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ON THE SPECTRAL RADII OF GRAPHS WITHOUT GIVEN CYCLES*

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Abstract. Let G be a graph with n vertices and $\rho(G)$ be the spectral radius of its adjacency matrix. Write C_l for the cycle of order l and let $g_l(n) = \max\{\rho(G) : |V(G)| = n$, neither C_l nor C_{l+1} is a subgraph of $G\}$. This paper obtains the exact value of $g_5(n)$ with the unique extremal graph.

Key words. Forbidden subgraph, Adjacency matrix, Spectral radius.

AMS subject classifications. 05C50.

1. Introduction. Let $V(G)$ be the vertex set of a graph G and

$$N^d(u) = \{v|v \in V(G), d_G(v,u) = d\},$$

where $d_G(v,u)$ is the distance between two vertices u and v. Denote by $d_G(u)$, the degree of u. A vertex of degree k is called a k-vertex. For a nonempty subset S of $V(G)$, the subgraph induced by S is denoted by $G[S]$. Let $A(G)$ be the adjacency matrix of G and $P(G,\lambda)$ be the characteristic polynomial of $A(G)$. The largest modulus of an eigenvalue of $A(G)$ is called the spectral radius of G and denoted by $\rho(G)$. It is valuable to study the relation between spectral radius and some kinds of subgraphs (such as, clique, path, cycle, complete bipartite subgraph, etc). See [2, 3, 4, 5, 6, 7, 8] for results along these lines.

We use C_r, P_r, $K_{1,r-1}$ and K_r to denote the cycle, path, star and complete graph of order r, respectively. In particular, $K_{1,0} \cong K_1$. For each positive integer $l \geq 3$, V. Nikiforov [7] defined a function $g_l(n)$ as follows.

$$g_l(n) = \max\{\rho(G) : |V(G)| = n, \text{ neither } C_l \text{ nor } C_{l+1} \text{ is a subgraph of } G\}.$$

Favaron, Mahéo and Saclé [1] showed that if a graph G of order n contains neither C_3 nor C_4, then $\rho(G) \leq \sqrt{n-1}$. Further, one can find that $g_3(n) = \sqrt{n-1}$ and $K_{1,n-1}$

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is the unique extremal. In [7], V. Nikiforov gave an estimate for the value of $g_l(n)$ and proposed the following conjecture:

CONJECTURE 1.1. ([7]) Let $k \geq 2$ and G be a graph of sufficiently large order n. Let $S_{n,k}$ be the graph obtained by joining each vertex of K_k to $n-k$ isolated vertices and $S^+_{n,k}$ be the graph obtained by adding one edge within the independent set of $S_{n,k}$.
(i) If G does not contain C_{2k+1} and C_{2k+2}, then $\rho(G) \leq \rho(S_{n,k})$ with equality if and only if $G \cong S_{n,k}$.
(ii) If G does not contain C_{2k+2}, then $\rho(G) \leq \rho(S^+_{n,k})$ with equality if and only if $G \cong S^+_{n,k}$.

Note that $S^+_{n,k}$ contains neither C_{2k+2} nor C_{2k+3}. If Conjecture 1.1 is true, then the exact value of $g_l(n)$ is completely obtained for $l \geq 5$ and sufficiently large n. This paper proves the following theorem, which implies the above conjecture is true for $k = 2$.

THEOREM 1.2. *Let* $n \geq 6$ *and*

$$\mathbb{G}_n = \{G : |V(G)| = n, \text{ neither } C_5 \text{ nor } C_6 \text{ is a subgraph of } G\}$$

and G^* *have maximal spectral radius among all graphs in* $\mathbb{G}_n$. *Then,* $G^* \cong S_{n,2}$.

2. Proof.

A graph is said to be trivial, if its edge set is empty. Straightforward calculation shows that

$$\rho^2 - \rho - 2(n-2) = 0. \tag{2.1}$$

for $\rho = \rho(S_{n,2})$. Let G^* have maximal spectral radius among all graphs in $\mathbb{G}_n$. Clearly, G^* is connected. Since G^* is C_5-free, for any vertex $u \in V(G^*)$, $G^*[N^1(u)]$ cannot contain P_4 as a subgraph. Further, we can observe the following properties.

LEMMA 2.1. *Let* u *be a vertex of* G^* *and* $w \in N^2(u)$.

(i) *Each component of* $G^*[N^1(u)]$ *is either a star* $K_{1,r}$ *for some* $r \geq 0$ *or a copy of* K_3.
(ii) *If* w *is adjacent to some nontrivial component of* $G^*[N^1(u)]$*, then this component is the unique one to which* w *is adjacent.*
(iii) *Particularly, if* w *is adjacent to some* $K_{1,r}$*-component for* $r \geq 2$ *or a* K_3*-component, then its neighbor in this component is also unique.*

Now we introduce some additional notation. Let $A = A(G^*)$ and $B = (b_{ij})_{n\times n} = A^2 - A - 2(n-2)I$. Given a vertex $u \in V(G^*)$, let $t_u(H)$ be the number of H-components of $G^*[N^1(u)]$ and $t'_u(H)$ be the number of vertices in $N^2(u)$ adjacent to an H-component of $G^*[N^1(u)]$. Let F_u be the bipartite subgraph induced by the

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edges from all the isolated vertices of $G^*[N^1(u)]$ to $N^2(u)$ and $e(F_u)$ be the number of edges in F_u.

LEMMA 2.2. *For any vertex $u \in V(G^*)$,*

$$\sum_{i=1}^{n} b_{ui} \leq 2 - 2\sum_{r\geq 0} t_u(K_{1,r}) - \sum_{r\geq 2} t'_u(K_{1,r}) - t'_u(K_3) - 2t'_u(K_{1,0}) + e(F_u).$$

Equality holds if and only if $N^3(u) = \emptyset$ and $N^1(v)\backslash\{w\} = N^1(w)\backslash\{v\}$ for any v, w within a same $K_{1,1}$-component of $G^*[N^1(u)]$.

Proof. Note that the (i, j)-element of A^k is the number of walks of length k from the vertex i to the vertex j in G^*. Clearly, $b_{uu} = d_{G^*}(u) - 2(n-2)$ and $b_{ui} = 0$ for any $i \in \cup_{d\geq 3} N^d(u)$. Further, we can observe that

$$\sum_{i\in N^1(u)} b_{ui} = 2[\sum_{r\geq 0} r t_u(K_{1,r}) + 3t_u(K_3)] - d_{G^*}(u).$$

By Lemma 2.1,

$$\sum_{i\in N^2(u)} b_{ui} \leq 2t'_u(K_{1,1}) + \sum_{r\geq 2} t'_u(K_{1,r}) + t'_u(K_3) + e(F_u). \tag{2.2}$$

Note that

$$|N^1(u)| = d_{G^*}(u) = \sum_{r\geq 0}(r+1)t_u(K_{1,r}) + 3t_u(K_3),$$

$$|N^2(u)| = \sum_{r\geq 0} t'_u(K_{1,r}) + t'_u(K_3)$$

and

$$|N^1(u)| + |N^2(u)| \leq n - 1. \tag{2.3}$$

We have

$$\begin{aligned}
\sum_{i=1}^{n} b_{ui} &\leq 2[\sum_{r\geq 0} r t_u(K_{1,r}) + 3t_u(K_3)] - 2(n-2) + \sum_{i\in N^2(u)} b_{ui} \\
&\leq 2[1 - \sum_{r\geq 0} t_u(K_{1,r}) - |N^2(u)|] + \sum_{i\in N^2(u)} b_{ui} \\
&= 2\{1 - \sum_{r\geq 0}[t_u(K_{1,r}) + t'_u(K_{1,r})] - t'_u(K_3)\} + 2t'_u(K_{1,1}) \\
&\quad + \sum_{r\geq 2} t'_u(K_{1,r}) + t'_u(K_3) + e(F_u) \\
&= 2 - 2\sum_{r\geq 0} t_u(K_{1,r}) - \sum_{r\geq 2} t'_u(K_{1,r}) - t'_u(K_3) - 2t'_u(K_{1,0}) + e(F_u).
\end{aligned}$$

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Equality holds if and only if both equality holds in both (2.2) and (2.3). This implies that $N^3(u) = \emptyset$ and $N^1(v)\backslash\{w\} = N^1(w)\backslash\{v\}$ for any v, w within a same $K_{1,1}$-component of $G^*[N^1(u)]$. ∎

Lemma 2.3. *Let $G = (X, Y)$ be a nontrivial bipartite graph of size $e(G)$. If G does not contain a path P_5 with both endpoints in X, then $e(G) \leq 2|Y| + |X| - 2$. Equality holds if and only if $G \cong K_{2,|Y|}$ or $G \cong K_{|X|,1}$.*

Proof. We may assume that G is connected. Now we use induction on $|Y|$. If $|Y| = 1$, then $G \cong K_{|X|,1}$ and $e(G) = |X| = 2|Y| + |X| - 2$. Suppose that $|Y| \geq 2$. If Y contains a vertex v of degree one, then $G - v$ is also connected. By the induction hypothesis, $e(G - v) \leq 2(|Y| - 1) + |X| - 2$, and hence,

$$e(G) \leq 2(|Y| - 1) + |X| - 2 + d_G(v) < 2|Y| + |X| - 2.$$

Next suppose that $d_G(v) \geq 2$ for any vertex $v \in Y$. Note that G is connected. If all of the vertices in Y have common neighborhood, then $G \cong K_{|X|,|Y|}$. Since G does not contain a copy of P_5 with both endpoints in X, $|X| \leq 2$ and the inequality holds. If not all the vertices in Y have common neighborhood, then there are two vertices $u, v \in Y$ with $N^1(u) \neq N^1(v)$ and $N^1(u) \cap N^1(v) \neq \emptyset$. Say $w_1, w_2 \in N^1(u)$ and $w_2, w_3 \in N^1(v)$. Then $w_1uw_2vw_3$ is a copy of P_5 with $w_1, w_3 \in X$, a contradiction. ∎

Let R_k be the graph obtained from k copies of K_4 by identifying a vertex of them. Let $R_{k,r}$ be the graph obtained from R_k and $S_{r,2}$ by identifying the central vertex of R_k with one of the $(r-1)$-vertices of $S_{r,2}$, where $k \geq 0$ and $r \geq 2$. In particular, $R_{0,r} \cong S_{r,2}$.

Claim 2.4. *For any vertex $u \in V(G^*)$, if $t_u(K_{1,0}) > 0$, then $\sum_{i=1}^{n} b_{ui} \leq 0$. Equality holds if and only if $G^* \cong R_{k,2}$ for some nonnegative integer $k = \frac{1}{3}(n-2)$.*

Proof. First assume that $t'_u(K_{1,0}) > 0$. If F_u contains a copy $P(v, w)$ of P_5 with both endpoints $v, w \in N^1(u)$, then $P(v, w) + uv + uw$ is a 6-cycle. Since G^* does not contain C_6, by Lemma 2.3, $e(F_u) \leq 2t'_u(K_{1,0}) + t_u(K_{1,0}) - 2$. Thus, by Lemma 2.2,

$$\sum_{i=1}^{n} b_{ui} \leq -2\sum_{r\geq 1} t_u(K_{1,r}) - \sum_{r\geq 2} t'_u(K_{1,r}) - t'_u(K_3) - t_u(K_{1,0}) < 0.$$

Now suppose that $t'_u(K_{1,0}) = 0$, then $e(F_u) = 0$. By Lemma 2.2, we have

$$\sum_{i=1}^{n} b_{ui} \leq 2 - 2\sum_{r\geq 0} t_u(K_{1,r}) - \sum_{r\geq 2} t'_u(K_{1,r}) - t'_u(K_3).$$

Note that $t_u(K_{1,0}) > 0$. Thus, $\sum_{i=1}^{n} b_{ui} \leq 0$. Equality holds if and only if $N^3(u) = \emptyset$,

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$\sum_{r\geq 0} t_u(K_{1,r}) = t_u(K_{1,0}) = 1$, and $t'_u(K_3) = 0$. This implies that $G^* \cong R_{k,2}$ for some $k = \frac{1}{3}(n-2)$. □

Claim 2.5. *For any vertex $u \in V(G^*)$, $\sum_{i=1}^{n} b_{ui} \leq 2$. Moreover,*

(i) $\sum_{i=1}^{n} b_{ui} = 2$ *if and only if $G^* \cong R_k$ for some positive integer $k = \frac{1}{3}(n-1)$;*

(ii) $\sum_{i=1}^{n} b_{ui} = 1$ *if and only if $G^* \cong R_k^+$ for some positive integer $k = \frac{1}{3}(n-2)$, where R_k^+ is the graph obtained from R_k by adding a pendant edge to some 3-vertex.*

Proof. According to Claim 2.4, we may assume that $t_u(K_{1,0}) = 0$. So $t'_u(K_{1,0}) = e(F_u) = 0$. By Lemma 2.2, we have

$$\sum_{i=1}^{n} b_{ui} \leq 2 - 2\sum_{r\geq 0} t_u(K_{1,r}) - \sum_{r\geq 2} t'_u(K_{1,r}) - t'_u(K_3).$$

Hence, $\sum_{i=1}^{n} b_{ui} \leq 2$. Moreover, $\sum_{i=1}^{n} b_{ui} = 2$ if and only if $\sum_{r\geq 0} t_u(K_{1,r}) = 0$ and $t'_u(K_3) = 0$. This implies that $G^* \cong R_k$ for some $k = \frac{1}{3}(n-1)$.

Similar to the above, $\sum_{i=1}^{n} b_{ui} = 1$ if and only if $\sum_{r\geq 0} t_u(K_{1,r}) = 0$ and $t'_u(K_3) = 1$. This implies that $G^* \cong R_k^+$ for some $k = \frac{1}{3}(n-2)$. □

The following lemma is an immediate consequence of Rayleigh's theorem applied for the adjacency matrices.

Lemma 2.6. *Let G be a connected graph in $\mathbb{G}_n$. If G has two cut vertices, then there exists a connected graph $G' \in \mathbb{G}_n$ such that $\rho(G) < \rho(G')$.*

Theorem 2.7. *If $n \geq 6$, then $\sum_{i=1}^{n} b_{ui} \leq 0$ for any $u \in V(G^*)$, and hence, $\rho(G^*) \leq \rho(S_{n,2})$.*

Proof. According to Claim 2.5, if $\sum_{i=1}^{n} b_{ui} = 1$, then $G^* \cong R_k^+$ for some $k \geq 2$. Now, G^* has two cut vertices. Since G^* has maximal spectral radius, by Lemma 2.6, we get a contradiction.

If $\sum_{i=1}^{n} b_{ui} = 2$, then $G^* \cong R_k$, where $n = |V(R_k)| = 3k+1 \geq 7$. Straightforward calculation shows that

$$\rho(R_k) = 1 + \sqrt{n} < \frac{1+\sqrt{1+8(n-2)}}{2} = \rho(S_{n,2})$$

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for $n \geq 7$, a contradiction. Thus, $\sum_{i=1}^{n} b_{ui} \leq 0$. Let X be a positive eigenvector of $A(G^*)$ corresponding to $\rho = \rho(G^*)$ such that $\sum_{i=1}^{n} x_i = 1$. Then

$$\rho^2 - \rho - 2(n-2) = \sum_{i=1}^{n} [\rho^2 - \rho - 2(n-2)] x_i = \sum_{i=1}^{n} (\sum_{j=1}^{n} b_{ij}) x_j \leq 0.$$

By (2.1), $\rho(G^*) \leq \rho(S_{n,2})$. □

Note that $S_{n,2} \in \mathbb{G}_n$. Theorem 2.7 implies $g_5(n) = \rho(S_{n,2}) = \frac{1+\sqrt{1+8(n-2)}}{2}$. Next we shall consider the uniqueness of the extremal graph G^*.

Let $T_{k,r}$ be the graph obtained from R_k and $S_{r,2}$ by identifying the central vertex of R_k with one of 2-vertices of $S_{r,2}$, where $k \geq 0$ and $r \geq 3$. Particularly, $T_{0,r} \cong S_{r,2}$ and $T_{k,3} \cong R_{k,3}$.

Lemma 2.8. *For any two positive integers k, r with $3k + r = n \geq 6$, G^* is not isomorphic to either $R_{k,r}$ or $T_{k,r}$.*

Proof. Assume to the contrary that $G^* \cong R_{k,r}$ for some $k \geq 1$ and $r \geq 2$. If $n = 6$, then $k = 1$ and $r = 3$. Straightforward calculation shows that $\rho(R_{1,3}) \doteq 3.2618 < \rho(S_{6,2})$, a contradiction.

Next let $n \geq 7$. By Theorem 2.7, $\sum_{i=1}^{n} b_{ui} \leq 0$ for any $u \in V(G^*)$. Let v be a 3-vertex in some K_4-copy of $R_{k,r}$. Clearly, $\sum_{r \geq 0} t_v(K_{1,r}) = 0$ and $t_v(K_3) = 1$. Moreover, $t'_v(K_3) = n-4 \geq 3$. According to Lemma 2.2, $\sum_{i=1}^{n} b_{vi} \leq 2-3 < 0$. Let X be a positive eigenvector of $A(G^*)$ corresponding to $\rho = \rho(G^*)$ such that $\sum_{i=1}^{n} x_i = 1$. Thus,

$$\rho^2 - \rho - 2(n-2) = \sum_{i=1}^{n} [\rho^2 - \rho - 2(n-2)] x_i = \sum_{i=1}^{n} (\sum_{j=1}^{n} b_{ij}) x_j = \sum_{j=1}^{n} (\sum_{i=1}^{n} b_{ji}) x_j < 0.$$

By (2.1), $\rho(G^*) < \rho(S_{n,2})$, a contradiction. So G^* is not isomorphic to $R_{k,r}$. Similarly, we can prove that G^* is not isomorphic to $T_{k,r}$ for any $k \geq 1$ and $r \geq 3$. □

Proof of Theorem 1.2. Suppose that G^* is not isomorphic to $S_{n,2}$. By Theorem 2.7, $\sum_{i=1}^{n} b_{ui} \leq 0$ for any $u \in V(G^*)$. Similar to the proof of Lemma 2.8, it suffices to show that there exists a vertex $v \in V(G^*)$ such that $\sum_{i=1}^{n} b_{vi} < 0$.

Select a vertex $v \in V(G^*)$ arbitrarily. Note that G^* is not isomorphic to $R_{k,r}$ for any $k \geq 1$ and $r \geq 2$. If $t_v(K_{1,0}) > 0$, then by Claim 2.4, $\sum_{i=1}^{n} b_{vi} < 0$.

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Next let $t_v(K_{1,0}) = 0$. Then by Lemma 2.2,

$$\sum_{i=1}^{n} b_{vi} \leq 2 - 2\sum_{r\geq 1} t_v(K_{1,r}) - \sum_{r\geq 2} t'_v(K_{1,r}) - t'_v(K_3). \tag{2.4}$$

If $t_v(K_{1,1}) > 0$, then $\sum\limits_{i=1}^{n} b_{vi} \leq 0$ with equality if and only if $\sum\limits_{r\geq 1} t_v(K_{1,r}) = t_v(K_{1,1}) = 1$ and $t'_v(K_3) = 0$. So the equality implies that $G^* \cong T_{k,r}$ for some $k \geq 1$ and $r \geq 3$ with $3k + r = n$, a contradiction. Thus, $\sum\limits_{i=1}^{n} b_{vi} < 0$.

Now let $t_v(K_{1,0}) = t_v(K_{1,1}) = 0$. Then (2.4) becomes

$$\sum_{i=1}^{n} b_{vi} \leq 2 - \sum_{r\geq 2}[2t_v(K_{1,r}) + t'_v(K_{1,r})] - t'_v(K_3).$$

If $\sum\limits_{r\geq 2} t_v(K_{1,r}) > 0$, then $\sum\limits_{i=1}^{n} b_{vi} \leq 0$, with equality if and only if $\sum\limits_{r\geq 2} t_v(K_{1,r}) = 1$ and $\sum\limits_{r\geq 2} t'_v(K_{1,r}) = t'_u(K_3) = 0$. So the equality implies that $G^* \cong R_{k,r}$ for some $k \geq 1$ and $r \geq 2$ with $3k + r = n$, a contradiction. Thus, $\sum\limits_{i=1}^{n} b_{vi} < 0$.

Finally, let $\sum\limits_{r\geq 0} t_v(K_{1,r}) = 0$. Then $t_v(K_3) > 0$. Since G^* is not isomorphic to $S_{n,2}$, $t'_v(K_3) > 0$. Note that $\sum\limits_{i=1}^{n} b_{vi} \leq 2 - t'_v(K_3)$. If $t'_v(K_3) > 2$, then $\sum\limits_{i=1}^{n} b_{vi} < 0$. It remains the case $t'_v(K_3) \in \{1, 2\}$. Now, if $t_v(K_3) > 1$, then G^* has at least two cut vertices, a contradiction by Lemma 2.6. So $t_v(K_3) = 1$. Since $n \geq 6$, $t'_v(K_3) = 2$ and $G^* \cong R_{1,3}$. This also induces a contradiction. $\square$

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