A NOTE ON POWER BOUNDED MATRICES*

R.E. HARTWIG[†] AND P. PATRICIO[‡]

Abstract. The concept of focal power is used to examine when a nonnegative power bounded matrix is periodic.

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1. Introduction. A complex $n \times n$ matrix is *power bounded* – shortened to PB – when there is a nonnegative M such that the inequalities $|(A^k)_{ij}| \leq M$ hold for all $i, j \in \{1, \dots, n\}$ and $k \in \mathbb{N}$.

A special subclass are the periodic matrices A , for which $A^{k+h} = A^k$ for some nonnegative integers h and k .

Given a block matrix $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ with A_{11} square, we say it is (1,1)-focused (as defined in [7]) if $A^k = \begin{bmatrix} A_{11}^k & ? \\ ? & ? \end{bmatrix}$ for all k , in which case A is said to have *focal power*.

We say A is *convergent* to the matrix $X = [x_{ij}]$ if the sequences $(A^k)_{ij}$ converge to x_{ij} as $k \rightarrow \infty$. A matrix is *convergent* if it is convergent to some matrix. This is a special case of convergence as defined in [4, Definition 10.1], for the matrix sequence $\{A, A^2, A^3, \dots\}$. In particular, we say A is zero convergent when A is convergent to the zero matrix. We distinguish between convergence and zero convergence (i.e., convergence to the zero matrix) of A . Needless to say, A can be power bounded without being convergent, as seen from the example $A = -I$.

We denote the rank, the range and the nullspace of a matrix A by $\text{rank}(A)$, $R(A)$ and $N(A)$ respectively. The spectrum and the spectral radius of A are denoted, respectively, by $\sigma(A)$ and $\rho(A)$. For $\lambda \in \sigma(A)$, its algebraic multiplicity is denoted by

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[†]Mathematics Department, NCSU Raleigh, NC 27695-8205, USA (hartwig@unity.ncsu.edu).

[‡]Departamento de Matemática e Aplicações, Universidade do Minho, 4710-057 Braga, Portugal (pedro@math.uminho.pt). Member of the research unit CMAT – Centro de Matemática da Universidade do Minho, Portugal.

$n(\lambda)$ and its index by $in(\lambda)$. The index $in(A)$ of A is defined as $in(0)$ and permutation similarity is denoted by $\approx_{\text{per}}$.

Given a matrix A with spectrum $\sigma(A) = \{\lambda_1, \dots, \lambda_s\}$, we define the spectral components of A as $Z_k^j = (A - \lambda_k I)^j (I - (A - \lambda_k I)(A - \lambda_k I)^D)^{\frac{1}{j!}}$, where X^D denotes the Drazin inverse of X (see [3]). When $in(A) = 1$, X^D becomes the *group* inverse of X and is denoted by $X^\#$. In particular, when $j = 0$, $Z_k^0(A) = I - (A - \lambda_k I)(A - \lambda_k I)^D$.

The question of when a matrix is PB is an old one, and has been examined by numerous authors, e.g., [9], [11], [10], [12], [13]. Likewise, periodic matrices have also been investigated, as for example in [6]. On the other hand, *focal power* has only recently been introduced [7], and appears naturally in the study of PB matrices.

In this paper, we first address the general complex case with aid of the Jordan form and its spectral properties. In addition a special canonical form is used. We subsequently derive results for block triangular matrices, and lastly, consider nonnegative matrices.

For a real nonnegative matrix A , we may combine the Frobenius normal form (FNF), or the irreducible normal form [8, page 506], with the critical normal form (see Section 4) to obtain a block upper triangular form for nonnegative PB matrices. We give a direct proof that avoids minimal polynomials or generalized eigenvectors. An alternative approach is to use the structure of the diagonal entries in the matrix, which is addressed at the end of this paper.

When A is nilpotent, then A is clearly periodic and PB. Recalling that a nilpotent matrix is characterized by the fact that all its powers have zero trace, we may conclude that if A is **not** nilpotent then there exists a power of A with nonzero trace. Moreover when, in addition, A is nonnegative then it has a FNF of the type

$$F_A = \begin{bmatrix} 0 & ? & ? \\ 0 & 0 & \\ & & \ddots \\ 0 & & 0 \end{bmatrix}.$$

An important matrix in applied matrix theory is the basic circulant matrix

$$\Omega = \begin{bmatrix} 0 & \cdots & 1 \\ 1 & 0 & \\ 0 & 1 & 0 & \vdots \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{bmatrix}_{n \times n}.$$

It is nonnegative, irreducible and bounded, but it is not convergent. It is periodic because $\Omega^n = I$.

For a nonnegative matrix A with $\rho(A) \leq 1$, the existence of positive right or left eigenvectors associated with the Perron root is not a necessary condition for a nonnegative matrix to be PB, as seen from the following example.

EXAMPLE 1.1. The matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$ with $\rho(A) = 1$, shows that A can be convergent (and thus, be PB) *without* having a positive left or right (Perron) eigenvector. Indeed, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a right and left eigenvector of A associated with the eigenvalue $\rho(A) = 1$.

As a special case, we examine nonnegative PB matrices whose powers have diagonal entries that are 0 or 1 exclusively. We use the concept of (1,1) Focal Power to show that these matrices must be periodic. Indeed, some power must be idempotent. This simplifies and extends in part some of the results of Yang and Zhang [14].

The stochastic matrix $P = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$ shows that for a matrix to be periodic, it does not suffice for it to be PB, nonnegative, with 0-1 diagonal entries. Indeed, because $P^3 = P^2 + (1/4)P - (1/4)I$, P is not periodic.

2. General results. For a square matrix A , we define its geometric and Cesaro sums respectively by

$$G_N(A) = I + A + \cdots + A^{N-1} \text{ and } \mathcal{C}_N(A) = \frac{1}{N}G_N(A)$$

The Jordan form can be used to establish the following theorem.

THEOREM 2.1. *Let A be a square complex matrix and $\lambda \in \sigma(A)$.*

1. *The following are equivalent, as $N \rightarrow \infty$:*
 - (a) A^N is bounded;
 - (b) $A^N/N \rightarrow 0$;
 - (c) for $\lambda \in \sigma(A)$, $|\lambda| \leq 1$ and $|\lambda| = 1 \Rightarrow \text{in}(\lambda) = 1$;
 - (d) The Cesaro sums $\mathcal{C}_N(A)$ converge.
2. *The following are equivalent, as $N \rightarrow \infty$:*
 - (a) $A^N \rightarrow 0$;
 - (b) The geometric sums $G_N(A)$ converge;
 - (c) $\rho(A) < 1$.
3. *The following are equivalent, as $N \rightarrow \infty$:*

- (a) A^N is bounded and $1 \notin \sigma(A)$;
 - (b) $G_N(A)$ is bounded;
 - (c) $\mathcal{C}_N(A) \rightarrow 0$;
 - (d) $|\lambda| \leq 1$, $\lambda \neq 1$ and $|\lambda| = 1 \Rightarrow \text{in}(\lambda) = 1$.
4. The following are equivalent:
- (a) A is convergent;
 - (b) $|\lambda| \leq 1$ and $|\lambda| = 1 \Rightarrow \lambda = 1$ with $\text{in}(\lambda) = 1$.

We next recall ([3, Lemma 1.2]) that if A is a matrix with minimal polynomial $\psi_A(\lambda) = (\lambda - 1)^s f(\lambda)$ such that $\gcd(\lambda - 1, f) = 1$, then

$$(2.1) \quad G_N(A) = \sum_{i=0}^{N-1} A^i = (I - A)^D(I - A^N) + \sum_{i=0}^{s-1} \binom{N}{i+1} (A - I)^i Z_1^0.$$

Now suppose A is PB, which ensures that the group inverse $(I - A)^\#$ exists (see [1]).

If $1 \notin \sigma(A)$ then $I - A$ has an inverse and

$$(I - A) \sum_{i=0}^{N-1} A^i = I - A^N \Rightarrow G_N = (I - A)^{-1}(I - A^N).$$

On the other hand, if $1 \in \sigma(A)$ then we have $m_1 = \text{in}(1) = 1$, ie, $\psi_A(\lambda) = (\lambda - 1)f(\lambda)$ with $f(1) \neq 0$, which gives

$$G_N(A) = (I - A)^\#(I - A^N) + NZ_1^0 = (I - A)^\#(I - A^N) + N[I - (I - A)(I - A)^\#].$$

Another tool to study PB matrices is the RCNZ (row-column-nonzero) canonical form

$$A = \begin{bmatrix} P & Q & 0 & 0 \\ 0 & 0 & 0 & 0 \\ R & S & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $\begin{bmatrix} P & Q \\ R & S \end{bmatrix}$ has no zero row or column. It is easily seen that

$$A^k = \begin{bmatrix} P^k & P^{k-1}Q & 0 & 0 \\ 0 & 0 & 0 & 0 \\ RP^{k-1} & RP^{k-2}Q & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and hence, that A and P have a similar behavior, namely,

- (i) A^N is (zero) convergent if and only if P^N is (zero) convergent,
- (ii) A is PB if and only if P is PB,
- (iii) $A^N/N \rightarrow 0$ if and only if $P^N/N \rightarrow 0$,
- (iv) $G_N(A)$ is bounded if and only if $G_N(P)$ is bounded,
- (v) $Tr(A^k) = Tr(P^k)$ etc.

Moreover, A is periodic, say $A^k = A^{k+h}$ for some nonnegative integers h and k , if and only if $P^k = P^{k+h}$, $P^{k-1}Q = P^{k-1+h}Q$, and $RP^k = RP^{k-1+h}$.

3. The block triangular case. Before we turn to the general nonnegative case let us give a few preliminary results dealing with block triangular matrices.

LEMMA 3.1. Let $M = \begin{bmatrix} A & ? & C \\ 0 & X & ? \\ 0 & 0 & D \end{bmatrix}$ with $\rho(A) < 1$ and $\rho(D) < 1$. Then, M is PB if and only if X is PB.

Proof. The necessity is clear, so suppose that X is PB. Then its minimal polynomial is given by $\psi_X(x) = f(x)g(x)$, where f only has distinct roots on the unit circle and all the roots of g are inside the unit circle. Now because

$$Lcm(\psi_A, \psi_X, \psi_D) | \psi_M | \psi_A \cdot \psi_X \cdot \psi_D,$$

we see that $f | \psi_M | f \cdot g \cdot \psi_A \cdot \psi_D$, and hence, $\psi_M = f(x) \cdot h(x)$, where all the roots of h lie inside the unit circle. This means that M has inherited the property that all eigenvalues on the unit circle have index 1, as desired. $\square$

Let us start with the simplest case where $M = \begin{bmatrix} A & C \\ 0 & D \end{bmatrix}$. It follows by induction that $M^N = \begin{bmatrix} A^N & \Gamma_N \\ 0 & D^N \end{bmatrix}$, where

$$\Gamma_N(A, C, D) = \sum_{k=0}^{N-1} A^{N-k-1} C D^k = A \Gamma_{N-1} + C D^{N-1}.$$

We note that M is PB if and only if A and D are PB and Γ_N is bounded for all N . In particular, Theorem 2.1 ensures that, for each $\lambda_A \in \sigma(A)$, $\lambda_D \in \sigma(D)$, we have $|\lambda_A| \leq 1$, $|\lambda_D| \leq 1$ with index at most one on the Perron circle $|\lambda| = 1$.

If in addition $\mathbf{u}^T A = \alpha \mathbf{u}^T$ and $D \mathbf{v} = \delta \mathbf{v}$ with $\alpha, \delta \neq 0$, then $\mathbf{u}^T \Gamma_N \mathbf{v} = (\mathbf{u}^T C \mathbf{v}) S_N$, where

$$S_N = \sum_{k=0}^{N-1} \alpha^{N-k-1} \delta^k = \begin{cases} N \alpha^{N-1} & \text{if } \alpha = \delta \\ \frac{\alpha^N - \delta^N}{\alpha - \delta} & \text{if } \alpha \neq \delta. \end{cases}$$

LEMMA 3.2. S_N is bounded for all N if and only if either $\alpha = \delta$ with $|\alpha| < 1$ or $\alpha \neq \delta$ with $|\alpha| \leq 1, |\delta| \leq 1$.

We may therefore state that:

if Γ_N is bounded for all N then $\mathbf{u}^T \Gamma_N \mathbf{v} = (\mathbf{u}^T C \mathbf{v}) S_N$ is bounded for all N .

This leads to

COROLLARY 3.3. $(\mathbf{u}^T C \mathbf{v}) S_N$ is bounded for all N if and only if either $\mathbf{u}^T C \mathbf{v} = 0$ with S_N arbitrary, or $\mathbf{u}^T C \mathbf{v} \neq 0$ and S_N is bounded for all N .

For the general block triangular case, suppose that

$$M = \left[\begin{array}{c|cc|c} A & & X_1 & C_1 \\ \hline 0 & ? & ? & \\ \hline 0 & 0 & ? & Y_1 \\ \hline 0 & 0 & & ? \\ \hline 0 & 0 & \cdots & 0 \\ \hline & & & D \end{array} \right] \quad \text{and} \quad M^N = \left[\begin{array}{c|cc|c} A^N & & X_N & C_N \\ \hline 0 & ? & ? & \\ \hline 0 & 0 & ? & Y_N \\ \hline 0 & 0 & & ? \\ \hline 0 & 0 & \cdots & 0 \\ \hline & & & D^N \end{array} \right].$$

We shall show by induction that $C_N = \Gamma_N(A, C, D) + Z_N$ for some Z_N . Indeed, first observe that it holds for $N = 2$ with $Z_2 = 0$. Now assuming it holds for N , we see that $C_{N+1} = AC_N + X_1 Y_N + CD^N = (A\Gamma_N + CD^N) + (AZ_N + X_1 Y_N) = \Gamma_{N+1}(A, C, D) + Z_{N+1}$.

We may now draw several easy conclusions for nonnegative matrices.

- Since $Z_N \geq 0$ we have
- $$(3.1) \quad C_N \geq \Gamma_N(A, C, D) = \sum_{k=0}^{N-1} A^{N-k-1} C D^k.$$
- If $\mathbf{u} > \mathbf{0}, \mathbf{v} > \mathbf{0}$ and $\Gamma_N \geq 0$, then $\mathbf{u}^T \Gamma_N \mathbf{v} = 0$ if and only if $\Gamma_N = 0$.
 - If we fix $k \geq 1$, then by applying (3.1) to $(M^k)^N$ we see that
- $$C_{N^k} \geq \Gamma_N(A^k, C_k, D^k), \text{ for } k = 1, 2, 3, \dots$$
- If M is PB so is M^k , and hence, C_{N^k} and $\Gamma_N(A^k, C_k, D^k)$ are bounded for all N .

This yields

THEOREM 3.4. Suppose $M = \begin{bmatrix} A & C \\ 0 & D \end{bmatrix} \geq 0$ and M is PB, with $\mathbf{u}^T A = \mathbf{u}^T$ and $D\mathbf{v} = \mathbf{v}$, where $\mathbf{u} > \mathbf{0}, \mathbf{v} > \mathbf{0}$. Then

- (i) M^k is PB and $\mathbf{u}^T A^k = \mathbf{u}^T$, $D^k \mathbf{v} = \mathbf{v}$ for $k = 1, 2, \dots$, and

(ii) $C_k = 0$.

Proof. Condition (ii) follows from the fact that we may apply Corollary 3.3 to the matrices M^k , A^k , and D^k . Indeed, since $\alpha^k = \delta^k = 1$, S_{N^k} **cannot** be bounded and we must have $\mathbf{u}^T C_k \mathbf{v} = 0$, which ensures that $C_k = 0$. $\square$

4. The Frobenius and critical normal forms. Let us now turn to the Frobenius normal form (FNF) of our matrix M . Recall that with a **nonnegative** matrix M we may associate the digraph $G_M = (V, \mathcal{E})$, where $V = \{S_1, \dots, S_n\}$ is a set of nodes (sites), and $\mathcal{E} = \{(S_{i_r}, S_{j_s})\}$ is the set of arcs (pairs), such that the pair $(S_i, S_j) \in \mathcal{E}$. That is, we draw an arc from S_i to S_j and write $S_i \rightarrow S_j$, exactly when $m_{ij} \neq 0$. If there is a path from S_i to S_j , we write $S_i \twoheadrightarrow S_j$ and if there is no path we indicate this by $S_i \not\rightarrow S_j$.

We define the equivalence relation on V , by $S_i \sim S_j$ if and only if $S_i = S_j$ or $S_i \twoheadrightarrow S_j$ and $S_j \twoheadrightarrow S_i$. The equivalence classes $\{V_1, \dots, V_r\}$ are called the “condensed nodes” of G . The condensed graph G_M^* of M is obtained by drawing an arc from V_p to V_q , $p \neq q$, if and only if there is **some** arc in G_M from a node in V_p to **some** node in V_q . The condensed graph has no cycles and the condensed nodes can be relabeled such that the matrix M is permutation similar to its FNF

$$F_M = \begin{bmatrix} A_{11} & & \cdots & & ? \\ 0 & A_{22} & & & \\ & & \ddots & & \\ 0 & 0 & 0 & A_{rr} & ? \\ & & & \ddots & \\ 0 & & & & A_{ww} \end{bmatrix} \quad (A_{ii} \text{ irreducible or } 0_{1 \times 1}).$$

Thus, the block A_{pq} of F_M is nonzero if and only if there is an arc from V_p to V_q . For convenience we shall shorten A_{ii} to A_i .

If M^N is bounded, so is F_M^N . Hence we know, for any eigenvalue λ of M , that $|\lambda| \leq 1$ or $|\lambda| = 1$ with $\text{in}(\lambda) = 1$.

This means that the diagonal blocks in F can be separated into “singular blocks” with $\rho(A_i) = 1$ and the remaining “nonsingular” blocks with $\rho(A_j) < 1$. Therefore, no nonsingular block contains any eigenvalue on the unit circle.

Consider two singular blocks A_p^T and A_q with $p < q$ with positive eigenvectors $\mathbf{u}$ and $\mathbf{v}$ associated with their spectral radius. Then the submatrix $M_{pq} =$

$\begin{bmatrix} A_p & ? & C_{pq} \\ 0 & ? & ? \\ 0 & 0 & A_q \end{bmatrix}$ has powers that are also bounded. Moreover, if

$$M_{pq}^k = \begin{bmatrix} A_p^k & ? & C_{pq}^{(k)} \\ 0 & ? & ? \\ 0 & 0 & A_q^k \end{bmatrix},$$

then by Theorem 3.4 we may conclude that $C_{pq}^{(k)} = 0$ for all $k = 1, 2, \dots$

In terms of the condensed graph of M , this means that there is **no k-step path** in G_M^* from a singular node V_p to any singular node V_q . Since k is arbitrary, that means that **there is no path** (no access) from V_p to V_q .

We now permute the FNF further to the “critical normal form”, in which we take the “critical nodes” V_i to be those nodes for which $\rho(A_i) = 1$.

Suppose that

$$M \underset{\text{per}}{\approx} F_M = \begin{matrix} & V_1 & V_2 & \cdots & V_w \\ \begin{matrix} V_1 \\ V_2 \\ \vdots \\ V_w \end{matrix} & \begin{bmatrix} A_1 & \cdots & & A_{1q} \\ 0 & A_2 & & \\ & & \ddots & \vdots \\ 0 & & & A_w \end{bmatrix} \end{matrix}$$

and that V_k is the *first* critical node in the sequence $(V_1, \dots, V_w)$. We now partition V into $V = V_k \cup V_k^- \cup V_k^\neq$, where $V_k^- = \{V_i; V_i \twoheadrightarrow V_k\}$ contains all the (condensed) nodes that have access to V_k and $V_k^\neq = \{V_i; V_i \not\rightarrow V_k\}$ is made up of all the nodes that have **no** access to V_k . If U and W are two subsets of nodes of the graph (i.e., of V), then we write $U \rightarrow W$, if $u \rightarrow w$ for some $u \in U, w \in W$, and $U \not\rightarrow W$ otherwise. We now note that:

- (i) $V_k^\neq \not\rightarrow V_k^-$ otherwise if $x \rightarrow y$ with $x \in V_k^\neq$ and $y \in V_k^-$, then $x \rightarrow y \twoheadrightarrow V_k$ which cannot be.
- (ii) $V_k \not\rightarrow V_k$ otherwise we have a loop in G^* .
- (iii) $V_k \in V_k^\neq$, since otherwise we have a cycle G^* .
- (iv) $V_k^\neq \not\rightarrow V_k$ by definition.
- (v) $V_k^- \subseteq \{V_1, \dots, V_{k-1}\}$ and $\{V_{k+1}, \dots, V_w\} \subseteq V_k^\neq$ because the later nodes have no access to V_k .

If we rearrange the nodes in V , keeping the indices in increasing order, we arrive at

$$M \underset{\text{per}}{\approx} \begin{array}{c} V_k^= \left\{ \begin{array}{c} V_{i_1} \\ \vdots \\ V_{i_{r-1}} \end{array} \right. \\ V_k \\ V_{i_{r+1}} \\ V_k^{\neq} \left\{ \begin{array}{c} \vdots \\ V_{i_w} \end{array} \right. \end{array} \left[\begin{array}{ccc|c|ccc} A_{i_1} & & & ? & & & \\ & \ddots & & & & ? & \\ 0 & & A_{i_{r-1}} & & & & \\ \hline & & 0 & A_k & & & \\ \hline 0 & & 0 & 0 & D_{i_{r+1}} & & \\ & & & & & \ddots & \\ 0 & & 0 & & & & D_{i_w} \end{array} \right].$$

We may repeat this by selecting a *first* critical node in the matrix D , while keeping the same increasing ordering, etc. Labeling the nodes according to the critical nodes, we end up with

$$M \underset{\text{per}}{\approx} F' = \left[\begin{array}{ccc|c|c} M_1 & & & ? & \\ 0 & M_2 & & & \\ & & \ddots & & \\ & & & M_s & ? \\ \hline 0 & & & & T \end{array} \right],$$

with $\rho(T) < 1$, where

$$M_i = \begin{array}{c} V_1^{(i)} \\ \vdots \\ V_{r_i-1}^{(i)} \\ V_{r_i}^{(i)} \end{array} \left[\begin{array}{ccc|c} A_1^{(i)} & & & ? \\ & \ddots & & \\ & & A_{r_i-1}^{(i)} & \\ \hline 0 & & & A_{r_i}^{(i)} \end{array} \right] = \begin{bmatrix} U_i & V_i \\ 0 & A_{r_i} \end{bmatrix}$$

is again block triangular for $i = 1, \dots, s$. In addition:

- (i) each corner block $A_{r_i}^{(i)} = A_{r_i}$ in each M_i , $i = 1, \dots, s$, corresponds to a critical node V_{r_i} , $i = 1, \dots, s$;
- (ii) each of the upper nodes $V_j^{(i)}$ with $j = 1, \dots, r_i - 1$, has access to corner node $V_{r_i}^{(i)}$;
- (iii) T is block triangular but contains no critical nodes.

Consider the form

$$F' = \left[\begin{array}{cc|cc|cc|cc|c} U_1 & V_1 & ? & & & & & & ? \\ 0 & A_{r_1} & X_{11} & X_{12} & X_{21} & X_{22} & \cdots & X_{s1} & X_{s2} \\ \hline & 0 & U_2 & V_2 & & & & & \\ & & 0 & A_{r_2} & Y_{21} & Y_{22} & \cdots & Y_{s1} & Y_{s2} \\ \hline & & & & U_3 & V_3 & & & \\ & & & & 0 & A_{r_3} & \cdots & Z_{s1} & S_{s2} \\ & & & & & & \ddots & & \\ \hline & & & & & & & U_s & V_s \\ & & & & & & & 0 & A_{r_s} \\ \hline 0 & & & & & & & & T \end{array} \right]$$

where the critical blocks A_{r_i} correspond to the critical nodes V_{r_i} .

Since the singular node V_{r_1} has no access to the other singular nodes we can conclude that $X_{i1} = 0$ and $X_{i2} = 0$. Likewise, all $Y_{ij} = 0, Z_{ij} = 0, \dots$. That is, the block rows containing the A_{r_i} do not contain any other nonzero blocks, except for the last block column. This means that we may permute the singular nodes, by first taking the odd block rows, except the last one, and then every odd block column, except the last one, followed by taking every even block row and then every even block column. This leads to

THEOREM 4.1. *A nonnegative matrix M is convergent (respectively, power bounded) if and only if it is permutation similar to a matrix of the form*

$$(4.1) \quad M' = \left[\begin{array}{cc|c} A & B & C \\ 0 & D & E \\ \hline 0 & 0 & T \end{array} \right],$$

where

- (i) A and T are block upper triangular;
- (ii) the diagonal blocks A_{ii} and T_{jj} are irreducible or $0_{1 \times 1}$;
- (iii) $\rho(A_{ii}) < 1$, $\rho(T_{ii}) < 1$;
- (iv) $D = \text{diag}(D_1, \dots, D_q)$, in which $D_i = A_{r_i}$ is irreducible, $\rho(D_i) = 1$ and D_i is convergent (respectively, power bounded);
- (v) B , C and E are arbitrary.

Since $D\mathbf{u} = 1 \cdot \mathbf{u}$, for some $\mathbf{u} > \mathbf{0}$, it is easily seen that, for all N , the convolution product $\sum_{k=0}^{N-1} A^{N-k-1} B D^k$ is bounded exactly when $\sum_{k=0}^{N-1} A^{N-k-1} B \mathbf{u}$ is bounded. This suffices to show that if $\rho(A) < 1$, then the matrix M' in (4.1) is also power bounded. $\square$

Given a nonnegative matrix A , let $\pi(A)$ be the number of eigenvalues of A (including multiplicities) on the Perron circle $\{\lambda : |\lambda| = \rho(A)\}$. We now may state

LEMMA 4.2. *The following are equivalent for $A \geq 0$:*

- (i) $A^N > 0$ for some N .
- (ii) A is irreducible and $\pi(A) = 1$.
- (iii) A is irreducible and $(A/\rho)^N$ converges to some $L > 0$.

In which case, $\lim_N (A/\rho)^N = \frac{\mathbf{u}\mathbf{v}^T}{\mathbf{v}^T\mathbf{u}}$, where $A\mathbf{u} = \rho\mathbf{u}$ and $A^T\mathbf{v} = \rho\mathbf{v}$.

Proof. This follows from [8, Theorem 8.5.1] and from [8, Theorem 8.5.2]. $\square$

For the irreducible central matrices D_i as in (4.1), we need the following result.

LEMMA 4.3. *Let D be nonnegative irreducible with $\rho(D) = 1$. Then*

(a) *The following are equivalent:*

- (i) D is convergent.
- (ii) $D^N > 0$ for some N .
- (iii) $\pi(D) = 1$.

In either case, $D^k \rightarrow \frac{\mathbf{u}\mathbf{v}^T}{\mathbf{v}^T\mathbf{u}}$ for some positive $\mathbf{u}, \mathbf{v}$.

(b) *If D is irreducible then D is PB if and only if $U^{-1}DU = P$, for some irreducible stochastic matrix P , and $U = dg(u_1, \dots, u_n)$ with $u_i > 0$*

Proof. (a) This follows from Lemma 4.2.

(b) By the Perron-Frobenius Theorem $D\mathbf{u} = 1 \cdot \mathbf{u}$, where $\mathbf{u} > \mathbf{0}$. Since $D\mathbf{e} = \mathbf{u}$, where $\mathbf{e} = [1, 1, \dots, 1]^T$, we now obtain $U^{-1}DU = P$ with $P\mathbf{e} = 1 \cdot \mathbf{e}$, as desired. $\square$

Example 1.1 shows that Theorem 4.1 is best possible.

We note in passing that for a nonnegative PB matrix, $a_{ii} \leq \rho(A) \leq 1$.

If M is periodic, then there is a *first* power of M , say M^{b+h} , that is equal to an earlier power M^b , and thus,

$$M^b = M^{b+h}.$$

This means that $x^b(x^h - 1)$ is the minimal polynomial for M with $\rho(M) = 1$ and h is the *Boolean period* of M . Consequently, the matrices A and T in the canonical form of equation (4.1) must be nilpotent and $D^b = D^{b+h}$. Let us next focus on one of the direct summands in $D = \text{diag}(D_1, \dots, D_q)$.

Consider a nonnegative, irreducible matrix D such that $\rho(D) = 1$ and $D^b = D^{b+h}$. It is known [5, pp 53–54], [2, Theorem 2.20] that D is permutation similar to its cyclic

form

$$D \underset{\text{per}}{\approx} E = \begin{bmatrix} 0 & X_1 & 0 & & 0 \\ 0 & 0 & X_2 & & \\ & & \ddots & & \\ 0 & 0 & 0 & & X_{f-1} \\ & & & \ddots & \\ X_f & 0 & & & 0 \end{bmatrix},$$

where the diagonal blocks are square and f will be referred to as the *Perron period* of D .

It further follows that $E^f = \text{diag}(B_0, B_1, \dots, B_{f-1})$, where each B_i is such that $B_i^{\varepsilon_i} > 0$, for some ε_i , and hence, $\pi(B_i) = 1$. If we set $\varepsilon = \max\{\varepsilon_i\}$, then $E^{f\varepsilon}$ is a direct sum of positive matrices. In addition $\rho(B_i) = \rho(B_0) = [\rho(E)]^f = 1$. This means that

$$(4.2) \quad \Delta_D(x) = x^{n_0}(x^f - 1) \prod_2^q (x^f - \gamma_k)^{n_k},$$

where n_0 is the algebraic multiplicity of the eigenvalue 0 and $0 < |\gamma_k| < 1$.

On the other hand, if $D^b = D^{b+h}$ then the minimal polynomial of D is given by $\psi(x) = x^b(x^h - 1)$, which has h distinct roots λ_i on the unit circle. This means that $\Delta_D(x)$ takes the form

$$(4.3) \quad \Delta_D(x) = x^{n_0}(x - \lambda_1)^{k_1} \cdots (x - \lambda_h)^{k_h}.$$

Identities (4.2) and (4.3) show that the γ_i must be absent and that all $k_i = 1$. It then follows that $f = h$, i.e., the Perron period and the Boolean period are **equal**.

Next, we observe that E is also periodic with $E^b = E^{b+hs}$ for $s = 1, 2, \dots$. Two cases can occur.

Case (I). If $b \leq h$ then multiplying through by E^{h-b} shows that $E^h = E^{2h}$. This means that $E^{h\varepsilon} = E^{2h\varepsilon} = \text{diag}(B_0^\varepsilon, \dots, B_{h-1}^\varepsilon)$, in which each B_i^ε is a *positive idempotent*.

Case (II). When $b > h$, we write $b = qh + r$, where $0 \leq r < h$. Then $E^b = (E^h)^q \cdot E^r = (E^h)^{q+1} \cdot E^r$. Post-multiplying by E^{h-r} then shows that $(E^h)^{q+1} = (E^h)^{q+2} = (E^h)^{q+3} = \dots$, and thus, $(E^{h\varepsilon})^{q+1} = (E^{h\varepsilon})^{2(q+2)}$. This means that we again have a block diagonal idempotent matrix with positive diagonal blocks. Thus, we have the following result:

COROLLARY 4.4. *A matrix is periodic, nonnegative irreducible with spectral radius one if and only if it has a power that is permutation similar to a direct sum of positive idempotents.*

We recall in passing that

LEMMA 4.5. *A positive idempotent matrix must have rank one.*

Proof. Let $E = E^2$ be positive $n \times n$. Then the Jordan form of E is of the form $I_r \oplus 0_{n-r}$. Since $E > 0$, it follows from the Perron-Frobenius theorem that $r = 1$. $\square$

A positive periodic matrix must have a Boolean period of one and a minimal polynomial of the form $x^b(x - 1)$, but it need not be idempotent. For example, consider any positive stochastic matrix with trace one and rank two.

EXAMPLE 4.6. The matrix $A = \begin{bmatrix} 1/6 & 1/3 & 1/3 \\ 1/6 & 1/3 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$ is positive, periodic, but not idempotent.

Let us now turn to a special nonnegative PB case, in which we capitalize on the fact that the two periods are equal.

5. Entries in nonnegative PB matrices.

5.1. Basics. We begin by listing some of the pertinent facts about a nonnegative matrix $M = [m_{ij}]$.

- (1) $\rho(M)$ is an eigenvalue of M and there exists a nonnegative eigenvector associated with $\rho(M)$.
- (2) $m_{ii} \leq \rho(M)$.
- (3) If $M = \begin{bmatrix} A & C \\ B & D \end{bmatrix} \geq 0$ and $M^N = \begin{bmatrix} A_N & C_N \\ B_N & D_N \end{bmatrix}$ then $A_N \geq A^N$.
- (4) $(M^N)_{ii} \geq (m_{ii})^N$ for $N = 1, 2, \dots$
- (5) $\sum_{i=1}^n m_{ii}^N \leq \text{Tr}(M^N)$ for $N = 1, 2, \dots$

Next, consider a nonnegative power bounded (nnPB) matrix M . First, some general results.

LEMMA 5.1.

- (i) If $A, B \geq 0$ and $A + B$ is PB, then so are A and B .
- (ii) If $M = \begin{bmatrix} A & C \\ B & D \end{bmatrix}$ is nnPB, then so are A and D .
- (iii) If M is nnPB then so are all principal submatrices of M .

Proof. (i) Clearly, $(A + B)^N \geq A^N$.

(ii) If $M = \begin{bmatrix} A & 0 \\ B & 0 \end{bmatrix} + \begin{bmatrix} 0 & C \\ 0 & D \end{bmatrix}$ is PB, then (by part (i)), so are its two components. It is clear that $\begin{bmatrix} A & 0 \\ B & 0 \end{bmatrix}^N = \begin{bmatrix} A^N & 0 \\ ? & 0 \end{bmatrix}$, which guarantees that A is PB.

(iii) Clear by permuting rows and columns. $\square$

Let us now turn to the diagonal entries of a nnPB matrix.

5.2. Diagonal entries of a nonnegative PB matrix. Let $M = [m_{ij}]$ be a nnPB matrix.

Using Theorem 2.1 and the nonnegativity, we may conclude that

$$(5.1) \quad m_{ii} \leq \rho(M) \leq 1.$$

This also follows from the fact that $(m_{ii})^N \leq (M^N)_{ii}$, which is bounded.

Our aim is to use the diagonal entries of M to investigate power boundedness and at the same time to relate the three concepts of PB, focal power, and convergence.

The two main cases to be considered are when some $m_{ii} = 1$ or when all $m_{ii} < 1$. Only in the former case does focal power play a role. In the latter case there is little we can say about the form of M , as seen from the example $M = \begin{bmatrix} .8 & .2 \\ .2 & .8 \end{bmatrix}$, which has eigenvalues $\{1, 0.6\}$.

We next have

LEMMA 5.2. *If M is nnPB, then, for all natural numbers k, N ,*

- (i) $m_{ii}^k \leq m_{ii}$,
- (ii) $\sum_{i=1}^n m_{ii}^k \leq \text{Tr}(M)$, and
- (iii) $m_{ii} = 1 \Rightarrow (M^N)_{ii} = 1$.

Proof. (i) and (ii) follow from the inequalities (5.1).

(iii). M^k is also PB with $(M^k)_{ii} \leq 1$. Hence, $1 = (m_{ii})^N \leq (M^N)_{ii} \leq 1$. $\square$

The matrix $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has the property that $(M^2)_{ii} = 1$ while $m_{ii} = 0$; thus, we cannot reverse the conditions of (iii).

We now introduce the parameters $u_k = \#\{i : (M^k)_{ii} = 1\}$ and $t_k = \text{Tr}(M^k)$.

Then it easily follows that

$$u_1 \leq u_k \leq t_k \leq n.$$

We note that the sequence $\{t_k\}$ need not be increasing, as seen from $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Also, if M is not nilpotent, then some t_k must be nonzero.

Suppose that $v = \max\{u_k\}$ and $w = \max\{t_k\}$ and let $\Delta = \{i : u_i = v\}$ and $\Gamma = \{i : \text{Tr}(A^i) = w\}$. Clearly, if $u_r = v$ then $u_{r^k} = v$ for all $k \geq 1$.

LEMMA 5.3.

- (i) If $M = I + X$ is nnPB, with $X \geq 0$, then $X = 0$.
- (ii) If $M = \begin{bmatrix} I + X & ? \\ ? & ? \end{bmatrix}$ is nnPB, with $X \geq 0$, then $X = 0$.
- (iii) If $M = \begin{bmatrix} I & C \\ B & D \end{bmatrix}$ is nnPB, then M is (1,1) focussed.

Proof. (i) $M^N = (I + X)^N \geq I + NX$. For M to be power bounded, we must have $X = 0$.

(ii) Since $I + X$ is nnPB, this follows from part (i).

(iii) M^N has the form $\begin{bmatrix} I + X & C_N \\ B_N & D_N \end{bmatrix} \geq 0$. By part (ii) X must vanish. $\square$

COROLLARY 5.4. If M is nnPB and all $m_{ii} = 1$, then $M = I$.

5.3. Unit diagonal entries. Suppose now that M is nnPB and has some diagonal elements $m_{ii} = 1$.

Applying Lemma 5.3 (ii), we may permute the rows and columns of M so that

$$M \underset{\text{per}}{\approx} M' = \begin{bmatrix} I_{u_1} & C \\ B & D \end{bmatrix},$$

where D is also nnPB and all $d_{ii} < 1$. From Lemma 5.3 (iii) we know that M' is (1,1) focussed and that $M'^N = \begin{bmatrix} I_{u_1} & C_N \\ B_N & D_N \end{bmatrix}$. Moreover, we also know, using [7, Theorem 1.1], that, for all N ,

$$(5.2) \quad \begin{aligned} (1) \quad CD^N B &= 0, \\ (2) \quad B_N &= G_N B = (I + D + \cdots + D^{N-1})B \text{ is bounded,} \\ (3) \quad C_N &= CG_N = C(I + D + \cdots + D^{N-1}) \text{ is bounded,} \\ (4) \quad D_N &= D^N + X_N \text{ where } X_N = \sum_{p+q \leq N-2} D^p B C D^q. \end{aligned}$$

For convenience we do not distinguish between M and M' .

It should be noted that:

- (i) Condition (1) implies that $CB_N = 0 = C_N B$.
- (ii) $X_N^2 = 0$ (by (1)).
- (iii) X_N is increasing with N .
- (iv) $X_N \leq G_N B C G_N$, which is bounded by (2) in (5.2). Therefore, X_N converges.
- (v) Taking traces shows that $Tr(D^p B C D^q) = Tr(C D^{p+q} B) = 0$, and hence, that

$$Tr(D_r) = Tr(D^r), \quad r = 1, 2, \dots$$

We next need

LEMMA 5.5. $G_N(F)B$ is bounded for all N if and only if $F^N B$ is bounded and $Z_1^0(F)B = 0$.

Proof. If $G_N(F)B$ is bounded for all N then so is $(I - F)G_N B = (I - F^N)B$, and hence, $F^N B$ is bounded as well. Next we post-multiply (2.1) by B giving

$$(5.3) \quad G_N(F)B = (I - F)^d (I - F^N)B + \sum_{j=0}^{m_1-1} \binom{N}{j+1} Z_1^j B,$$

in which the left hand side and the first term on the right hand side are bounded. Now pre-multiply by $Z_1^{m_1-1}$. This leaves us the bounded term $N Z_1^{m_1-1} B$. This cannot be, unless $Z_1^{m_1-1} B = 0$. Next we multiply by $Z_1^{m_1-2}$ etc., implying eventually that $Z_1^0(F)B = 0$.

Conversely, if $Z_1^0(F)B = 0$ then the second term on the right hand side of (5.3) vanishes, while the first term is bounded because $F^N B$ is bounded. $\square$

Let us now apply the previous Lemma to the block matrix M .

PROPOSITION 5.6. The matrix $M = \begin{bmatrix} I_{u_1} & C \\ B & D \end{bmatrix} \geq 0$ is PB if and only if M is (1,1) focussed, D is PB and $Z_1^0 B = 0 = C Z_1^0$, where $Z_1^0 = I - (I - D)(I - D)^\#$.

Proof. We have seen that if M is PB then necessarily, D is PB, M is (1,1) focussed, and $G_N B$ and $C G_N$ are bounded, for all N . The latter ensure that $Z_1^0 B = 0 = C Z_1^0$ by (5.3).

Conversely, if M is (1,1) focussed then $M^N = \begin{bmatrix} I_{u_1} & C_N \\ B_N & D_N \end{bmatrix}$ in which B_N and C_N are bounded by Lemma 5.5, and $D_N = D^N + X_N$ is bounded because D is PB and (iii) $X_N \leq G_N B C G_N$ is also bounded, for all N . $\square$

A special subcase is when $\rho(D) < 1$, which includes the case where D is nilpotent and $\rho(D) = 0$. In this case $D^N \rightarrow 0$, $B_N \rightarrow (I - D)^{-1}B$, and $C_N \rightarrow C(I - D)^{-1}$. In addition, $D_N = D^N + X_N$ in which $D^N \rightarrow 0$ and $X_N = \sum_{p+q \leq N-2} D^p B C D^q$ is monotone increasing and is bounded above by $(I - D)^{-1} B C (I - D)^{-1}$ and as such, converges.

This leads to

COROLLARY 5.7. *If $M = \begin{bmatrix} I & C \\ B & D \end{bmatrix}$ is $(1,1)$ focussed, $Z_1^0(D)B = 0 = CZ_1^0(D)$, and $\rho(D) < 1$, then M^N converges to $Z_1^0(M)$.*

5.4. No unit diagonal entries. Let M be PB with all $m_{ii} < 1$, as was the case with the matrix D in the previous setting. It is not clear if a power of M can have unit diagonal entries. We may or may not have $\rho(M) = 1$. For example $M = \begin{bmatrix} .5 & .1 \\ .1 & .5 \end{bmatrix}$ has eigenvalue of $\{.6, .4\}$, and thus, is zero convergent.

We may even have a zero diagonal, as seen from the example $M = \begin{bmatrix} 0 & c \\ b & 0 \end{bmatrix}$ with $|bc| \leq 1$. Indeed, $M^{2k} = (bc)^k I$. This example also shows that for a nnPB matrix some of its entries may be larger than 1.

Let us now turn to a special class of PB matrices where we keep the diagonal entries under control and can effectively employ the trace concept. This in turn allows us to bring the previous “diagonal” argument to a more transparent conclusion.

6. The semigroup case of nonnegative PB matrices. Consider a nonnegative PB matrix M for which $(M^k)_{ii} = 0$ or 1 (**only**), for all powers $k = 1, 2, \dots$. Our aim is to show that such a matrix must be **periodic**, and that a power of M is permutation similar to an idempotent matrix.

If M is nilpotent, then M is periodic and the idempotent is zero. So let us assume that M is nilpotent in what follows.

First we need some simple observations.

If M is nnPB and $m_{ii} = 0$ or 1, then $m_{ii}^k = m_{ii}$ and $u_k = t_k$ for all $k = 1, 2, \dots$, and hence, we can combine the two parts of Lemma 5.2 in

$$(6.1) \quad \text{Tr}(M^k) \geq \sum_{i=1}^n m_{ii}^k = \sum_{i=1}^n m_{ii} = \text{Tr}(M) \quad \text{for } k = 1, 2, \dots,$$

i.e., $t_1 \leq t_k$ for all k . In general, for a nnPB matrix, we saw that $0 \leq m_{ii} \leq 1$ and hence the assumption that m_{ii} can only equal 0 or 1 is an **extremal** condition.

Because of the assumption that $(M^k)_{ii} = 1$ or $(M^k)_{ii} = 0$, we see that for every $k = 0, 1, \dots$

$$M^k \underset{\text{per}}{\approx} \begin{bmatrix} I_{t_k} + X_k & ? \\ ? & N_k \end{bmatrix}$$

where $t_k = \text{Tr}(M^k)$, $X_k \geq 0$, $(X_k)_{ii} = 0$ and $(N_k)_{ii} = 0$. Lemma 5.3 ensures that $X_k = 0$ and we may state that for all $k = 0, 1, \dots$

$$M^k \underset{\text{per}}{\approx} \begin{bmatrix} I_{t_k} & ? \\ ? & N_k \end{bmatrix},$$

with $(N_k)_{ii} = 0$. Using the same argument, we see that for $r = 1, 2, \dots$

$$(6.2) \quad (M^k)^r \underset{\text{per}}{\approx} \begin{bmatrix} I_{t_k} & ? \\ ? & ? \end{bmatrix},$$

which has $\text{Tr}(M^{kr}) \geq t_k$ and has at least t_k ones on its diagonal.

Next we return to the case where $k = 1$, $M \underset{\text{per}}{\approx} M' = \begin{bmatrix} I_t & C \\ B & D \end{bmatrix}$, $t = t_1 = \text{Tr}(M)$, $B = B_1$, $C = C_1$, and $(D)_{ii} = 0$. Then from (6.2), we may conclude that

$$(6.3) \quad M^r \underset{\text{per}}{\approx} (M')^r = \begin{bmatrix} I_t & C_r \\ B_r & D_r \end{bmatrix},$$

which is (1,1) focussed and has $\text{Tr}(M^r) \geq t$.

In addition we know that

- (i) $\text{Tr}(D^r) = \text{Tr}(D_r) = \text{Tr}(M^r) - \text{Tr}(M) \leq n - t$,
- (ii) $\text{Tr}(D^r)$ may oscillate, as seen from $D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ with $D^2 = I$.

Since M is nonnegative non-nilpotent, there exists a power of M , say M^u , with $\text{Tr}(M^u) > 0$ (recall that M is nilpotent if and only if all powers have zero trace!).

We now focus on

- (a) $w = \max\{\text{Tr}(M^k); k \in \mathbb{N}\} = \max\{t_k\}$ and (b) $\Gamma = \{i; \text{Tr}(M^i) = w\}$.

The set Γ must be nonempty. Now suppose that $r \in \Gamma$, i.e., $\text{Tr}(M^r) = w$. By (6.1) applied to M^r , we see that

$$w = \text{Tr}(M^r) \leq \text{Tr}(M^{rs}) \leq w \text{ for all } s,$$

and thus, $rs \in \Gamma$ for $s = 1, 2, \dots$. This means that

$$(6.4) \quad \Gamma + \Gamma \subseteq \Gamma.$$

Thus, Γ is an **additive subsemigroup** of $\mathbb{N}$.

Next, let $M^r \underset{\text{per}}{\approx} Y = \begin{bmatrix} I_w & L \\ K & \eta \end{bmatrix}$, where $(\eta)_{ii} = 0$. By (6.3) we see that

$$(6.5) \quad Y^i = \begin{bmatrix} I_w & L_i \\ K_i & \eta_i \end{bmatrix}$$

so that $\text{Tr}(Y^i) = \text{Tr}(M^{ri}) = w + \text{Tr}(\eta_i)$ for all $i = 1, 2, \dots$. By the maximality of w we conclude that $\text{Tr}(\eta_i) = 0$ for all $i = 1, 2, \dots$. In addition, we see from (5.2) that

$$(6.6) \quad \begin{aligned} (i) \quad L\eta^i K &= 0 \text{ for all } i = 0, 1, \dots, \\ (ii) \quad K_i &= (I + \eta + \dots + \eta^{i-1})K, \\ (iii) \quad L_i &= L(I + \eta + \eta^2 + \dots + \eta^{i-1}), \end{aligned}$$

as well as

$$(6.7) \quad \eta_i = \eta^i + \sum_{p+q \leq i-2} \eta^p K L \eta^q.$$

The identity (6.7) implies that $0 = \text{Tr}(\eta_i) = \text{Tr}(\eta^i)$ for $i = 0, 1, \dots$. That is, η is nilpotent, say of index $\text{in}(\eta) = s$.

We now claim that matrices of the form (6.5) are in fact periodic. Indeed, it is clear that $K_i \rightarrow (I - \eta)^{-1}K$ and $L_i \rightarrow L(I - \eta)^{-1}$ so both sequences are bounded. Moreover, from (6.7), we see that if $i = 2s + t + 2$, then we may split the sum into

$$\sum_{p+q \leq 2s} \eta^p K L \eta^q + \sum_{2s < p+q \leq 2s+t} \eta^p K L \eta^q.$$

In the second summand, either $p \geq s$ or $q \geq s$, and hence, this term vanishes. Thus, η_i reduces to just a finite (bounded) sum. Let us now take $i = s$ and partition Y^s as

$$Y^s = \begin{bmatrix} I_w & L(I - \eta)^{-1} \\ (I - \eta)^{-1}K & \eta_s \end{bmatrix}$$

where $\eta_s = \sum_{p+q \leq s-2} \eta^p K L \eta^q$. Using the fact that $L\eta^j K = 0$ for all j , we see that $(\eta_s)^2 = 0$ in addition to $L(I - \eta)^{-1}\eta_s = 0$. Likewise if we use (6.6) then it follows that $\eta_s(I - \eta)^{-1}K = 0$ and $L(I - \eta)^{-2}K = 0$. We now apply the following trivial result

LEMMA 6.1. If $G = \begin{bmatrix} I & Q \\ P & R \end{bmatrix}$ with $R^2 = 0, QP = 0, RP = 0$ and $QR = 0$, then G^2 is idempotent.

It now follows that for $s = \text{in}(\eta)$, Y^{2s} is idempotent and hence is periodic. We have thus shown that if M is nonnegative power bounded, with $(M^k)_{ii} = 0, 1$, then M^{2rs} is idempotent.

Conversely, if M^m is idempotent then it is trivially power bounded. Moreover, it follows that M is also PB. Indeed, for any $k \in \mathbb{N}$, let $k = tm + \alpha$ with $0 \leq \alpha < m$. Further let $|(M^{tm})_{ij}| \leq K_1$ for all i, j, t , and let $K_2 = \max_{j=1}^n \{(M^\alpha)_{pj}; j = 1, \dots, m, \alpha = 0, 1, \dots, m-1\}$. Then $|(M^k)_{ij}| \leq K_1 K_2$ for all i, j, k .

7. Remarks.

- (a) To accommodate “large” entries besides (possible) zero entries, in a nnPB matrix M , it is not surprising that M has to be periodic.
- (b) One instance where our matrices do have 0-1 power diagonals is furnished when the entries in M come from a sub-semigroup S^* of $\mathbb{R}^+$ for which the following implication holds [14]:
 (i) $S^* + S^* \subseteq S^*$, (ii) $S^* \cdot S^* \subseteq S^*$, (iii) $0, 1 \in S^*$, and (iv) $x \in S^*, x \neq 0 \Rightarrow x \geq 1$.
- (c) The eigenvalues of a periodic matrix are either zero or lie on the unit circle. On the other hand, the eigenvalues of an irreducible nonnegative matrix lie symmetrically around the unit circle. This means that the index of primitivity $\pi(A)$ equals the periodicity.
- (d) Can the zero-one structure of the diagonals of the powers be used to obtain a canonical form for PB matrices?

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REFERENCES

- [1] A. Ben-Israel and T.N.E. Greville. *Generalized Inverses: Theory and Applications*. John Wiley and Sons, New York, 1974.
- [2] A. Berman and R.J. Plemmons. *Nonnegative Matrices in the Mathematical Sciences*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 1994.
- [3] X. Chen and R.E. Hartwig. The Picard iteration and its application. *Linear Multilinear Algebra*, 54(5):329–341, 2006.
- [4] D.T. Finkbeiner. *Introduction to Matrices and Linear Transformations*, second edition. W.H. Freeman and Co., San Francisco, 1966.
- [5] F.R. Gantmacher. *The Theory of Matrices* vol. 2. Translated from the Russian by K.A. Hirsch. Reprint of the 1959 translation. AMS Chelsea Publishing, Providence, RI, 1998.
- [6] R.E. Hartwig. A note on periodic matrices. *Indust. Math.*, 27:51–55, 1977.
- [7] R.E. Hartwig and J. Maroulas. Focal power. *Electron. J. Linear Algebra*, 13:56–71, 2005.

- [8] R. Horn and C.R. Johnson. *Matrix Analysis*. Cambridge University Press, Cambridge, 1990.
- [9] A. Ostrowski. Über Normen von Matrizen. *Math. Z.*, 63:2–18, 1955.
- [10] M. Neumann and R.J. Plemmons. Convergent nonnegative matrices and iterative methods for consistent linear systems. *Numer. Math.*, 31:265–279, 1978/79.
- [11] M. Neumann and R.J. Plemmons. Generalized inverse-positivity and splittings of M -matrices. *Linear Algebra Appl.*, 23:21–35, 1979.
- [12] U. Rothblum. A rank characterization of the number of final classes of a nonnegative matrix. *Linear Algebra Appl.*, 23:65–68, 1979.
- [13] U. Rothblum. Expansions of sums of matrix powers. *SIAM Rev.*, 23:143–164, 1981.
- [14] S. Yang and R. Zhang. Green's relations in the matrix semigroup $M_n(S)$. *Linear Algebra Appl.*, 222:63–76, 1995.