**Electronic Journal of Linear Algebra ISSN 1081-3810**
**A publication of the International Linear Algebra Society**
**Volume 23, pp. 586-598, July 2012**

**ELA**

# PERTURBATION ANALYSIS OF $A_{T,S}^{(2)}$ ON BANACH SPACES*

FAPENG DU† AND YIFENG XUE‡

**Abstract.** In this paper, the perturbation problems of $A_{T,S}^{(2)}$ are considered. By virtue of the gap between subspaces, we derive conditions that make the perturbation of $A_{T,S}^{(2)}$ stable when $T, S$ and $A$ have suitable perturbations. At the same time, explicit formulas for perturbation of $A_{T,S}^{(2)}$ and new results on perturbation bounds are obtained.

**Key words.** Gap, Subspaces, Banach space, Group inverse.

**AMS subject classifications.** 15A09, 47A55.

**1. Introduction.** In recent years, there have been many fruitful results concerning the quantitative analysis of the perturbation of the Moore–Penrose inverses on Hilbert spaces and Drazin inverses on Banach spaces. For example, G. Chen, M. Wei and Y. Xue gave an estimation of perturbation bounds of the Moore–Penrose inverse on Hilbert spaces under stable perturbation of operators, which is a generalization of the rank–preserving perturbation of matrices in [4, 15]. Meanwhile, many perturbation analysis results of the Drazin inverse on Banach spaces have been obtained in [1, 2, 3] and [9], respectively, by means of the gap–function. Recently, G. Chen and Y. Xue gave some estimations of the perturbations of the Drazin inverse on a Banach space and a Banach algebra in [13] and [16], respectively, under stable perturbations.

Let $X$, $Y$ be Banach spaces and let $B(X, Y)$ denote the set of bounded linear operators from $X$ to $Y$. For an operator $A \in B(X, Y)$, let $R(A)$ and $N(A)$ denote the range and kernel of $A$, respectively. Let $T$ be a closed subspace of $X$ and $S$ be a closed subspace of $Y$. Recall that $A_{T,S}^{(2)}$ is the unique operator $G$ satisfying

$$GAG = G, \quad R(G) = T, \quad N(G) = S. \tag{1.1}$$

It is known that (1.1) is equivalent to the following condition:

$$N(A) \cap T = \{0\}, \quad AT \dotplus S = Y$$

*Received by the editors on October 17, 2011. Accepted for publication on June 9, 2012. Handling Editor: Oskar Maria Baksalary.

†Department of mathematics, East China Normal University, Shanghai 200241, P.R. China and School of Mathematical & Physical Sciences, Xuzhou Institute of Technology, Xuzhou 221008, Jiangsu Province, P.R. China (jsdfp@163.com).

‡Department of mathematics, East China Normal University, Shanghai 200241, P.R. China (yfxue@math.ecnu.edu.cn).

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 586-598, July 2012

ELA



(cf. [6, 7]). It is well–known that the five common kinds of generalized inverses (the Moore–Penrose inverse $A^+$, the weighted Moore–Penrose inverse $A^+_{MN}$, the Drazin inverse $A^D$, the group inverse $A^g$ and the Bott–Duffin inverse $A^{(-1)}_{(L)}$) can be reduced to an $A^{(2)}_{T,S}$ for certain choices of $T$ and $S$ (cf. [5, 6, 7]).

The perturbation analysis of $A^{(2)}_{T,S}$ has been studied by several authors (see [12, 11], [17, 18]) when $X$ and $Y$ are finite–dimensional. A lot of results pertaining to error bounds have been obtained. But when $X$ and $Y$ are infinite–dimensional, there is little known about the perturbation of $A^{(2)}_{T,S}$ if $T$, $S$ and $A$ have small perturbations respectively. In this paper, using the gap–function $\hat{\delta}(\cdot,\cdot)$ of two closed subspaces, we give upper bounds of $\|\bar{A}^{(2)}_{T',S'}\|$ and $\|\bar{A}^{(2)}_{T',S'} - A^{(2)}_{T,S}\|$ respectively. The main result is the following:

Let $A, \bar{A} = A + E \in B(X,Y)$ and let $T \subset X$, $S \subset Y$ be closed subspaces such that $A^{(2)}_{T,S}$ exists. Let $T' \subset X$, $S' \subset Y$ be closed subspaces with $\hat{\delta}(T,T') < \dfrac{1}{(1+\kappa)^2}$ and $\hat{\delta}(S,S') < \dfrac{1}{3+\kappa}$. Suppose that $\|A^{(2)}_{T,S}\|\|E\| < \dfrac{2\kappa}{(1+\kappa)(4+\kappa)}$. Then $\bar{A}^{(2)}_{T',S'}$ exists and

$$\|\bar{A}^{(2)}_{T',S'}\| \leq \frac{(1+\hat{\delta}(S',S))\|A^{(2)}_{T,S}\|}{1-(1+\kappa)\hat{\delta}(T,T') - \kappa\hat{\delta}(S',S) - (1+\hat{\delta}(S',S))\|A^{(2)}_{T,S}\|\|E\|},$$

$$\frac{\|\bar{A}^{(2)}_{T',S'} - A^{(2)}_{T,S}\|}{\|A^{(2)}_{T,S}\|} \leq \frac{(1+\kappa)(\hat{\delta}(T,T') + \hat{\delta}(S',S) + (1+\hat{\delta}(S',S))\|A^{(2)}_{T,S}\|\|E\|}{1-(1+\kappa)\hat{\delta}(T,T') - \kappa\hat{\delta}(S',S) - (1+\hat{\delta}(S',S))\|A^{(2)}_{T,S}\|\|E\|},$$

where $\kappa = \|A\|\|A^{(2)}_{T,S}\|$ is called the condition number of $A^{(2)}_{T,S}$. These results improve Theorem 4.4.5 of [14].

**2. Preliminaries.** Let $Z$ be a complex Banach space. Let $M$, $N$ be two closed subspaces in $Z$. Set

$$\delta(M,N) = \begin{cases} \sup\{dist(x,N) \,|\, x \in M,\ \|x\| = 1\}, & M \neq \{0\}, \\ 0, & M = \{0\}, \end{cases}$$

where $dist(x,N) = \inf\{\|x-y\| \,|\, y \in N\}$. The gap $\hat{\delta}(M,N)$ of $M$, $N$ is given by $\hat{\delta}(M,N) = \max\{\delta(M,N), \delta(N,M)\}$. For convenience, we list some properties about $\delta(M,N)$ and $\hat{\delta}(M,N)$ which come from [8] as follows.

Proposition 2.1. *Let $M$, $N$ be closed subspaces in a Banach space $Z$. Then*

(1) $\delta(M,N) = 0$ *if and only if* $M \subset N$*;*
(2) $\hat{\delta}(M,N) = 0$ *if and only if* $M = N$*;*
(3) $\hat{\delta}(M,N) = \hat{\delta}(N,M)$*;*

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 586-598, July 2012

ELA



(4) $0 \leq \delta(M,N) \leq 1$, $0 \leq \hat{\delta}(M,N) \leq 1$.

An operator $A \in B(Z,Z)$ is group invertible if there is $B \in B(Z,Z)$ such that

$$ABA = A, \quad BAB = B, \quad AB = BA.$$

The operator $B$ is called the group inverse of $A$ and is denoted by $A^g$. Clearly, $R(A^g) = R(A)$ and $N(A^g) = N(A)$.

LEMMA 2.2. *Let $A \in B(X,Y)$. Let $T \subset X$, $S \subset Y$ be closed subspaces such that $A^{(2)}_{T,S}$ exists. Let $G \in B(Y,X)$ be an operator with $R(G) = T$ and $N(G) = S$. Then*

(1) *$R(AG) = AT$, $N(AG) = S$ and $R(GA) = T$, $N(GA) \cap T = \{0\}$;*
(2) *$GA$ and $AG$ are group invertible and $A^{(2)}_{T,S} = (GA)^g G = G(AG)^g$.*

*Proof.* (1) Using $AT \dot{+} S = Y$ and $N(A) \cap T = \{0\}$, we can obtain the assertion.

(2) The assertion follows from [5, Lemma 3.1]. □

LEMMA 2.3 ([10, Theorem 11, pp. 100]). *Let $M$ be a complemented subspace of $X$. Let $P \in B(X,X)$ be an idempotent operator with $R(P) = M$. Let $M'$ be a closed subspace of $H$ satisfying $\hat{\delta}(M,M') < \dfrac{1}{1+\|P\|}$. Then $M'$ is complemented, that is, $H = R(I-P) \dot{+} M'$.*

Let $A \in B(X,Y)$. Let $T \subset X$ and $S \subset Y$ be closed subspaces such that $A^{(2)}_{T,S}$ exists. Put $\kappa = \|A\|\|A^{(2)}_{T,S}\|$. The symbol $\kappa$ will be used throughout the paper.

LEMMA 2.4. *Let $A \in B(X,Y)$. Let $T \subset X$ and $S \subset Y$ be closed subspaces such that $A^{(2)}_{T,S}$ exists. Let $T'$ be a closed subspace of $X$ such that $\hat{\delta}(T,T') < \dfrac{1}{1+\kappa}$. Then*

(1) *$\hat{\delta}(AT,AT') \leq \dfrac{\kappa\,\hat{\delta}(T,T')}{1-(1+\kappa)\hat{\delta}(T,T')}$;*
(2) *$N(A) \cap T' = \{0\}$.*

*Proof.* (1) First we show $\delta(AT,AT') \leq \|A\|\|A^{(2)}_{T,S}\|\delta(T,T') \leq \kappa\,\hat{\delta}(T,T')$.

Let $x \in T$. Then $x = A^{(2)}_{T,S}Ax$ and $\|x\| \leq \|A^{(2)}_{T,S}\|\|Ax\|$. For any $y \in T'$, we have $\|Ax - Ay\| \leq \|A\|\|x - y\|$. So

$$\begin{aligned} dist(Ax,AT') &= \inf_{y\in T'} \|Ax - Ay\| \leq \|A\| \inf_{y\in T'} \|x - y\| \\ &= \|A\| dist(x,T') \leq \|A\|\|x\|\delta(T,T') \\ &\leq \|A\|\|A^{(2)}_{T,S}\|\|Ax\| dist(T,T'). \end{aligned}$$

This means that $\delta(AT,AT') \leq \kappa\,\delta(T,T') \leq \kappa\,\hat{\delta}(T,T')$.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 586-598, July 2012



Next we show $\delta(AT', AT) \leq \dfrac{\kappa\,\hat{\delta}(T,T')}{1-(1+\kappa)\hat{\delta}(T,T')}$ when $\hat{\delta}(T,T') < \dfrac{1}{1+\kappa}$.

For each $x' \in T'$ and $x \in T$, we have

$$\begin{aligned}\|Ax'\| &= \|A(x'-x+x)\| \geq \|Ax\| - \|A\|\|x'-x\| \\ &\geq \|A^{(2)}_{T,S}\|^{-1}\|x\| - \|A\|\|x'-x\| \\ &\geq \|A^{(2)}_{T,S}\|^{-1}\|x'\| - \|A^{(2)}_{T,S}\|^{-1}\|x'-x\| - \|A\|\|x'-x\| \\ &\geq \|A^{(2)}_{T,S}\|^{-1}\|x'\| - (\|A^{(2)}_{T,S}\|^{-1} + \|A\|)\|x'-x\|,\end{aligned}$$

Thus,

$$(\|A^{(2)}_{T,S}\|^{-1} + \|A\|)\|x'-x\| \geq \|A^{(2)}_{T,S}\|^{-1}\|x'\| - \|Ax'\|$$

and consequently,

$$\|A^{(2)}_{T,S}\|^{-1}\|x'\| - \|Ax'\| \leq \|x'\|(\|A^{(2)}_{T,S}\|^{-1} + \|A\|)\delta(T',T),$$

that is,

$$\|A^{(2)}_{T,S}\|\|Ax'\| \geq \big[1 - (1 + \|A\|\|A^{(2)}_{T,S}\|)\delta(T',T)\big]\|x'\|. \tag{2.1}$$

Therefore,

$$\begin{aligned}dist(Ax', AT) &\leq \|A\|dist(x',T) \leq \|A\|\|x'\|\delta(T',T) \\ &\leq \frac{\|A\|\|Ax'\|\|A^{(2)}_{T,S}\|\hat{\delta}(T,T')}{1-(1+\|A\|\|A^{(2)}_{T,S}\|)\hat{\delta}(T,T')},\end{aligned}$$

i.e., $\delta(AT', AT) \leq \dfrac{\kappa\,\hat{\delta}(T,T')}{1-(1+\kappa)\hat{\delta}(T,T')}$ when $\hat{\delta}(T,T') < \dfrac{1}{1+\kappa}$.

The final assertion follows from above arguments.

(2) From (2.1), we get that $N(A) \cap T' = \{0\}$. ◻

## 3. Main results.

LEMMA 3.1. *Let $A \in B(X,Y)$ and let $T \subset X$, $S \subset Y$ be closed subspaces such that $A^{(2)}_{T,S}$ exists. Let $T'$ be closed subspace in $X$ with $\hat{\delta}(T,T') < \dfrac{1}{(1+\kappa)^2}$. Then $A^{(2)}_{T',S}$ exists and the following hold:*

(1) $A^{(2)}_{T',S} = A^{(2)}_{T,S} + (I - A^{(2)}_{T,S}A)F(I + (AG)^g AF)^{-1}(AG)^g$, *where $G$, $H \in B(Y,X)$ are arbitrary operators such that*

$$R(G) = T,\ R(H) = T',\ N(G) = N(H) = S \text{ and } F = H - G.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 586-598, July 2012



(2) $\|A^{(2)}_{T',S} - A^{(2)}_{T,S}\| \le \dfrac{(1+\kappa)\hat{\delta}(T,T')}{1-(1+\kappa)\hat{\delta}(T,T')}\|A^{(2)}_{T,S}\|$.

(3) $\|A^{(2)}_{T',S}\| \le \dfrac{\|A^{(2)}_{T,S}\|}{1-(1+\kappa)\hat{\delta}(T,T')}$.

*Proof.* Put $P_{AT,S} = AA^{(2)}_{T,S}$. Then $P_{AT,S}$ is an idempotent operator onto $AT$ along $S$. By Lemma 2.4 (1), we have

$$\hat{\delta}(AT, AT') \le \frac{\kappa\,\hat{\delta}(T,T')}{1-(1+\kappa)\hat{\delta}(T,T')} < \frac{1}{1+\kappa} \le \frac{1}{1+\|P_{AT,S}\|},$$

when $\hat{\delta}(T,T') < \dfrac{1}{(1+\kappa)^2}$. So $AT'$ is complemented and $AT' \dotplus S = Y$ by Lemma 2.3. Consequently, $A^{(2)}_{T',S}$ exists by Lemma 2.4 (2).

Let $G$, $H \in B(Y,X)$ with $R(G) = T$, $N(G) = N(H) = S$ and $R(H) = T'$. Then by Lemma 2.2, we have

$$A^{(2)}_{T,S} = G(AG)^g = (GA)^g G, \quad A^{(2)}_{T',S} = H(AH)^g = (HA)^g H.$$

Put $F = H - G$. Then $S \subseteq N(F)$.

Now we show that $I + (AG)^g AF$ is invertible. Let $y \in N(I + (AG)^g AF)$. Then

$$y = -(AG)^g AFy \in R((AG)^g) = R(AG) = AT.$$

Hence,

$$AA^{(2)}_{T,S}y = y = -(AG)^g AFy = AA^{(2)}_{T,S}y - (AG)^g AHy.$$

So $(AG)^g AHy = 0$. This indicates that

$$AHy \in R(AH) \cap N((AG)^g) = AT' \cap S = \{0\}.$$

From $AHy = 0$, we get that $y \in N(AH) \cap AT = S \cap AT = \{0\}$, i.e., $y = 0$. Therefore, $I + (AG)^g AF$ is injective.

Note that $N((AG)^g) = S$ and $AT' \dotplus S = Y$. So

$$AT = R(AG) = R((AG)^g) = (AG)^g AT' = R((AG)^g AH),$$

and consequently, for any $y \in Y = S \dotplus AT$, there is $y_1 \in S$ and $y_2 \in R((AG)^g AH)$ such that $y = y_1 + y_2$. Choose $z \in Y$ such that $y_2 = (AG)^g AHz$. Write $z = z_1 + z_2$ where $z_1 \in AT$ and $z_2 \in S$. Since $N(H) = S$, $y_2 = (AG)^g AHz_1$. Set $\xi = y_1 + z_1$. Then

$$\begin{aligned}(I + (AG)^g AF)\xi &= (I - AA^{(2)}_{T,S} + (AG)^g AH)\xi \\ &= (I - AA^{(2)}_{T,S})\xi + (AG)^g AH\xi = y_1 + (AG)^g AHz_1 \\ &= y_1 + y_2 = y,\end{aligned}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 586-598, July 2012

**ELA**



that is, $I+(AG)^gAF$ is surjective. Therefore, $I+(AG)^gAF$ is invertible and $I+AF(AG)^g$ is invertible too.

Put

$$D = A^{(2)}_{T,S} + (I - A^{(2)}_{T,S}A)F(I + (AG)^gAF)^{-1}(AG)^g.$$

It is easy to verify that $DAD = D$ and $N(D) = S$. Since $(I+(AG)^gAF)^{-1}(AG)^g = (AG)^g(I+AF(AG)^g)^{-1}$ and

$$\begin{aligned} D &= A^{(2)}_{T,S} + (I - A^{(2)}_{T,S}A)F(I + (AG)^gAF)^{-1}(AG)^g \\ &= G(AG)^g + (I - G(AG)^gA)F(I + (AG)^gAF)^{-1}(AG)^gAG(AG)^g \\ &= (G + G(AG)^gAF + F - G(AG)^gAF)(I + (AG)^gAF)^{-1}(AG)^g \\ &= H(I + (AG)^gAF)^{-1}(AG)^g \\ &= H(AG)^g(I + AF(AG)^g)^{-1}, \end{aligned}$$

by Lemma 2.2 (2), we have that

$$R(D) = R(H(AG)^g) = H(AT) = H(AT \dotplus S) = R(H) = T'.$$

Thus, $A^{(2)}_{T',S} = D$.

Put $W = A^{(2)}_{T',S} - A^{(2)}_{T,S}$. For any $\xi \in Y = AT' \dotplus S$, there is $u \in AT'$ and $u' \in S$ such that $\xi = u + u'$. Choose $x \in Y$ such that $u = AA^{(2)}_{T',S}x$. Since $dist(A^{(2)}_{T',S}x, T) \le \|A^{(2)}_{T',S}x\|\delta(T', T)$, for every $\epsilon > 0$, we can find $y \in Y$ such that

$$\|A^{(2)}_{T',S}x - A^{(2)}_{T,S}y\| < \|A^{(2)}_{T',S}x\|\delta(T', T) + \epsilon.$$

Set $v = AA^{(2)}_{T,S}y$. Then

$$\|u - v\| = \|AA^{(2)}_{T',S}x - AA^{(2)}_{T,S}y\| < \|A\|\|A^{(2)}_{T',S}x\|\delta(T', T) + \|A\|\epsilon.$$

Consequently,

$$\begin{aligned} \|W\xi\| &= \|Wu\| = \|A^{(2)}_{T',S}u - A^{(2)}_{T,S}u\| \\ &\le \|A^{(2)}_{T',S}u - A^{(2)}_{T,S}v\| + \|A^{(2)}_{T,S}u - A^{(2)}_{T,S}v\| \\ &\le \|A^{(2)}_{T',S}x - A^{(2)}_{T,S}y\| + \|A^{(2)}_{T,S}\|\|u - v\| \\ &\le (1+\kappa)\|A^{(2)}_{T',S}x\|\delta(T', T) + (1+\kappa)\epsilon. \qquad (3.1) \end{aligned}$$

Since

$$\|A^{(2)}_{T',S}x\| = \|A^{(2)}_{T',S}u\| = \|W\xi + A^{(2)}_{T,S}\xi\| \le \|A^{(2)}_{T,S}\|\|\xi\| + \|W\xi\|, \qquad (3.2)$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 586-598, July 2012

**ELA**



it follows from (3.1) and (3.2) that

$$\|W\xi\| \le (1+\kappa)(\|A^{(2)}_{T,S}\|\|\xi\| + \|W\xi\|)\delta(T',T) + (1+\kappa)\epsilon,$$

and hence, $\|W\xi\| \le \dfrac{(1+\kappa)\delta(T',T)}{1-(1+\kappa)\delta(T',T)}\|A^{(2)}_{T,S}\|\|\xi\|$ by letting $\epsilon \to 0^+$. Therefore,

$$\|A^{(2)}_{T',S} - A^{(2)}_{T,S}\| \le \frac{(1+\kappa)\hat{\delta}(T',T)}{1-(1+\kappa)\hat{\delta}(T',T)}\|A^{(2)}_{T,S}\|.$$

Furthermore,

$$\begin{aligned}\|A^{(2)}_{T',S}\| &= \|W + A^{(2)}_{T,S}\| \le \|W\| + \|A^{(2)}_{T,S}\| \\ &\le \frac{(1+\kappa)\hat{\delta}(T',T)}{1-(1+\kappa)\hat{\delta}(T',T)}\|A^{(2)}_{T,S}\| + \|A^{(2)}_{T,S}\| \\ &= \frac{\|A^{(2)}_{T,S}\|}{1-(1+\kappa)\hat{\delta}(T',T)}. \quad \square\end{aligned}$$

LEMMA 3.2. *Let $A \in B(X,Y)$ and let $T \subset X$, $S \subset Y$ be closed subspaces such that $A^{(2)}_{T,S}$ exists. Let $S'$ be a closed subspace in $Y$ such that $\hat{\delta}(S,S') < \dfrac{1}{2+\kappa}$. Then $A^{(2)}_{T,S'}$ exists and the following hold:*

(1) $A^{(2)}_{T,S'} = A^{(2)}_{T,S} + A^{(2)}_{T,S}(I+(AG)^g AF)^{-1}(AG)^g AF(I - AA^{(2)}_{T,S})$, *where $F = H - G$ and $G$, $H \in B(Y,X)$ are arbitrary with $R(G) = R(H) = T$, $N(G) = S$ and $N(H) = S'$.*

(2) $\|A^{(2)}_{T,S} - A^{(2)}_{T,S'}\| \le \dfrac{(1+\kappa)\hat{\delta}(S',S)}{1-\kappa\,\hat{\delta}(S',S)}\|A^{(2)}_{T,S}\|$.

(3) $\|A^{(2)}_{T,S'}\| \le \dfrac{1+\hat{\delta}(S',S)}{1-\kappa\,\hat{\delta}(S',S)}\|A^{(2)}_{T,S}\|$.

*Proof.* Let $P_{S,AT} = I - AA^{(2)}_{T,S}$ be an idempotent operator from $Y$ onto $S$ along $AT$. Since $\|P_{S,AT}\| \le 1 + \|A\|\|A^{(2)}_{T,S}\| = 1+\kappa$, we have $\hat{\delta}(S,S') \le \dfrac{1}{1+\|P_{S,AT}\|}$. So, $Y = AT \dotplus S'$ by Lemma 2.3. Noting that $N(A)\cap T = \{0\}$, we get that $A^{(2)}_{T,S'}$ exists.

Using the facts:

$$AT \dotplus S = Y = AT \dotplus S', \quad N(A)\cap T = \{0\}$$

and the similar method appeared in the proof of Lemma 3.1, we can deduce that $I + (AG)^g AF$ is invertible and so is the operator $I + AF(AG)^g$.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 586-598, July 2012



Put $D = A^{(2)}_{T,S}(I + (AG)^g AF)^{-1}(AG)^g AH$. Then $R(D) \subset T$, $S' \subset N(D)$ and

$$
\begin{aligned}
&A^{(2)}_{T,S} + A^{(2)}_{T,S}(I + (AG)^g AF)^{-1}(AG)^g AF(I - AA^{(2)}_{T,S})\\
&\quad = A^{(2)}_{T,S} + A^{(2)}_{T,S}(I + (AG)^g AF)^{-1}[I + (AG)^g AF - I](I - AA^{(2)}_{T,S})\\
&\quad = A^{(2)}_{T,S}(I + (AG)^g AF)^{-1}[I + (AG)^g AF - (I - AA^{(2)}_{T,S})]\\
&\quad = A^{(2)}_{T,S}(I + (AG)^g AF)^{-1}(AG)^g AH. \qquad (3.3)
\end{aligned}
$$

Clearly, $DAD = D$ by (3.3). In order to obtain $A^{(2)}_{T,S} = D$, we need only to prove that $T \subset R(D)$ and $S' \supset N(D)$.

Since $AT \dotplus S = Y$ and $N((AG)^g) = S$, $R(H) = T$, it follows that

$$R((AG)^g) = (AG)^g AT = R((AG)^g AH),$$

and hence,

$$
\begin{aligned}
R(D) &= A^{(2)}_{T,S}(I + (AG)^g AF)^{-1}R((AG)^g) = R(A^{(2)}_{T,S}(AG)^g(I + AF(AG)^g)^{-1})\\
&= A^{(2)}_{T,S}R((AG)^g) = A^{(2)}_{T,S}AT = A^{(2)}_{T,S}(AT \dotplus S) = R(A^{(2)}_{T,S}) = T.
\end{aligned}
$$

Now let $x \in N(D)$ and put $y = (I + (AG)^g AF)^{-1}(AG)^g AHx$. Then $y \in S$ and $y \in R((AG)^g) = AT$. So $y = 0$ and consequently, $(AG)^g AHx = 0$. But this means that $AHx \in AT \cap N((AG)^g) = AT \cap S = \{0\}$. Thus, $AHx = 0$ and $Hx = 0$. Since $N(A) \cap T = \{0\}$, it follows that $x \in N(H) = S'$. Therefore,

$$A^{(2)}_{T,S} = A^{(2)}_{T,S}(I + (AG)^g AF)^{-1}(AG)^g AH.$$

Put $B' = I - AA^{(2)}_{T,S'}$, $B = I - AA^{(2)}_{T,S}$. Note that

$$W = A^{(2)}_{T,S} - A^{(2)}_{T,S'} = A^{(2)}_{T,S} - A^{(2)}_{T,S}AA^{(2)}_{T,S'} = A^{(2)}_{T,S}(AA^{(2)}_{T,S} - AA^{(2)}_{T,S'}).$$

So, $W = A^{(2)}_{T,S}(AA^{(2)}_{T,S} - AA^{(2)}_{T,S'}) = A^{(2)}_{T,S}(B' - B)$. Since $B'\xi \in S'$, $\forall \xi \in Y$, we have $dist(B'\xi, S) \le \delta(S', S)\|B'\xi\|$. Thus, for any $\epsilon > 0$, there is $u \in Y$ such that $\|B'\xi - Bu\| \le \delta(S', S)\|B'\xi\| + \epsilon$ and so that

$$\|A^{(2)}_{T,S}(B'\xi - Bu)\| \le \delta(S', S)\|B'\xi\|\|A^{(2)}_{T,S}\| + \|A^{(2)}_{T,S}\|\epsilon.$$

Noting that $A^{(2)}_{T,S}B = 0$, we have

$$
\begin{aligned}
\|W\xi\| &= \|A^{(2)}_{T,S}(B'\xi - B\xi)\| = \|A^{(2)}_{T,S}(B'\xi - Bu)\|\\
&\le \delta(S', S)\|B'\xi\|\|A^{(2)}_{T,S}\| + \|A^{(2)}_{T,S}\|\epsilon.
\end{aligned}
$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 586-598, July 2012

ELA



But $\|B'\xi\| \le \|\xi\| + \|A\|\|A^{(2)}_{T,S}\xi - W\xi\| \le (1+\kappa)\|\xi\| + \|A\|\|W\xi\|$. Thus,

$$\|W\xi\| \le \delta(S',S)\|A^{(2)}_{T,S}\|((1+\kappa)\|\xi\| + \|A\|\|W\xi\|) + \|A^{(2)}_{T,S}\|\epsilon. \tag{3.4}$$

(3.4) indicates that $\|W\| \le \dfrac{(1+\kappa)\hat{\delta}(S',S)}{1-\kappa\hat{\delta}(S',S)}\|A^{(2)}_{T,S}\|$ and

$$\|A^{(2)}_{T,S'}\| \le \|A^{(2)}_{T,S}\| + \|W\| \le \frac{1+\hat{\delta}(S',S)}{1-\kappa\hat{\delta}(S',S)}\|A^{(2)}_{T,S}\|. \quad \square$$

We now present our main result of this paper as follows.

Theorem 3.3. *Let $A \in B(X,Y)$ and let $T \subset X$, $S \subset Y$ be closed subspaces such that $A^{(2)}_{T,S}$ exists. Let $T' \subset X$, $S' \subset Y$ be closed subspaces such that $\hat{\delta}(T,T') < \dfrac{1}{(1+\kappa)^2}$ and $\hat{\delta}(S,S') < \dfrac{1}{3+\kappa}$ respectively. Then $A^{(2)}_{T',S'}$ exists and the following hold:*

(1) $$\begin{aligned} A^{(2)}_{T',S'} &= A^{(2)}_{T,S} + (I - A^{(2)}_{T,S}A)F(I+(AG)^gAF)^{-1}(AG)^g \\ &\quad + \{A^{(2)}_{T,S} + (I - A^{(2)}_{T,S}A)F(I+(AG)^gAF)^{-1}(AG)^g\} \\ &\quad \times (I+(A\tilde{G})^gA\tilde{F})^{-1}(A\tilde{G})^gA\tilde{F}(I - AA^{(2)}_{T,S})(I+AF(AG)^g)^{-1}. \end{aligned}$$

(2) $$\|A^{(2)}_{T',S'} - A^{(2)}_{T,S}\| \le \frac{(1+\kappa)(\hat{\delta}(T,T') + \hat{\delta}(S',S))}{1-(1+\kappa)\hat{\delta}(T,T') - \kappa\hat{\delta}(S',S)}\|A^{(2)}_{T,S}\|.$$

(3) $$\|A^{(2)}_{T',S'}\| \le \frac{1+\hat{\delta}(S',S)}{1-(1+\kappa)\hat{\delta}(T,T') - \kappa\hat{\delta}(S',S)}\|A^{(2)}_{T,S}\|.$$

*where $G, \tilde{G}, \tilde{H} \in B(Y,X)$ are such that $R(G) = T$, $R(\tilde{G}) = R(\tilde{H}) = T'$, $N(G) = N(\tilde{G}) = S$, $N(\tilde{H}) = S'$ and $F = \tilde{G} - G$, $\tilde{F} = \tilde{H} - \tilde{G}$.*

*Proof.* Since $\hat{\delta}(T,T') < \dfrac{1}{(1+\kappa)^2}$, it follows from Lemma 3.1 that $A^{(2)}_{T',S}$ exists and

$$\begin{aligned} \|A\|\|A^{(2)}_{T',S}\| &\le \frac{\kappa}{1-(1+\kappa)\hat{\delta}(T,T')} < 1+\kappa, \\ \hat{\delta}(S,S') &< \frac{1}{2+1+\kappa} < \frac{1}{2+\|A\|\|A^{(2)}_{T',S}\|}. \end{aligned}$$

Thus, by Lemma 3.2 we have that $A^{(2)}_{T',S'}$ exists and

$$A^{(2)}_{T',S'} = A^{(2)}_{T',S} + A^{(2)}_{T',S}(I+(A\tilde{G})_gA\tilde{F})^{-1}(A\tilde{G})_gA\tilde{F}(I - AA^{(2)}_{T',S}) \tag{3.5}$$

by Lemma 3.2, where $\tilde{G}$, $\tilde{H} \in B(Y,X)$ with $R(\tilde{G}) = T'$, $N(\tilde{G}) = S$ and $R(\tilde{H}) = T'$, $N(\tilde{H}) = S'$ and $\tilde{F} = \tilde{H} - \tilde{G}$. By Lemma 3.1, we have

$$A^{(2)}_{T',S} = A^{(2)}_{T,S} + (I - A^{(2)}_{T,S}A)F(I+(AG)^gAF)^{-1}(AG)^gAA^{(2)}_{T,S}. \tag{3.6}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 586-598, July 2012



Combining (3.5) with (3.6), we get that

$$\begin{aligned} A^{(2)}_{T',S'} &= A^{(2)}_{T,S} + (I - A^{(2)}_{T,S}A)F(I + (AG)^g AF)^{-1}(AG)^g \\ &\quad + \{A^{(2)}_{T,S} + (I - A^{(2)}_{T,S}A)F(I + (AG)^g AF)^{-1}(AG)^g\} \\ &\quad \times (I + (A\tilde{G})^g A\tilde{F})^{-1}(A\tilde{G})^g A\tilde{F}(I - AA^{(2)}_{T,S})(I + AF(AG)^g)^{-1}. \end{aligned}$$

By Lemma 3.1 and Lemma 3.2, we have

$$\begin{aligned} \|A^{(2)}_{T',S'} - A^{(2)}_{T,S}\| &\le \|A^{(2)}_{T',S'} - A^{(2)}_{T',S}\| + \|A^{(2)}_{T',S} - A^{(2)}_{T,S}\| \\ &\le \frac{(1 + \|A\|\|A^{(2)}_{T',S}\|)\hat{\delta}(S',S)}{1 - \|A\|\|A^{(2)}_{T',S}\|\hat{\delta}(S',S)}\|A^{(2)}_{T',S}\| + \frac{(1+\kappa)\hat{\delta}(T,T')}{1-(1+\kappa)\hat{\delta}(T,T')}\|A^{(2)}_{T,S}\| \\ &\le \Big[\frac{(1+\kappa)\hat{\delta}(S',S)(1-\hat{\delta}(T,T'))}{\{1-(1+\kappa)\hat{\delta}(T,T')-\kappa\hat{\delta}(S',S)\}\{1-(1+\kappa)\hat{\delta}(T,T')\}} \\ &\quad + \frac{(1+\kappa)\hat{\delta}(T,T')}{1-(1+\kappa)\hat{\delta}(T,T')}\Big]\|A^{(2)}_{T,S}\| \\ &= \frac{(1+\kappa)(\hat{\delta}(T,T')+\hat{\delta}(S',S))}{1-(1+\kappa)\hat{\delta}(T,T')-\kappa\hat{\delta}(S',S)}\|A^{(2)}_{T,S}\|, \end{aligned}$$

$$\begin{aligned} \|A^{(2)}_{T',S'}\| &\le \|A^{(2)}_{T',S'} - A^{(2)}_{T,S}\| + \|A^{(2)}_{T,S}\| \\ &\le \frac{(1+\kappa)(\hat{\delta}(T,T')+\hat{\delta}(S',S))}{1-(1+\kappa)\hat{\delta}(T,T')-\kappa\hat{\delta}(S',S)}\|A^{(2)}_{T,S}\| + \|A^{(2)}_{T,S}\| \\ &\le \frac{1+\hat{\delta}(S',S)}{1-(1+\kappa)\hat{\delta}(T,T')-\kappa\hat{\delta}(S',S)}\|A^{(2)}_{T,S}\|. \quad \square \end{aligned}$$

LEMMA 3.4. *Let $A$, $\bar{A} = A + E \in B(X,Y)$ and $T \subset X, S \subset Y$ be closed subspaces such that $A^{(2)}_{T,S}$ exists. If $\|A^{(2)}_{T,S}\|\|E\| < 1$, then*

$$\bar{A}^{(2)}_{T,S} = (I + A^{(2)}_{T,S}E)^{-1}A^{(2)}_{T,S} = A^{(2)}_{T,S}(I + EA^{(2)}_{T,S})^{-1},$$

*and*

$$\|\bar{A}^{(2)}_{T,S}\| \le \frac{\|A^{(2)}_{T,S}\|}{1 - \|A^{(2)}_{T,S}\|\|E\|}, \quad \|\bar{A}^{(2)}_{T,S} - A^{(2)}_{T,S}\| \le \frac{\|A^{(2)}_{T,S}\|^2\|E\|}{1 - \|A^{(2)}_{T,S}\|\|E\|}.$$

*Proof.* $\|A^{(2)}_{T,S}\|\|E\| < 1$ implies that $(I + A^{(2)}_{T,S}E)^{-1}$ exists. Since

$$(I + A^{(2)}_{T,S}E)A^{(2)}_{T,S} = A^{(2)}_{T,S}(I + EA^{(2)}_{T,S}),$$

**Electronic Journal of Linear Algebra ISSN 1081-3810**
**A publication of the International Linear Algebra Society**
**Volume 23, pp. 586-598, July 2012**



we have

$$(I + A^{(2)}_{T,S}E)^{-1}A^{(2)}_{T,S} = A^{(2)}_{T,S}(I + EA^{(2)}_{T,S})^{-1}.$$

Put $B = (1 + A^{(2)}_{T,S}E)^{-1}A^{(2)}_{T,S}$. Then $R(B) = R(A^{(2)}_{T,S}) = T$, $N(B) = N(A^{(2)}_{T,S}) = S$ and $B(A + E)B = B$. Therefore, $\bar{A}^{(2)}_{T,S} = (I + A^{(2)}_{T,S}E)^{-1}A^{(2)}_{T,S}$ with

$$\|\bar{A}^{(2)}_{T,S}\| \le \|(I + A^{(2)}_{T,S}E)^{-1}\|\|A^{(2)}_{T,S}\| \le \frac{\|A^{(2)}_{T,S}\|}{1 - \|A^{(2)}_{T,S}\|\|E\|} \quad \text{and}$$

$$\|\bar{A}^{(2)}_{T,S} - A^{(2)}_{T,S}\| = \| - (I + A^{(2)}_{T,S}E)^{-1}A^{(2)}_{T,S}EA^{(2)}_{T,S}\| \le \frac{\|A^{(2)}_{T,S}\|^2\|E\|}{1 - \|A^{(2)}_{T,S}\|\|E\|}. \quad \square$$

We close this section by giving the perturbation analysis for $A^{(2)}_{T,S}$ when $T$, $S$ and $A$ all have small perturbations.

THEOREM 3.5. *Let $A, \bar{A} = A+E \in B(X,Y)$ and let $T \subset X$, $S \subset Y$ be closed subspaces such that $A^{(2)}_{T,S}$ exists. Let $T' \subset X$, $S' \subset Y$ be closed subspaces with $\hat{\delta}(T,T') < \dfrac{1}{(1+\kappa)^2}$ and $\hat{\delta}(S,S') < \dfrac{1}{3+\kappa}$. Suppose that $\|A^{(2)}_{T,S}\|\|E\| < \dfrac{2\kappa}{(1+\kappa)(4+\kappa)}$. Then*

$$\begin{aligned}
(1)\ \bar{A}^{(2)}_{T',S'} = {} & [1 + A^{(2)}_{T,S}E + (I - A^{(2)}_{T,S}A)F(I + (AG)^gAF)^{-1}(AG)^gE \\
& + \{A^{(2)}_{T,S} + (I - A^{(2)}_{T,S}A)F(I + (AG)^gAF)^{-1}(AG)^g\} \\
& \times (I + (A\tilde{G})^gA\tilde{F})^{-1}(A\tilde{G})^gA\tilde{F}(I - AA^{(2)}_{T,S})(I + AF(AG)^g)^{-1}E]^{-1} \\
& \times [A^{(2)}_{T,S} + (I - A^{(2)}_{T,S}A)F(I + (AG)^gAF)^{-1}(AG)^g \\
& + \{A^{(2)}_{T,S} + (I - A^{(2)}_{T,S}A)F(I + (AG)^gAF)^{-1}(AG)^g\} \\
& \times (I + (A\tilde{G})^gA\tilde{F})^{-1}(A\tilde{G})^gA\tilde{F}(I - AA^{(2)}_{T,S})(I + AF(AG)^g)^{-1}];
\end{aligned}$$

$$(2)\ \|\bar{A}^{(2)}_{T',S'}\| \le \frac{(1 + \hat{\delta}(S',S))\|A^{(2)}_{T,S}\|}{1 - (1+\kappa)\hat{\delta}(T,T') - \kappa\hat{\delta}(S',S) - (1 + \hat{\delta}(S',S))\|A^{(2)}_{T,S}\|\|E\|};$$

$$(3)\ \frac{\|\bar{A}^{(2)}_{T',S'} - A^{(2)}_{T,S}\|}{\|A^{(2)}_{T,S}\|} \le \frac{(1+\kappa)(\hat{\delta}(T,T') + \hat{\delta}(S',S) + (1 + \hat{\delta}(S',S))\|A^{(2)}_{T,S}\|\|E\|}{1 - (1+\kappa)\hat{\delta}(T,T') - \kappa\hat{\delta}(S',S) - (1 + \hat{\delta}(S',S))\|A^{(2)}_{T,S}\|\|E\|}.$$

*where, $F = \tilde{G} - G$, $\tilde{F} = \tilde{H} - \tilde{G}$ and $G, \tilde{G}, \tilde{H} \in B(Y,X)$ are arbitrary such that $R(G) = T$, $R(\tilde{G}) = R(\tilde{H}) = T'$, $N(G) = N(\tilde{G}) = S$ and $N(\tilde{H}) = S'$.*

*Proof.* We have $A^{(2)}_{T',S'}$ exists and $\|A^{(2)}_{T',S'}\| \le \dfrac{(1 + \hat{\delta}(S',S))\|A^{(2)}_{T,S}\|}{1 - (1+\kappa)\hat{\delta}(T,T') - \kappa\hat{\delta}(S',S)}$ by Theorem 3.3. Thus, $\|A^{(2)}_{T',S'}\|\|E\| < \dfrac{(1+\kappa)(4+\kappa)\|A^{(2)}_{T,S}\|\|E\|}{2\kappa} < 1$ and hence $\bar{A}^{(2)}_{T',S'}$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 586-598, July 2012



exists with $\bar{A}_{T',S'}^{(2)} = (I + A_{T',S'}^{(2)}E)^{-1}A_{T',S'}^{(2)}$ by Lemma 3.4. It follows from Lemma 3.1 and 3.2 that

$$\begin{aligned}
\bar{A}_{T',S'}^{(2)} = {} & [I + A_{T,S}^{(2)}E + (I - A_{T,S}^{(2)}A)F(I + (AG)^g AF)^{-1}(AG)^g E \\
& + \{A_{T,S}^{(2)} + (I - A_{T,S}^{(2)}A)F(I + (AG)^g AF)^{-1}(AG)^g\} \\
& \times (I + (A\tilde{G})^g A\tilde{F})^{-1}(A\tilde{G})^g A\tilde{F}(I - AA_{T,S}^{(2)})(I + AF(AG)^g)^{-1}E]^{-1} \\
& \times [A_{T,S}^{(2)} + (I - A_{T,S}^{(2)}A)F(I + (AG)^g AF)^{-1}(AG)^g \\
& + \{A_{T,S}^{(2)} + (I - A_{T,S}^{(2)}A)F(I + (AG)^g AF)^{-1}(AG)^g\} \\
& \times (I + (A\tilde{G})^g A\tilde{F})^{-1}(A\tilde{G})^g A\tilde{F}(I - AA_{T,S}^{(2)})(I + AF(AG)^g)^{-1}].
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\|\bar{A}_{T',S'}^{(2)}\| & \leq \frac{\|A_{T',S'}^{(2)}\|}{1 - \|A_{T',S'}^{(2)}\|\|E\|} \\
& \leq \frac{(1 + \hat{\delta}(S', S))\|A_{T,S}^{(2)}\|}{1 - (1 + \kappa)\hat{\delta}(T, T') - \kappa\hat{\delta}(S', S) - (1 + \hat{\delta}(S', S))\|A_{T,S}^{(2)}\|\|E\|}.
\end{aligned}$$

Notice that

$$\begin{aligned}
\bar{A}_{T',S'}^{(2)} - A_{T,S}^{(2)} & = (I + A_{T',S'}^{(2)}E)^{-1}A_{T',S'}^{(2)} - A_{T,S}^{(2)} \\
& = (I + A_{T',S'}^{(2)}E)^{-1}(A_{T',S'}^{(2)} - (I + A_{T',S'}^{(2)}E)A_{T,S}^{(2)}) \\
& = (I + A_{T',S'}^{(2)}E)^{-1}(A_{T',S'}^{(2)} - A_{T,S}^{(2)} - A_{T',S'}^{(2)}EA_{T,S}^{(2)}).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\|\bar{A}_{T',S'}^{(2)} - A_{T,S}^{(2)}\| & \leq \|(I + A_{T',S'}^{(2)}E)^{-1}\|(\|A_{T',S'}^{(2)} - A_{T,S}^{(2)}\| + \|A_{T',S'}^{(2)}EA_{T,S}^{(2)}\|) \\
& \leq \frac{1}{1 - \|A_{T',S'}^{(2)}\|\|E\|}(\|A_{T',S'}^{(2)} - A_{T,S}^{(2)}\| + \|A_{T',S'}^{(2)}\|\|E\|\|A_{T,S}^{(2)}\|) \\
& \leq \frac{(1+\kappa)(\hat{\delta}(T, T') + \hat{\delta}(S', S) + (1+\hat{\delta}(S', S))\|A_{T,S}^{(2)}\|\|E\|}{1-(1+\kappa)\hat{\delta}(T, T') - \kappa\hat{\delta}(S', S) - (1+\hat{\delta}(S', S))\|A_{T,S}^{(2)}\|\|E\|}\|A_{T,S}^{(2)}\|. \quad \square
\end{aligned}$$

**Acknowledgement.** The authors thank to referees for their helpful comments and suggestions.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 586-598, July 2012

ELA



REFERENCES

[1] N. Castro–González and J.J. Koliha. Perturbation of Drazin inverse for closed linear operators. *Integral Equations Operator Theory*, 36:92–106, 2000.

[2] N. Castro–González, J.J. Koliha, and V. Rakočević. Continuity and general perturbation of Drazin inverse for closed linear operators. *Abstr. Appl. Anal.*, 7:355–347, 2002.

[3] N. Castro–González, J.J. Koliha, and Y. Wei. Error bounds for perturbation of the Drazin inverse of closed operators with equal spectral idempotents. *Appl. Anal.*, 81:915–928, 2002.

[4] G. Chen, M. Wei, and Y. Xue. Perturbation analysis of the least square solution in Hilbert spaces. *Linear Alebra Appl.*, 244: 69–80, 1996.

[5] D. Djordjević. Unified approach to the reverse order rule for generalized inverses. *Acta Sci. Math. (Szeged)*, 167:761–776, 2001.

[6] D. Djordjević and P. Stanimirović. Splitting of operators and generalized inverses. *Publ. Math. Debrecen*, 59:147–159, 2001.

[7] D. Djordjević, P. Stanimirović, and Y. Wei. The representation and approximations of outer generalized inverses. *Acta Math. Hungar.*, 104(1–2):1–26, 2004.

[8] T. Kato. *Perturbation Theory for Linear Operators.* Springer-Verlag, New York, 1984.

[9] J.J. Koliha. Error bounds for a general perturbation of Drazin inverse. *Appl. Math. Comput.*, 126:181–185, 2002.

[10] V. Müller. *Spectral Theory of Linear Operators and Spectral Systems in Banach Algebras*, second edition. Birkhäuser, Basel, 2007.

[11] Y. Wei and G. Wang. Perturbation theory for the generalized inverse $A^{(2)}_{T,S}$. *J. Fudan Univ. Nat. Sci.*, 39(5):482–488, 2000.

[12] Y. Wei and H. Wu. On the perturbation and subproper splittings for the generalized inverse $A^{(2)}_{T,S}$ of rectangular matrix A. *J. Comput. Appl. Math.*, 137:317–329, 2001.

[13] Y. Xue. Stable perturbation in Banach algebras. *J. Aust. Math. Soc.*, 83:1–14, 2007.

[14] Y. Xue. *Stable Perturbation of Operators and Related Topics.* World Scientific, 2012.

[15] Y. Xue and G. Chen. Some equivalent conditions of stable perturbation of operators in Hilbert spaces. *Appl. Math. Comput.*, 147:765–772, 2004.

[16] Y. Xue and G. Chen. Perturbation analysis for the Drazin inverse under stable perturbation in Banach space. *Missouri J. Math. Sci.*, 19(2):106–120, 2007.

[17] N. Zhang and Y. Wei. Perturbation bounds for the generalized inverse $A^{(2)}_{T,S}$ with application to constrained linear system. *Appl. Math. Comput.*, 142:63–78, 2003.

[18] N. Zhang and Y. Wei. A note on the perturbation of an outer inverse. *Calcolo*, 45(4):263–273, 2008.