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ON THE NUMERICAL RANGES OF THE WEIGHTED SHIFT OPERATORS WITH GEOMETRIC AND HARMONIC WEIGHTS*

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Abstract. In this paper, an exact formula for det $(tI_n - (Q_n + Q_n^*))$ is obtained. This formula yields a simple computation of the numerical ranges of the geometric weighted shift operator Q_n and the harmonic weighted shift operator H_n for $n = 3, 4$.

Key words. Numerical range, Weighted shift operators, Geometric weights, Harmonic weights.

AMS subject classifications. 15A60, 47A12.

1. Introduction. The numerical range of an $n \times n$ matrix T is defined as the set

$$W(T) = \{\langle Tx, x\rangle : \|x\| = 1\}.$$

where $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ denote the standard inner product and its associated norm in $\mathbb{C}^n$. It is known that $W(T)$ is a nonempty convex subset of $\mathbb{C}$; see for example [3]. The numerical radius $w(T)$ of a matrix T is given by

$$w(T) = \sup\{|\lambda| : \lambda \in W(T)\}.$$

For its other properties, see [3].

A shift matrix

$$T = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ a_1 & 0 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & 0 & \dots & 0 \\ 0 & 0 & a_3 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & a_{n-1} & 0 \end{bmatrix}, \tag{1.1}$$

and a diagonal matrix

$$U = \mathrm{diag}\,(1, \exp(i\theta), \exp(2i\theta)), \exp(3i\theta), \dots, \exp((n-1)i\theta))$$

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satisfy the equation

$$UTU^* = \exp(i\theta)T,$$

and hence, $\exp(i\theta)W(T) = W(T)$ for $0 \le \theta \le 2\pi$. For a shift matrix, the numerical radius $w(T)$ is characterized as the maximum root of the characteristic polynomial

$$P(x) = \det\left(xI_n - \frac{1}{2}(T + T^*)\right).$$

In [2], the value

$$M(\theta) = \max\{\Re(z\exp(-i\theta)) : z \in W(T)\}$$

for a matrix T is characterized as the maximum eigenvalue of a hermitian matrix

$$\frac{1}{2}(\exp(i\theta)T + \exp(-i\theta)T^*)$$

$(0 \le \theta \le 2\pi)$. If T is a shift matrix, then the numerical range $W(T)$ is a closed circular disc with center at the origin, and hence, $w(T)$ is the maximum eigenvalue of a hermitian matrix $(T + T^*)/2$.

We consider a weighted shift operator A on the Hilbert space $\ell^2(\mathbf{N})$ defined by

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ a_1 & 0 & 0 & 0 & \dots \\ 0 & a_2 & 0 & 0 & \dots \\ 0 & 0 & a_3 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix} \tag{1.2}$$

where $\{a_n\}$ is a bounded sequence. The numerical range is also defined for Hilbert space operators. It is known that $W(A)$ is a circular disk centered at the origin [5]. In particular, if the weights are geometric $a_n = q^{n-1}$ for some $0 < q < 1$ and $n \in \mathbb{N}$, then the numerical range of T_n is closed disc centered at the origin [1]. Furthermore, the authors of [1] found upper and lower bounds for $w(T)$. However, we do not use their result and we develop a simple and different method to solve it. Consider the following two finite operators

$$Q_n = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & q & 0 & 0 & \dots & 0 \\ 0 & 0 & q^2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & q^{n-2} & 0 \end{bmatrix}, \tag{1.3}$$

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where $0 < q < 1$ and

$$H_n = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & \frac{1}{n-1} & 0 \end{bmatrix}. \tag{1.4}$$

In this paper, we study the numerical ranges of matrices defined in (2.10) and (1.4). We give a general exact formula for $\det(tI_n - (Q_n + Q_n^*))$. Using this exact formula for $n = 3, 4$, we verify that $W(Q_n)$ and $W(H_n)$ are closed disks centered at the origin.

2. Geometric weights.

THEOREM 2.1. *Let*

$$f_m = \det(zI_m - (Q_m + Q_m^*)). \tag{2.1}$$

Then we have

$$f_m(z) = z^m + \sum_{k=1}^{\left[\frac{m}{2}\right]} (-1)^k z^{m-2k} p^{k(k-1)} \prod_{i=1}^{k} \frac{1 - p^{m-2k+i}}{1 - p^i}, \tag{2.2}$$

where $p = q^2$, for $m \geq 2$.

Proof. Let $q^2 = p$. Assume that $f_0(z) = 1, f_1(z) = z$. Then we have $f_2(z) = z^2 - 1, f_3(z) = z^3 - z(1 + p)$. Expanding on the last row of the matrix (2.1) leads to the recurrence formula

$$f_{k+2}(z) = zf_{k+1}(z) - p^k f_k(z). \tag{2.3}$$

Now we prove (2.2) by induction method. We prove the formula (2.2) for the case $m = 2n$ and the case $m = 2n + 1$ can be done in an analogous way. $m = 2$ is trivial. Now assume that (2.2) is holds for $m = 2, 3, \dots, n, n + 1$ then we prove that for $m = n + 2$.

$$f_m(z) = z^{2n} + \sum_{k=1}^{n} (-1)^k z^{2n-2k} p^{k(k-1)} \prod_{i=1}^{k} \frac{1 - p^{2n-2k+i}}{1 - p^i}, \tag{2.4}$$

$$f_{m+1}(z) = z^{2n+1} + \sum_{k=1}^{n} (-1)^k z^{2n+1-2k} p^{k(k-1)} \prod_{i=1}^{k} \frac{1 - p^{2n+1-2k+i}}{1 - p^i}. \tag{2.5}$$

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Substituting (2.4) and (2.5) into the (2.3), we have

$$\begin{aligned}
f_{m+2} &= zf_{m+1}(z) - p^m f_m(z) \\
&= z^{2n+2} + \sum_{k=1}^{n}(-1)^k z^{2n+2-2k} p^{k(k-1)} \prod_{i=1}^{k} \frac{1-p^{2n+1-2k+i}}{1-p^i} \\
&\quad -p^{2n}z^{2n} - p^{2n}\sum_{k=1}^{n}(-1)^k z^{2n-2k}p^{k(k-1)}\prod_{i=1}^{k}\frac{1-p^{2n-2k+i}}{1-p^i}.
\end{aligned} \tag{2.6}$$

On the other hand, we have

(2.7)

$$\begin{aligned}
&(-1)^k z^{2n+2-2k} p^{k(k-1)} \prod_{i=1}^{k} \frac{1-p^{2n+1-2k+i}}{1-p^i} \\
&-p^{2n}(-1)^{k-1} z^{2n-2k+2} p^{(k-1)(k-2)} \cdot \prod_{i=1}^{k-1} \frac{1-p^{2n-2k+2+i}}{1-p^i} \\
&= (-1)^k z^{2n-2k+2} p^{k(k-1)} \frac{(1-p^{2n-2k+3})(1-p^{2n-2k+4})\cdots(1-p^{2n-k+1})}{(1-p)(1-p^2)\cdots(1-p^{n-1})} \\
&\quad \cdot \left[\frac{1-p^{2n-2k+2}}{1-p^k} + p^{2-2k+2n}\right] \\
&= (-1)^k z^{2n-2k+2} p^{k(k-1)} \frac{(1-p^{2n-2k+3})(1-p^{2n-2k+4})\cdots(1-p^{2n-k+1})(1-p^{2n-k+2})}{(1-p)(1-p^2)\cdots(1-p^{n-1})(1-p^n)} \\
&= (-1)^k z^{2n+2-2k} p^{k(k-1)} \prod_{i=1}^{k} \frac{1-p^{2n+1-2k+i}}{1-p^i}
\end{aligned}$$

Also

$$\begin{aligned}
-z^{2n}\cdot p^0 \cdot \frac{1-p^{2n}}{1-p} - p^{2n}z^{2n} - p^{2n}z^{2n} &= -z^{2n}\left(\frac{1-p^{2n}}{1-p} + p^{2n}\right) \\
&= -z^{2n}\cdot\frac{1-p^{2n+1}}{1-p},
\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}
-p^{2n}\cdot(-1)^n\cdot z^0\cdot p^{n(n-1)}\prod_{i=1}^{n}\frac{1-p^i}{1-p^i} &= -p^{2n}(-1)^n\cdot p^{n^2-n} = (-1)^{n+1}\cdot p^{n^2+n} \\
&= (-1)^{n+1} z^{2n+2-2(n+1)}\cdot p^{n(n+1)}\prod_{i=1}^{n+1}\frac{1-p^i}{1-p^i}
\end{aligned} \tag{2.9}$$

From (2.7), (2.8) and (2.9), it follows

$$f_{m+2}(z) = z^{2n+2} + \sum_{k=1}^{n+1}(-1)^k z^{2n+2-2k} p^{k(k-1)} \prod_{i=1}^{k} \frac{1-p^{2n+2-2k+i}}{1-p^i}.$$

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Hence, (2.2) is proved. □

Now we give a simple proof of a well known result; see for example [4].

THEOREM 2.2. *Let S be the shift matrix*

$$S = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & 1 & 0 \end{bmatrix}, \tag{2.10}$$

then the numerical range of S is a closed disc with centered at origin and $w(S) = \cos\left(\frac{\pi}{n+1}\right)$.

Proof. In Theorem2.1, we set $q = 1$, Then we have

$$f_n(z) = z^n + \sum_{k=1}^{[\frac{n}{2}]} (-1)^k C_{n-k}^k z^{n-2k}. \tag{2.11}$$

Recalling the Chebyshev polynomials of the second kind, $U_n(x)$, we have

$$U_n(x) = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k C_{n-k}^k (2x)^{n-2k} = \prod_{k=1}^{n} \left(x - \cos\left(\frac{k\pi}{n+1}\right)\right).$$

If we substitute $x = \frac{z}{2}$, then we have

$$\begin{aligned} U_n(z/2) = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k C_{n-k}^k (z)^{n-2k} &= \prod_{k=1}^{n} \left(\frac{z}{2} - \cos\left(\frac{k\pi}{n+1}\right)\right) \\ &= \frac{1}{2^n} \prod_{k=1}^{n} \left(z - 2\cos\left(\frac{k\pi}{n+1}\right)\right). \end{aligned} \tag{2.12}$$

Now from (2.11) and (2.12), it follows

$$f_n(z) = \frac{1}{2^n} \prod_{k=1}^{n} \left(z - 2\cos\left(\frac{k\pi}{n+1}\right)\right).$$

Hence, as we mentioned Section 1 and from [2], we have

$$w(S) = \cos\left(\frac{\pi}{n+1}\right)$$

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and the numerical range of S is circular disc with centered at origin. □

PROPOSITION 2.3. *Let Q_3 be the operator in $\mathbb{C}^3$ defined by the matrix*

$$Q_3 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & q & 0 \end{bmatrix}, \quad 0 < q < 1. \tag{2.13}$$

Then the numerical range of Q_3 is a closed disk centered at the origin and the radius is $\frac{\sqrt{1+q^2}}{2}$, i.e.,

$$W(Q_3) = \overline{\mathbb{D}}\left(0; \frac{\sqrt{1+q^2}}{2}\right). \tag{2.14}$$

Proof. Setting $m = 3$ in (2.2) yields $f_3(z) = z^3 - z(1+p)$. The maximum root of the equation $f_3(z) = 0$ is $\sqrt{1+q^2}$. Then, as we mentioned above in Section 1 (see [2]), it is easy to see that $W(Q_3) = \overline{\mathbb{D}}\left(0; \frac{\sqrt{1+q^2}}{2}\right)$. □

PROPOSITION 2.4. *Let Q_4 be the operator in $\mathbb{C}^4$ defined by the matrix*

$$Q_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & q^2 & 0 \end{bmatrix}, \quad 0 < q < 1. \tag{2.15}$$

Then the numerical range of Q_3 is a closed disk centered at the origin and radius is

$$\frac{1}{2}\sqrt{\frac{1}{2}\left((1+q^2+q^4) + \sqrt{(1-q^2+q^4)(1+3q^2+q^4)}\right)},$$

i.e.,

$$W(Q_4) = \overline{\mathbb{D}}\left(0; \frac{1}{2}\sqrt{\frac{1}{2}\left((1+q^2+q^4) + \sqrt{(1-q^2+q^4)(1+3q^2+q^4)}\right)}\right).$$

Proof. Setting $m = 4$ in (2.2) yields $f_4(z) = z^4 - z^2(1+p+p^2) + p^2$. The maximum root of the equation $f_4(z) = 0$ is

$$\sqrt{\frac{1}{2}\left((1+q^2+q^4) + \sqrt{(1-q^2+q^4)(1+3q^2+q^4)}\right)}.$$

Then, as we mentioned above in Section 1 (see [2]), it is easy to see that

$$W(Q_4) = \overline{\mathbb{D}}\left(0; \frac{1}{2}\sqrt{\frac{1}{2}\left((1+q^2+q^4) + \sqrt{(1-q^2+q^4)(1+3q^2+q^4)}\right)}\right). \quad \square$$

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3. Harmonic weights. In this section, we find $W(H_n)$ for $n = 3, 4$. We have

$$H_n + H_n^* = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \frac{1}{2} & 0 & \dots & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{3} & \dots & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & \frac{1}{n-1} \\ 0 & 0 & 0 & 0 & \dots & \frac{1}{n-1} & 0 \end{bmatrix}, \tag{3.1}$$

and let

$$P_n(x) = \det\left(xI_n - (H_n + H_n^*)\right). \tag{3.2}$$

We can assume that $P_0(x) = 1, P_1(x) = x$. Then we have

$$P_2(x) = 4(x^2 - 1), P_3(x) = 9(4x^3 - 5x).$$

Expanding on the last row of the matrix (3.2) leads to the recurrence formula

$$P_n(x) = n^2\left(xP_{n-1} - P_{n-2}(x)\right), \quad n \geq 2. \tag{3.3}$$

Now we find the numerical range of H_n for $n = 3, 4$ by using the recurrence formula 3.3.

Proposition 3.1. *In $\mathbb{C}^3$ let H_3 be the operator defined by the matrix*

$$H_3 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}. \tag{3.4}$$

Then the numerical range of H_3 is a closed disk centered at the origin and radius is $\frac{\sqrt{5}}{4}$, i.e.,

$$W(H_3) = \overline{\mathbb{D}}\left(0; \frac{\sqrt{5}}{4}\right). \tag{3.5}$$

Proof. In (3.3), we set $n = 3$. Then we have $P_3(x) = 9(4x^3 - 5x)$. The maximum root of the equation $P_3(x) = 0$ is $\frac{\sqrt{5}}{2}$, Then as we mentioned above in Section 1 (see [2]), it is easy to see that $W(H_3) = \overline{\mathbb{D}}\left(0; \frac{\sqrt{5}}{4}\right)$. $\square$

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PROPOSITION 3.2. *In $\mathbb{C}^4$, let H_4 be the operator defined by the matrix*

$$H_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \end{bmatrix}. \tag{3.6}$$

Then, the numerical range of H_4 is a closed disk centered at the origin with radius equal to $\frac{1}{2}\sqrt{\frac{49+5\sqrt{73}}{72}}$, i.e.,

$$W(H_4) = \overline{\mathbb{D}}\left(0; \frac{1}{2}\sqrt{\frac{49+5\sqrt{73}}{72}}\right). \tag{3.7}$$

Proof. In (3.3), we set $n = 4$. Then we have $P_4(x) = 16(36x^4 - 49x^2 + 4)$, The maximum root of the equation $P_4(x) = 0$ is $\sqrt{\frac{49+5\sqrt{73}}{72}}$. Then, as we mentioned above in Section 1 (see [2]), it is easy to see that

$$W(H_4) = \overline{\mathbb{D}}\left(0; \frac{1}{2}\sqrt{\frac{49+5\sqrt{73}}{72}}\right). \qquad \square$$

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