 (ii) and Proposition 5.9 (iv) of [3].

THEOREM 2.3. *Let I be an infinite set, $p \geq 1$ and $T : \ell^p(I) \rightarrow \ell^p(I)$ be a bounded linear map.*

- (i) For the case $p > 1$, T is a preserver of majorization if and only if for all $j_1, j_2 \in I$, $Te_{j_1} \sim Te_{j_2}$, and for each $i \in I$ there is at most one $j \in I$ with $(Te_j)(i) \neq 0$.
- (ii) In the case $p = 1$, T is a preserver of majorization if and only if for all $j_1, j_2 \in I$, $Te_{j_1} \sim Te_{j_2}$, and for each $i \in I$, either there exists exactly one $j \in I$ with $(Te_j)(i) \neq 0$, or the set $\{(Te_j)(i) \mid j \in I\}$ is a singleton.

According to this theorem, the proof of our claim depends on the value of p . We begin with the case $p > 1$.

THEOREM 2.4. For each $p > 1$, the set $\mathcal{M}_{Pr}(\ell^p(I))$ is a norm closed subset of $\mathcal{B}(\ell^p(I))$.

Proof. Let $(T_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{M}_{Pr}(\ell^p(I))$ converging to some bounded linear map $T : \ell^p(I) \rightarrow \ell^p(I)$. On the one hand, for distinct $j_1, j_2 \in I$, since $e_{j_1} \sim e_{j_2}$,

$$T_n e_{j_1} \sim T_n e_{j_2},$$

for each $n \in \mathbb{N}$. On the other hand, according to part (i) of Theorem 2.3,

$$(2.3) \quad \forall i \in I, \quad T_n e_{j_1}(i) T_n e_{j_2}(i) = 0.$$

Hence, by Lemma 2.2, $Te_{j_1} \sim Te_{j_2}$. If there exists $i \in I$ and distinct $j_1, j_2 \in I$ such that $Te_{j_1}(i) \neq 0$ and $Te_{j_2}(i) \neq 0$ then, since $T_n \rightarrow T$, there exists $n \in \mathbb{N}$ with both $T_n e_{j_1}(i)$ and $T_n e_{j_2}(i)$ non-zero which contradicts (2.3). Therefore, for each $i \in I$, there is at most one $j \in I$ such that $Te_j(i) \neq 0$. By part (i) of Theorem 2.3, $T \in \mathcal{M}_{Pr}(\ell^p(I))$. $\square$

We now turn our attention towards the case $p = 1$. We begin with the following lemma.

LEMMA 2.5. If the bounded linear map $T : \ell^1(I) \rightarrow \ell^1(I)$ belongs to the closure of the set $\mathcal{M}_{Pr}(\ell^1(I))$ then for each $i \in I$, either there exists exactly one $j \in I$ with $Te_j(i) \neq 0$, or the set $\{Te_j(i) \mid j \in I\}$ is a singleton.

Proof. Let $(T_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{M}_{Pr}(\ell^1(I))$ which converges in the norm topology to T . For $i \in I$, suppose there are distinct $j_1, j_2 \in I$ with $Te_{j_1}(i) \neq 0$ and $Te_{j_2}(i) \neq 0$. Since $T_n e_{j_k}(i) \rightarrow Te_{j_k}(i)$, for $k = 1, 2$, the values $T_n e_{j_1}(i)$ and $T_n e_{j_2}(i)$ are both non-zero for n large enough. But then, according to part (ii) of Theorem 2.3, $T_n e_{j_1}(i) = T_n e_{j_2}(i) = T_n e_j(i)$, for all $j \in I$, and for these values of n . Hence, $Te_{j_1}(i) = Te_{j_2}(i) = Te_j(i)$ for all $j \in I$, and therefore, the set $\{Te_j(i) \mid j \in I\}$ is a singleton. $\square$

THEOREM 2.6. The set $\mathcal{M}_{Pr}(\ell^1(I))$ is closed in the norm topology of $\mathcal{B}(\ell^1(I))$.

Proof. Let $T : \ell^1(I) \rightarrow \ell^1(I)$ be in the closure of the set $\mathcal{M}_{pr}(\ell^1(I))$. According to the previous lemma and part (ii) of Theorem 2.3, it suffices to prove that $Te_{j_1} \sim Te_{j_2}$ for all $j_1, j_2 \in I$. Suppose $(T_n)_{n \in \mathbb{N}}$ is a sequence in $\mathcal{M}_{pr}(\ell^1(I))$ with $T_n \rightarrow T$ and let $f_n := T_n e_{j_1}$, $g_n := T_n e_{j_2}$, $f := Te_{j_1}$ and $g := Te_{j_2}$. Then, $f_n \rightarrow f$, $g_n \rightarrow g$ and $f_n \sim g_n$ for each $n \in \mathbb{N}$. Hence, by Lemma 2.1, there is a bijection $\theta_1 : \text{supp}(f) \rightarrow \text{supp}(g)$ with $f(i) = g(\theta_1(i))$ for each $i \in \text{supp}(f)$. This shows that $\text{Im}(f) \setminus \{0\} = \text{Im}(g) \setminus \{0\}$, and that for each $x \in \text{Im}(f) \setminus \{0\}$, the two finite sets $f^{-1}(\{x\})$ and $g^{-1}(\{x\})$ have the same cardinality. Hence, the two sets $f^{-1}(\{x\}) \setminus g^{-1}(\{x\})$ and $g^{-1}(\{x\}) \setminus f^{-1}(\{x\})$ are in one-to-one correspondence. Let $\theta_x : f^{-1}(\{x\}) \setminus g^{-1}(\{x\}) \rightarrow g^{-1}(\{x\}) \setminus f^{-1}(\{x\})$ be a bijection. Note that if $i \in f^{-1}(\{x\}) \setminus g^{-1}(\{x\})$ then $Te_{j_1}(i) = f(i) = x$, while $Te_{j_2}(i) = g(i) \neq x$ which, according to Lemma 2.5, indicates that $g(i) = Te_{j_2}(i) = 0$. Hence, the map $\theta_0 : I_f^0 \rightarrow I_g^0$ given by

$$\forall i \in I_f^0, \quad \theta_0(i) = \begin{cases} i & g(i) = 0, \\ \theta_x^{-1}(i) & g(i) = x \neq 0. \end{cases}$$

is well-defined and $f(i) = 0 = g(\theta_0(i))$ for each $i \in I_f^0$. It is easily proved that θ_0 is also a bijection. Now, using the two bijections $\theta_1 : \text{supp}(f) \rightarrow \text{supp}(g)$ and $\theta_0 : I_f^0 \rightarrow I_g^0$ with the above mentioned properties, one can easily define a bijection $\theta : I \rightarrow I$ with $f(i) = g(\theta(i))$ for each $i \in I$. Hence, $f = Te_{j_1} \sim g = Te_{j_2}$. $\square$

REMARK 2.7. The proofs of Theorem 2.4, Lemma 2.5 and Theorem 2.6 remain valid if we replace the operator norm topology of $\mathcal{B}(\ell^p(I))$ with the strong operator topology. Therefore, for each $p \geq 1$, the set $\mathcal{M}_{pr}(\ell^p(I))$ is still a closed subset of $\mathcal{B}(\ell^p(I))$ in this coarser topology.

There is another notion of majorization on $\ell^1(\mathbb{N})$ which generalizes naturally this notion on $\mathbb{R}^n$. See, for example, [6] and [8]. In [6], where this notion has been called strong majorization and has been denoted by $\preceq$, it is proved that for non-negative sequences $f, g \in \ell^1(\mathbb{N})$, if $f \preceq g$ then there is an orthostochastic infinite matrix Q such that $f = Qg$ [6, Theorem 3.9]. A matrix $Q = (q_{ij})$ is called orthostochastic if there exists a unitary matrix with real entries $L = (l_{ij})$ for which $q_{ij} = |l_{ij}|^2$ for all $i, j \in \mathbb{N}$. Hence, every orthostochastic matrix is doubly stochastic. The converse is also true, even for a wider case, i.e., for non-negative $f, g \in \ell^1(\mathbb{N})$, if $f = Qg$ for some column stochastic matrix Q , then $f \preceq g$ (Lemma 2.10, *ibid*). Therefore, in the case of non-negative sequences in $\ell^1(\mathbb{N})$ this notion of majorization coincides with that of us. However, the authors of this paper do not know whether these two notions agree on all sequences in $\ell^1(\mathbb{N})$. Thus, the results of this paper are not known to be true for this notion of majorization.

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