Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 562-577, June 2012

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PALINDROMIC LINEARIZATIONS OF A MATRIX POLYNOMIAL OF ODD DEGREE OBTAINED FROM FIEDLER PENCILS WITH REPETITION*

M.I. BUENO† AND S. FURTADO‡

Abstract. Many applications give rise to structured, in particular T-palindromic, matrix polynomials. In order to solve a polynomial eigenvalue problem $P(\lambda)x = 0$, where $P(\lambda)$ is a T-palindromic matrix polynomial, it is convenient to use palindromic linearizations to ensure that the symmetries in the eigenvalues, elementary divisors, and minimal indices of $P(\lambda)$ due to the palindromicity are preserved. In this paper, new T-palindromic strong linearizations valid for all palindromic matrix polynomials of odd degree are constructed. These linearizations are formulated in terms of Fiedler pencils with repetition, a new family of companion forms that was obtained recently by Antoniou and Vologiannidis.

Key words. Matrix polynomials, Linearization, Fiedler pencils with repetition, T-Palindromic linearizations, Companion form, Polynomial eigenvalue problem.

AMS subject classifications. 65F15, 15A18, 15A22.

1. Introduction. Let $\mathbb{F}$ be a field and denote by $M_n(\mathbb{F})$ the set of $n \times n$ matrices over $\mathbb{F}$. Let

$$P(\lambda) = A_k\lambda^k + A_{k-1}\lambda^{k-1} + \cdots + A_0, \tag{1.1}$$

where $A_i \in M_n(\mathbb{F})$, be a matrix polynomial of degree $k \geq 2$ (that is, $A_k \neq 0$). The matrix polynomial $P(\lambda)$ is said to be *regular* if $\det(P(\lambda)) \not\equiv 0$. Otherwise, $P(\lambda)$ is said to be *singular*.

For regular matrix polynomials, the polynomial eigenvalue problem consists of finding scalars $\lambda_0 \in \mathbb{F}$ and nonzero vectors $x, y \in \mathbb{F}^n$ satisfying $P(\lambda_0)x = 0$ and $y^T P(\lambda_0) = 0$. These λ_0's are the finite eigenvalues of P. Matrix polynomials may also have infinite eigenvalues [8, 10]. For singular matrix polynomials, other magnitudes such as minimal indices and minimal bases are of interest [7].

A standard way of solving polynomial eigenvalue problems is by using linearizations. A matrix pencil $L(\lambda) = \lambda L_1 - L_0$, with $L_1, L_0 \in M_{nk}(\mathbb{F})$, is a linearization of $P(\lambda)$ (see [9]) if there exist two unimodular matrix polynomials (matrix polynomials with constant nonzero

*Received by the editors on December 30, 2011. Accepted for publication on May 26, 2012. Handling Editor: Christian Mehl.

†Department of Mathematics, University of California, Santa Barbara, CA, USA (mbueno@math.ucsb.edu).

‡Faculdade de Economia do Porto, Rua Dr. Roberto Frias 4200-464 Porto, Portugal (sbf@fep.up.pt). This work was done within the activities of Centro de Estruturas Lineares e Combinatórias da Universidade de Lisboa.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 562-577, June 2012

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determinant), $U(\lambda)$ and $V(\lambda)$ such that

$$U(\lambda)L(\lambda)V(\lambda) = \begin{bmatrix} I_{(k-1)n} & 0 \\ 0 & P(\lambda) \end{bmatrix}.$$

Here and hereafter, I_m denotes the $m \times m$ identity matrix. Also, I_0 denotes the empty block.

If $P(\lambda)$ is a regular matrix polynomial, a linearization $L(\lambda)$ of $P(\lambda)$ is also regular and the eigenvalues and eigenvectors of $L(\lambda)$ can be computed by well-known algorithms for matrix pencils. Note that $P(\lambda)$ and $L(\lambda)$ share the finite eigenvalues but not necessarily the infinite eigenvalues. If $P(\lambda)$ is singular, linearizations can also be used to compute the minimal indices of $P(\lambda)$ [4, 5].

The *reversal* of the matrix polynomial $P(\lambda)$ in (1.1) is the matrix polynomial obtained by reversing the order of the coefficient matrices, that is,

$$\operatorname{rev} P(\lambda) := \sum_{i=0}^{k} \lambda^i A_{k-i}.$$

A linearization $L(\lambda)$ is called a *strong linearization* of a polynomial $P(\lambda)$ if $\operatorname{rev} L(\lambda)$ is also a linearization of $\operatorname{rev} P(\lambda)$. Observe that strong linearizations of $P(\lambda)$ have the same finite and infinite elementary divisors [8] as $P(\lambda)$. Moreover, if $P(\lambda)$ is regular, any linearization with the same infinite elementary divisors as $P(\lambda)$ is a strong linearization.

The matrix polynomial $P(\lambda)$ is said to be *T-palindromic* [15] if $A_i^T = A_{k-i}$, for $i = 0, \ldots, k$. T-palindromic matrix polynomials appear in numerous applications as in the vibrational analysis of railroad tracks excited by high speed trains [11, 13, 14, 15], or in the mathematical modelling and numerical simulation of the behavior of periodic surface acoustic wave filters [12, 18], among others.

When the polynomial $P(\lambda)$ is structured, it is convenient, both from the theoretical and computational point of view, to use linearizations with the same structure as $P(\lambda)$ to preserve any spectral symmetries. For example, when $P(\lambda)$ is T-palindromic, the elementary divisors corresponding to the eigenvalues $\lambda_0 \neq \pm 1$ always come in pairs $(\lambda - \lambda_0)^s$, $(\lambda - 1/\lambda_0)^s$.

Here, we are particularly interested in finding companion-like T-palindromic strong linearizations for T-palindromic matrix polynomials (regular or singular), that is, companion forms that are T-palindromic when the matrix polynomial is. An $nk \times nk$ matrix pencil $L_P(\lambda) = \lambda L_1 - L_0$ is said to be *a companion form* for general $n \times n$ matrix polynomials $P(\lambda)$ of degree k of the form (1.1) if $L_P(\lambda)$ is a strong linearization for every $P(\lambda)$ and each $n \times n$ block of L_1 and L_0 is either 0_n, I_n, or $\pm A_i$, for $i = 0, 1, \ldots, k$, when L_1 and L_0 are viewed as $k \times k$ block matrices. In [6], a family of companion-like T-palindromic linearizations for each odd degree $k \geq 3$ was constructed. These linearizations were obtained from generalized Fiedler pencils, introduced in [1, 2]. In this paper we construct a new family of T-palindromic companion forms based on the Fiedler pencils with repetition (FPR),

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 562-577, June 2012

which were recently presented in [17]. We consider a particular subfamily of the FPR that we call reverse-FPR of type 1 to obtain our T-palindromic linearizations. In [17], it was proven that symmetric linearizations for a symmetric matrix polynomial can be constructed from a subfamily of FPR that share with the subfamily we are considering the "type 1" property.

The T-palindromic linearizations we give in this paper have particular interest because, as it is shown in the preprint [3], it is easy to recover the eigenvectors and minimal bases of a T-palindromic matrix polynomial from them.

When the degree k of the matrix polynomial is even, there exist T-palindromic linearizations only if the elementary divisors of the matrix polynomial satisfy some conditions [16]. For this reason, the even case requires a separate treatment and we postpone such a study for a later paper.

The paper is organized as follows: In Section 2, we introduce some background that we use in the rest of the paper. In particular, we define a subfamily of the FPR from which we construct the T-palindromic linearizations. In Section 3, we prove the main result of the paper (Theorem 3.3), which describes how to construct our T-palindromic linearizations. Finally, in Section 4, we find strong T-anti-palidromic linearizations for T-anti-palindromic matrix polynomials of odd degree as a corollary of the main result obtained in Section 3.

2. Basic definitions and results. In this section, we introduce some definitions and results that will be used to prove our main theorem.

2.1. The matrices M_i. In Subsection 2.3 we will introduce the family of strong linearizations of $P(\lambda)$ from which we will obtain our linearizations that are T-palindromic when $P(\lambda)$ is. This family is constructed using the matrices $M_i(P)$, depending on the coefficients of the polynomial P, which we define below.

Note that in [17] the FPR are constructed using the matrices $A_i = RM_iR$, where R is the matrix in (2.3). However, if we multiply our linearizations on the left and on the right by the matrix R, we get linearizations of the form described there.

The matrices M_i that we now define are presented as block matrices partitioned into $k \times k$ blocks of size $n \times n$. Here we consider a polynomial $P(\lambda)$ of the form (1.1). Unless the context makes it ambiguous, we will denote these matrices by M_i instead of $M_i(P)$.

$$M_0 := \left[\begin{array}{c|c} I_{(k-1)n} & 0 \\ \hline 0 & -A_0 \end{array}\right], \quad M_{-k} := \left[\begin{array}{c|c} A_k & 0 \\ \hline 0 & I_{(k-1)n} \end{array}\right], \quad \text{and}$$

$$M_i := \left[\begin{array}{c|cc|c} I_{(k-i-1)n} & 0 & 0 & 0 \\ \hline 0 & -A_i & I_n & 0 \\ 0 & I_n & 0 & 0 \\ \hline 0 & 0 & 0 & I_{(i-1)n} \end{array}\right], \quad i = 1, \ldots, k-1. \tag{2.1}$$

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A publication of the International Linear Algebra Society
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The matrices M_i in (2.1) are always invertible and their inverses are given by

$$M_{-i} := M_i^{-1} = \left[\begin{array}{c|cc|c} I_{(k-i-1)n} & 0 & 0 & 0 \\ \hline 0 & 0 & I_n & 0 \\ 0 & I_n & A_i & 0 \\ \hline 0 & 0 & 0 & I_{(i-1)n} \end{array}\right].$$

The matrices M_0 and M_{-k} are invertible if and only if A_0 and A_k, respectively, are. If M_0 is invertible, we denote M_0^{-1} by M_{-0}; if M_{-k} is invertible, we denote M_{-k}^{-1} by M_k.

It is easy to check that the commutativity relations

$$M_i(P)M_j(P) = M_j(P)M_i(P), \quad \text{for any } P(\lambda) \text{ with degree } k, \tag{2.2}$$

hold if and only if $||i| - |j|| \neq 1$.

In this paper, we are concerned with pencils constructed from products of matrices M_i and M_{-i}. In our analysis, the order in which these matrices appear in the products is relevant. For this reason, we will associate an index tuple with each of these products to simplify our developments. We also introduce some additional concepts defined in [17] which are related to this notion. We will use boldface letters $(\mathbf{t}, \mathbf{q}, \mathbf{z} \ldots)$ for ordered tuples of indices (called *index tuples* in the following).

Let $\mathbf{t} = (i_1, i_2, \ldots, i_r)$ be an index tuple containing indices from $\{0, 1, \ldots, k, -0, -1, \ldots, -k\}$. We denote

$$M_{\mathbf{t}} := M_{i_1} M_{i_2} \cdots M_{i_r}.$$

When $-0 \in \mathbf{t}$ (resp., $k \in \mathbf{t}$), we assume that M_0 (resp., M_{-k}) is invertible.

DEFINITION 2.1. Let $\mathbf{t}_1$ and $\mathbf{t}_2$ be two index tuples with indices from $\{0, 1, \ldots, k, -0, -1, \ldots, -k\}$. We say that $\mathbf{t}_1$ is *equivalent* to $\mathbf{t}_2$, and write $\mathbf{t}_1 \sim \mathbf{t}_2$, if $M_{\mathbf{t}_1}(P) = M_{\mathbf{t}_2}(P)$ for any matrix polynomial $P(\lambda)$ of the form (1.1).

Notice that $\sim$ is an equivalence relation and, if $M_{\mathbf{t}_2}$ is obtained from $M_{\mathbf{t}_1}$ by the repeated application of the commutativity relations (2.2), then $\mathbf{t}_1$ is equivalent to $\mathbf{t}_2$.

We finish this subsection with a result concerning the matrices M_i associated with a T-palindromic matrix polynomial which will be used in the proof of our main result. The quasi-identity matrices, which we now define, will play a crucial role.

DEFINITION 2.2. We say that $S \in M_{nk}(\mathbb{F})$ is a *quasi-identity matrix* if $S = \epsilon_1 I_n \oplus \cdots \oplus \epsilon_k I_n$ for some $\epsilon_i \in \{1, -1\}$, $i = 1, \ldots, k$. We call $(\epsilon_1, \ldots, \epsilon_k)$ the *parameters* of S. For $i = 1, \ldots, k$, we denote by $S(i,i)$ the ith diagonal block $\epsilon_i I_n$ of S. Also, by S_i we denote the quasi-identity matrix whose only negative parameter is ϵ_i. Moreover, we denote by S_0 and S_{k+1} the identity matrix I_{nk}.

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A publication of the International Linear Algebra Society
Volume 23, pp. 562-577, June 2012

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Note that a quasi-identity matrix S is exactly the product of the matrices S_i for which ϵ_i is a negative parameter of S.

We consider the following $nk \times nk$ matrix partitioned into $k \times k$ blocks of size $n \times n$:

(2.3) $$R := \begin{bmatrix} 0 & & I_n \\ & \iddots & \\ I_n & & 0 \end{bmatrix} \in M_{nk}(\mathbb{F}).$$

Note that $R^2 = I$.

Taking into account that $RS_iR = S_{k+1-i}$, for $i = 0, \ldots, k+1$, the next result can be easily obtained.

PROPOSITION 2.3. *Suppose that the matrix polynomial $P(\lambda)$ defined in (1.1) is T-palindromic. Then,*

$$RM_{-i}R = S_{i+1}M_{k-i}^T S_i,\ i = 1, \ldots, k,$$

or, equivalently,

$$RM_{k-i}R = S_{k+1-i}M_{-i}^T S_{k-i},\ i = 1, \ldots, k.$$

If M_0 is invertible, both equalities hold for $i = 0$, with $M_{-i} = M_{-0}$.

2.2. Simple and type 1 index tuples. We start with a definition for general index tuples that will be useful throughout the paper.

DEFINITION 2.4. Given an index tuple $\mathbf{t} = (i_1, \ldots, i_r)$, we define the *reversal* tuple of $\mathbf{t}$ as $\operatorname{rev} \mathbf{t} := (i_r, \ldots, i_1)$.

Let $\mathbf{t}_1$ and $\mathbf{t}_2$ be two index tuples. Some immediate properties of the reversal operation are:

- $rev(rev(\mathbf{t}_1)) = \mathbf{t}_1$,
- $rev(\mathbf{t}_1, \mathbf{t}_2) = (rev(\mathbf{t}_2), rev(\mathbf{t}_1))$.

DEFINITION 2.5. Let $\mathbf{q} = (i_1, i_2, \ldots, i_r)$ be an index tuple of integers. We say that $\mathbf{q}$ is *simple* if $i_j \neq i_l$ for all $j, l \in \{1, 2, \ldots, r\}$, $j \neq l$.

We will refer to a simple index tuple consisting of consecutive integers as a *string*. We will use the notation $(a : b)$ for the string of integers from a to b, that is,

$$(a : b) := \begin{cases} (a, a+1, \ldots, b), & \text{if } a \leq b, \\ \emptyset, & \text{if } a > b. \end{cases}$$

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A publication of the International Linear Algebra Society
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We now focus on tuples of nonnegative indices. The definitions and results presented can be extended to tuples of negative indices [17] but we don't need them for our purposes.

DEFINITION 2.6. Let $\mathbf{q}$ be a permutation of $\{0, 1, \ldots, h\}$. Then, $\mathbf{q}$ is said to be in *column standard form* if

$$\mathbf{q} = (t_w + 1 : h, t_{w-1} + 1 : t_w, \ldots, t_2 + 1 : t_3, t_1 + 1 : t_2, 0 : t_1)$$

for some positive integers $0 \leq t_1 < t_2 < \cdots < t_w < t_{w+1} = h$. We call each subinterval of consecutive integers $(t_{i-1} + 1 : t_i)$, for $i = 1, \ldots, w + 1$, with $t_0 = -1$, a *string* in $\mathbf{q}$.

DEFINITION 2.7. Let $\mathbf{q}$ be a simple index tuple. Then, we call the *column standard form of* $\mathbf{q}$ the unique index tuple in column standard form equivalent to $\mathbf{q}$ and we denote it by $csf(\mathbf{q})$.

The next definition, which was introduced in [5], is crucial in our work.

DEFINITION 2.8. [5] Let $\mathbf{q}$ be a simple index tuple with all its elements from $\{0, 1, \ldots, h\}$. We say that $\mathbf{q}$ has a *consecution* at j if both $j, j + 1 \in \mathbf{q}$ and j is to the left of $j + 1$ in $\mathbf{q}$. We say that $\mathbf{q}$ has an *inversion* at j if both $j, j + 1 \in \mathbf{q}$ and j is to the right of $j + 1$ in $\mathbf{q}$.

EXAMPLE 2.9. Let $\mathbf{q} = (10 : 13, 9, 5 : 8, 4, 3, 0 : 2)$. This tuple has consecutions at $0, 1, 5, 6, 7, 10, 11$ and 12. It has inversions at $2, 3, 4, 8,$ and 9.

Note that two equivalent simple index tuples have the same inversions and consecutions.

We now present some definitions for tuples with repeated indices that, in particular, allow us to associate a simple tuple with a tuple with repetitions. These definitions will play a central role in the description of our T-palindromic linearizations.

DEFINITION 2.10. (Type 1 indices relative to a simple index tuple) Let h be a nonnegative integer and $\mathbf{q}$ be a permutation of $\{0, 1, \ldots, h\}$. Let s be an index in $\{0, 1, \ldots, h - 1\}$.

- s is said to be a *right index of type 1 relative to* $\mathbf{q}$ if there is a string $(t_{d-1} + 1 : t_d)$ in $csf(\mathbf{q})$ such that $s = t_{d-1} + 1 < t_d$.
- s is said to be a *left index of type 1 relative to* $\mathbf{q}$ if s is a right index of type 1 relative to $rev(\mathbf{q})$.

Note that if s is a right index of type 1 relative to $\mathbf{q}$, then $(\mathbf{q}, s) \sim (s, \mathbf{q}')$ where $\mathbf{q}'$ is also a simple tuple. This observation justifies the following definition.

DEFINITION 2.11. (Associated simple tuple) Let h be a nonnegative integer and $\mathbf{q}$ be a permutation of $\{0, 1, \ldots, h\}$. Let $csf(\mathbf{q}) = (\mathbf{b}_{w+1}, \mathbf{b}_w, \ldots, \mathbf{b}_1)$, where $\mathbf{b}_i = (t_{i-1} + 1 : t_i)$, $i = 1, \ldots, w + 1$, are the strings of $csf(\mathbf{q})$. We say that the *simple tuple associated with* $\mathbf{q}$ is $csf(\mathbf{q})$ and denote it by $z_r(\mathbf{q})$. If s is a right index of type 1 with respect to $\mathbf{q}$, say $s = t_{d-1} + 1 < t_d$, then the *simple tuple associated with* $(\mathbf{q}, s)$ is the simple tuple:

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A publication of the International Linear Algebra Society
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- $z_r(\mathbf{q}, s) := \left(\mathbf{b}_{w+1}, \mathbf{b}_w, \ldots, \mathbf{b}_{d+1}, \widetilde{\mathbf{b}}_d, \widetilde{\mathbf{b}}_{d-1}, \mathbf{b}_{d-2}, \ldots, \mathbf{b}_1\right)$, where

$$\widetilde{\mathbf{b}}_d = (t_{d-1} + 2 : t_d) \qquad \text{and} \qquad \widetilde{\mathbf{b}}_{d-1} = (t_{d-2} + 1 : t_{d-1} + 1)$$

  if $s \neq 0$.
- $z_r(\mathbf{q}, s) := \left(\mathbf{b}_{w+1}, \mathbf{b}_w, \ldots, \mathbf{b}_d, \ldots, \widetilde{\mathbf{b}}_1, \widetilde{\mathbf{b}}_0\right)$, where

$$\widetilde{\mathbf{b}}_1 = (1 : t_1) \qquad \text{and} \qquad \widetilde{\mathbf{b}}_0 = (0)$$

  if $s = 0$.

Notice that, if s is a right index of type 1 relative to $\mathbf{q}$, then $(s, z_r(\mathbf{q}, s)) \sim (\mathbf{q}, s)$. Moreover, the simple tuple associated with a right index is, by definition, in column standard form.

DEFINITION 2.12. (Right and left index tuple of type 1) Let h be a nonnegative integer, $\mathbf{q}$ be a permutation of $\{0, 1, \ldots, h\}$, and $\mathbf{r}_q$ and $\mathbf{l}_q$ be tuples with indices from $\{0, 1, \ldots, h-1\}$, possibly with repetitions.

- We say that $\mathbf{r}_q = (s_1, \ldots, s_r)$, where s_i is the ith index of $\mathbf{r}_q$, is a *right index tuple of type 1* relative to $\mathbf{q}$ if, for $i = 1, \ldots, r$, s_i is a right index of type 1 with respect to $z_r(\mathbf{q}, (s_1, \ldots, s_{i-1}))$, where $z_r(\mathbf{q}, (s_1, \ldots, s_{i-1})) := z_r(z_r(\mathbf{q}, (s_1, \ldots, s_{i-2})), s_{i-1})$ for $i > 2$.
- We say that $\mathbf{l}_q$ is a *left index tuple of type 1* relative to $\mathbf{q}$ if $rev(\mathbf{l}_q)$ is a right index tuple of type 1 relative to $rev(\mathbf{q})$. Moreover, if $\mathbf{l}_q$ is a left index tuple of type 1 relative to $\mathbf{q}$, we define $z_l(\mathbf{l}_q, \mathbf{q}) := rev(z_r(rev(\mathbf{q}), rev(\mathbf{l}_q)))$.

We observe that, if $\mathbf{r}_q$ is a right index tuple of type 1 relative to $\mathbf{q}$, then $(\mathbf{q}, \mathbf{r}_q) \sim (\mathbf{r}_q, z_r(\mathbf{q}, \mathbf{r}_q))$.

Note that if $\mathbf{l}_q = (s_r, \ldots, s_1)$, $r > 1$, we have $z_l(\mathbf{l}_q, \mathbf{q}) = z_l(s_r, z_l((s_{r-1}, \ldots, s_1), \mathbf{q}))$.

EXAMPLE 2.13. Let $\mathbf{q} = (10 : 13, 9, 5 : 8, 4, 3, 0 : 2)$ and let $\mathbf{r}_q = (10 : 12, 5 : 6, 9, 0 : 1, 0)$. Observe that, while $\mathbf{q}$ is a simple tuple, $\mathbf{r}_q$ contains repeated indices. Note that 10 is a right index of type 1 relative to $\mathbf{q}$. The simple tuple associated with $(\mathbf{q}, 10)$ is $z_r(\mathbf{q}, (10)) = (11 : 13, 9 : 10, 5 : 8, 4, 3, 0 : 2)$. Also, 11 is a right index of type 1 relative to $z_r(\mathbf{q}, 10)$, therefore, $z_r(\mathbf{q}, (10, 11)) = (12 : 13, 9 : 11, 5 : 8, 4, 3, 0 : 2)$. It is easy to check that $\mathbf{r}_q$ is a right index tuple of type 1 and $z_r(\mathbf{q}, \mathbf{r}_q) = (13, 10 : 12, 7 : 9, 4 : 6, 2, 1, 0)$.

2.3. Reverse Fiedler pencils with repetition of type 1. Here, we focus on a particular class of pencils associated with matrix polynomials of odd degree k that we call the *reverse-FPR* of type 1, and from which we obtain our T-palindromic linearizations. This class is contained in the family of FPR introduced in [17]. We do not give its definition here as it involves some concepts that are not needed for our purposes. However, we observe that FPR

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A publication of the International Linear Algebra Society
Volume 23, pp. 562-577, June 2012

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are companion-like [17] since the coefficients of these pencils can be viewed as $k \times k$ block matrices of sizes $n \times n$ which are either 0_n, I_n, or $\pm A_i$ for some $i \in \{0, 1, \ldots, k\}$; also any FPR associated with a matrix polynomial $P(\lambda)$ is a strong linearization of $P(\lambda)$, as proven in [17].

DEFINITION 2.14. (Reverse-FPR of type 1) Let $P(\lambda)$ be a matrix polynomial of odd degree $k \geq 3$ as in (1.1) and $h = \frac{k-1}{2}$. Let $\mathbf{q}$ be a permutation of $\{0, 1, \ldots, h\}$. Assume that $\mathbf{r}_q$ and $\mathbf{l}_q$, if nonempty, are, respectively, a right and a left index tuple of type 1 relative to $\mathbf{q}$. Then, the pencil given by

$$\lambda M_{-k+rev(\mathbf{r}_q)} M_{\mathbf{l}_q} M_{-k+rev(\mathbf{q})} M_{\mathbf{r}_q} M_{-k+rev(\mathbf{l}_q)} - M_{-k+rev(\mathbf{r}_q)} M_{\mathbf{l}_q} M_{\mathbf{q}} M_{\mathbf{r}_q} M_{-k+rev(\mathbf{l}_q)}$$

is called a *reverse-Fiedler pencil with repetition (reverse-FPR) of type 1* associated with $P(\lambda)$ and is denoted by $F_{\mathbf{l}_q,\mathbf{q},\mathbf{r}_q}(\lambda)$.

Note that the matrices $M_{-k+rev(\mathbf{r}_q)}$ and $M_{\mathbf{l}_q}$ (resp., $M_{-k+rev(\mathbf{l}_q)}$ and $M_{\mathbf{r}_q}$) commute.

We now show that a reverse-FPR of type 1 is an FPR. For that purpose we just need to show that $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ in Definition 2.14 satisfies the SIP property, whose definition we include next, as all the other conditions in the definition of FPR are clearly satisfied.

DEFINITION 2.15. [17] Let $\mathbf{t} = (i_1, i_2, \ldots, i_r)$ be an index tuple with elements from $\{0, 1, \ldots, h\}$. Then, $\mathbf{t}$ is said to satisfy the *Successor Infix Property (SIP)* if for every pair of indices $i_a, i_b \in \mathbf{t}$, with $0 \leq a < b \leq r$, satisfying $i_a = i_b$, there exists at least one index $i_c = i_a + 1$ with $a < c < b$.

The next lemma allows us to conclude that the "type 1" property for index tuples implies the SIP property required in the definition of a general FPR. Thus, as already mentioned, a reverse-FPR of type 1 is a Fiedler pencil with repetition.

LEMMA 2.16. *Let* $\mathbf{q}$ *be a permutation of* $\{0, 1, \ldots, h\}$. *Suppose that* $\mathbf{r}_q$ *is a right index tuple of type* 1 *relative to* $\mathbf{q}$ *and* $\mathbf{l}_q$ *is a left index tuple of type* 1 *relative to* $\mathbf{q}$. *Then* $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ *satisfies the SIP.*

Proof. It is enough to prove that $(\mathbf{q}, \mathbf{r}_q)$ and $(\mathbf{l}_q, \mathbf{q})$ satisfy the SIP as, because $\mathbf{q}$ is a permutation of $\{0, 1, \ldots, h\}$, this implies that $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ satisfies the SIP. We show the first claim. The proof is by induction on the number r of indices of $\mathbf{r}_q$. Clearly, the result holds if $r = 0$. Suppose that $r > 0$ and let $\mathbf{r}_q = (s_1, \ldots, s_r)$. Then, $\mathbf{r}'_q = (s_1, \ldots, s_{r-1})$ is a right index tuple of type 1 relative to $\mathbf{q}$ and s_r is a right index of type 1 relative to $z_r(\mathbf{q}, (s_1, \ldots, s_{r-1}))$. By the induction hypothesis, $(\mathbf{q}, \mathbf{r}'_q)$ satisfies the SIP. Also, there is a string $(t_i + 1 : t_{i+1})$ in $z_r(\mathbf{q}, (s_1, \ldots, s_{r-1}))$ such that $s_r = t_i + 1 < t_{i+1}$. By Definitions 2.11 and 2.12, this means that $s_r + 1$ is to the right of the last index (from left to right) equal to s_r in $(\mathbf{q}, \mathbf{r}'_q)$. Therefore, $(\mathbf{q}, \mathbf{r}'_q, s_r)$ satisfies the SIP which implies the result.

Since $\mathbf{l}_q$ is a left index tuple of type 1 relative to $\mathbf{q}$, $rev(\mathbf{l}_q)$ is a right index tuple of type 1 relative to $rev(\mathbf{q})$. By the previous case, $rev(\mathbf{l}_q, \mathbf{q}) = (rev(\mathbf{q}), rev(\mathbf{l}_q))$ satisfies the SIP.

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A publication of the International Linear Algebra Society
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Then, $(\mathbf{l}_q, \mathbf{q}) = rev(rev(\mathbf{q}), rev(\mathbf{l}_q))$ satisfies the SIP as well. Note that if an index tuple satisfies the SIP, its reversal also satisfies the SIP. $\square$

3. T-palindromic linearizations from reverse-FPR of type 1. In this section, we prove Theorem 3.3, which is the main result in this paper, and give some corollaries of it.

T-palindromic linearizations obtained from a particular class of companion-like pencils have already been considered in [6]. Here we focus on a different class of companion-like pencils, the reverse-Fiedler pencils with repetition of type 1.

The next lemma is crucial in our proofs.

LEMMA 3.1. *Let $P(\lambda)$ be the matrix polynomial defined in (1.1). Suppose that $P(\lambda)$ is T-palindromic. If $T(\lambda)$ is a T-palindromic strong linearization of $P(\lambda)$ and $L(\lambda) = Q_1 T(\lambda) Q_2$ for some constant nonsingular matrices $Q_1, Q_2 \in M_{nk}(\mathbb{F})$, then $Q_2^T Q_1^{-1} L(\lambda)$ is a T-palindromic strong linearization of $P(\lambda)$.*

Proof. Clearly, $Q_2^T Q_1^{-1} L(\lambda)$ is strictly equivalent to $T(\lambda)$ and, therefore, is a strong linearization of $P(\lambda)$. To see that $Q_2^T Q_1^{-1} L(\lambda)$ is T-palindromic, note that

$$Q_2^T Q_1^{-1} L(\lambda) = Q_2^T T(\lambda) Q_2.$$

Let $T(\lambda) = \lambda T_1 - T_0$. Since $T_1 = -T_0^T$, we have $Q_2^T T_1 Q_2 = -Q_2^T T_0^T Q_2$, implying that $Q_2^T Q_1^{-1} L(\lambda)$ is T-palindromic. $\square$

From now on we assume that $P(\lambda)$ is a T-palindromic polynomial with odd degree $k \geq 3$ as in (1.1). We consider R the matrix defined in (2.3).

LEMMA 3.2. *Let $P(\lambda)$ be a (regular or singular) T-palindromic matrix polynomial of odd degree $k \geq 3$. Let $L(\lambda) = F_{\emptyset, \mathbf{q}, \mathbf{r}_q}(\lambda)$ be a reverse-FPR of type 1 associated with $P(\lambda)$. Then, $SRL(\lambda)$ is a T-palindromic strong linearization of $P(\lambda)$, where S is the quasi-identity matrix defined as follows:*

$$S(i,i) = -I \quad \textit{if and only if} \quad \begin{cases} \mathbf{q} \textit{ has an inversion at } i-1, \\ z_r(\mathbf{q}, \mathbf{r}_q) \textit{ has a consecution at } k-i. \end{cases} \tag{3.1}$$

Proof. We prove the result by induction on the number r of indices of $\mathbf{r}_q$. If $r = 0$, the result follows from [6]. Now suppose that $r > 0$. Assume that the result is true when $\mathbf{r}_q$ contains $r-1$ indices. Suppose that $\mathbf{r}_q = (s_1, \ldots, s_r)$, where s_i denotes the ith index in $\mathbf{r}_q$ and let $\mathbf{r}'_q = (s_1, \ldots, s_{r-1})$. Note that $\mathbf{r}'_q$ is a right index tuple of type 1 relative to $\mathbf{q}$. Consider the reverse-FPR of type 1, $L'(\lambda) = F_{\emptyset, \mathbf{q}, \mathbf{r}'_q}$, associated with $P(\lambda)$. By the induction hypothesis, $S'RL'(\lambda)$ is a T-palindromic strong linearization of $P(\lambda)$, where S' is the quasi-identity matrix satisfying

$$S'(i,i) = -I \quad \text{if and only if} \quad \begin{cases} \mathbf{q} \text{ has an inversion at } i-1, \\ z_r(\mathbf{q}, \mathbf{r}'_q) \text{ has a consecution at } k-i. \end{cases}$$

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A publication of the International Linear Algebra Society
Volume 23, pp. 562-577, June 2012

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Note that $L(\lambda) = M_{-k+s_r} RS'(S'RL'(\lambda))M_{s_r}$. By Lemma 3.1, $SRL(\lambda)$ is a T-palindromic strong linearization of $P(\lambda)$, where $S := M_{s_r}^T S'RM_{-k+s_r}^{-1} R$. We next show that S is the quasi-identity matrix satisfying (3.1). Since s_r is a right index of type 1 relative to $(\mathbf{q}, \mathbf{r}'_q)$, there exists a string in $z_r(\mathbf{q}, \mathbf{r}'_q)$ of the form $(t_i + 1 : t_{i+1})$ with $s_r = t_i + 1 < t_{i+1}$. Then,

$$S = M_{t_i+1}^T S'RM_{k-t_i-1}R,$$

or, equivalently, by using Proposition 2.3 (recall that $S_{k+1} = I_n$),

$$S = M_{t_i+1}^T S' S_{k-t_i} M_{-t_i-1}^T S_{k-t_i-1}. \tag{3.2}$$

(Here $M_{-t_i-1} = M_{-0}$ if $s_r = 0$.) Assume $s_r = t_i + 1 > 0$. As we show next, the matrices $M_{t_i+1}^T$ and $S'S_{k-t_i}$ commute. Thus,

$$S = S'S_{k-t_i}S_{k-t_i-1}. \tag{3.3}$$

Let us show that $M_{t_i+1}^T$ and $S'S_{k-t_i}$ commute. Since M_{t_i+1} has the form $I_{k-t_i-2} \oplus [*] \oplus I_{t_i}$, where $[*]$ is a 2×2 block, it is enough to note that both the $(k - t_i - 1)th$ and the $(k - t_i)th$ parameters of $S'S_{k-t_i}$ have the same sign, which follows because the parameters of S' in positions $k - t_i - 1$ is -1 and the one in position $k - t_i$ is 1. To see this, note that, because $k - t_i - 1 > h = (k - 1)/2$, the parameter of S' in position $k - t_i - 1$ is -1 if and only if $z_r(\mathbf{q}, \mathbf{r}'_q)$ has a consecution at $t_i + 1$. Also, the parameter of S' in position $k - t_i$ is -1 if and only if $z_r(\mathbf{q}, \mathbf{r}'_q)$ has a consecution at t_i. It can be easily verified that $z_r(\mathbf{q}, \mathbf{r}'_q)$ has a consecution at $t_i + 1$ and has an inversion at t_i. Assume that

$$z_r(\mathbf{q}, \mathbf{r}'_q) = (t_w + 1 : h, \ldots, t_i + 1 : t_{i+1}, t_{i-1} + 1 : t_i, \ldots, 0 : t_1),$$

where $0 \le t_1 < t_2 < \cdots < t_w < h$. Then, by Definition 2.12, we get

$$z_r(\mathbf{q}, \mathbf{r}_q) = (t_w{+}1 : h, t_{w-1}{+}1 : t_w, \ldots, t_i{+}2 : t_{i+1}, t_{i-1}{+}1 : t_i{+}1, \ldots, t_1{+}1 : t_2, 0 : t_1).$$

Note that the inversions and consecutions that occur in $z_r(\mathbf{q}, \mathbf{r}'_q)$ are the same as those that occur in $z_r(\mathbf{q}, \mathbf{r}_q)$, except at t_i, where $z_r(\mathbf{q}, \mathbf{r}'_q)$ has an inversion and $z_r(\mathbf{q}, \mathbf{r}_q)$ has a consecution, and at $t_i + 1$, where $z_r(\mathbf{q}, \mathbf{r}'_q)$ has a consecution and $z_r(\mathbf{q}, \mathbf{r}_q)$ has an inversion. Also, from (3.3), the parameters of the quasi-identity matrices S and S' coincide, except those in positions $k - t_i$ and $k - t_i - 1$. Since the $(k - t_i)th$ and the $(k - t_i - 1)th$ parameters of S' are 1 and -1, respectively, we just need to note that $z_r(\mathbf{q}, \mathbf{r}_q)$ has a consecution at t_i and an inversion at $t_i + 1$. Then, it follows that $M_{t_i+1}^T$ and $S'S_{k-t_i}$ commute, which implies that S has the desired form.

Suppose $s_r = 0$. Then, M_0 is invertible, and from (3.2), $S = M_0^T S' M_{-0}^T S_k$. Clearly, M_0^T and S' commute. Thus, $S = S'S_k$. Using the notation above for $z_r(\mathbf{q}, \mathbf{r}'_q)$, we have

$$z_r(\mathbf{q}, \mathbf{r}_q) = (t_w + 1 : h, t_{w-1} + 1 : t_w, \ldots, t_1 + 1 : t_2, 1 : t_1, 0).$$

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A publication of the International Linear Algebra Society
Volume 23, pp. 562-577, June 2012

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Similar arguments show that S satisfies (3.1). □

The next result, which generalizes Lemma 3.2, is the main result in this paper. Note that, although the proofs of both results use similar arguments, one cannot be deduced from the other.

THEOREM 3.3. *Let $P(\lambda)$ be a (regular or singular) T-palindromic matrix polynomial of odd degree $k \geq 3$. Let $L(\lambda) = F_{\mathbf{l}_q,\mathbf{q},\mathbf{r}_q}(\lambda)$ be a reverse-FPR of type 1 associated with $P(\lambda)$. Then, $SRL(\lambda)$ is a T-palindromic strong linearization of $P(\lambda)$, where S is the quasi-identity matrix given by*

$$(3.4) \qquad S(i,i) = -I \quad \textit{if and only if} \quad \begin{cases} z_l(\mathbf{l}_q, \mathbf{q}) \textit{ has an inversion at } i-1, \\ z_r(\mathbf{q}, \mathbf{r}_q) \textit{ has a consecution at } k-i. \end{cases}$$

Proof. We prove the result by induction on the number r of indices of $\mathbf{l}_q$. If $r = 0$, the result follows from Lemma 3.2. Now suppose that the result is true when $\mathbf{l}_q$ contains $r-1$ indices. Suppose $\mathbf{l}_q = (s_r, \ldots, s_1)$, where s_i denotes an index in $\mathbf{l}_q$, and let $\mathbf{l}'_q = (s_{r-1}, \ldots, s_1)$. Note that $\mathbf{l}'_q$ is a left index tuple of type 1 relative to $\mathbf{q}$. Consider the reverse-FPR of type 1 $L'(\lambda) = F_{\mathbf{l}'_q,\mathbf{q},\mathbf{r}_q}$ associated with $P(\lambda)$. By the induction hypothesis, $S'RL'(\lambda)$ is a T-palindromic strong linearization of $P(\lambda)$, where S' is the quasi-identity matrix satisfying

$$S'(i,i) = -I \quad \text{if and only if} \quad \begin{cases} z_l(\mathbf{l}'_q, \mathbf{q}) \text{ has an inversion at } i-1, \\ z_r(\mathbf{q}, \mathbf{r}_q) \text{ has a consecution at } k-i. \end{cases}$$

Note that $L(\lambda) = M_{s_r} RS'(S'RL'(\lambda))M_{-k+s_r}$. By Lemma 3.1, $SRL(\lambda)$ is a T-palindromic strong linearization of $P(\lambda)$, where $S := M^T_{-k+s_r} S'RM^{-1}_{s_r} R$. We next show that S is the quasi-identity matrix satisfying (3.4). Since s_r is a left index of type 1 relative to $z_l(\mathbf{l}'_q, \mathbf{q})$, there exists a string in $rev(z_l(\mathbf{l}'_q, \mathbf{q}))$ of the form $(t_i + 1 : t_{i+1})$ with $s_r = t_i + 1 < t_{i+1}$. Then,

$$S = M^T_{-k+t_i+1} S'RM_{-t_i-1}R,$$

or, equivalently, by using Proposition 2.3,

$$(3.5) \qquad S = M^T_{-k+t_i+1} S' S_{t_i+2} M^T_{k-t_i-1} S_{t_i+1}.$$

Suppose that $s_r = t_i + 1 > 0$. As we show next, the matrices $M^T_{k-t_i-1}$ and $S'S_{t_i+2}$ commute. Thus,

$$(3.6) \qquad S = S' S_{t_i+2} S_{t_i+1}.$$

Let us show that $M^T_{k-t_i-1}$ and $S'S_{t_i+2}$ commute. Since M_{k-t_i-1} has the form $I_{t_i} \oplus [*] \oplus I_{k-t_i-2}$, where $[*]$ is a 2×2 block, it is enough to note that both the $(t_i + 1)th$ and the

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A publication of the International Linear Algebra Society
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$(t_i+2)th$ parameters of $S'S_{t_i+2}$ have the same sign, which follows because the parameter of S' in position t_i+1 is 1 and the one in position t_i+2 is -1. To see this, note that, because $k-t_i-2>h=(k-1)/2$, the parameter of S' in position t_i+2 is -1 if and only if $z_l(\mathbf{l}'_q,\mathbf{q})$ has an inversion at t_i+1. Also, the parameter of S' in position t_i+1 is -1 if and only if $z_l(\mathbf{l}'_q,\mathbf{q})$ has an inversion at t_i. As $rev(z_l(\mathbf{l}'_q,\mathbf{q}))$ has a consecution at t_i+1 and an inversion at t_i, $z_l(\mathbf{l}'_q,\mathbf{q})$ has an inversion at t_i+1 and a consecution at t_i.

Note that $rev(z_l(\mathbf{l}'_q,\mathbf{q}))$ is in column standard form. Also,

$$rev(z_l(\mathbf{l}_q,\mathbf{q}))=z_r(rev(\mathbf{q}),rev(\mathbf{l}_q))=$$

$$z_r(z_r(rev(\mathbf{q}),rev(\mathbf{l}'_q)),s_r)=z_r(rev(z_l(\mathbf{l}'_q,\mathbf{q})),s_r). \tag{3.7}$$

Assume that

$$rev(z_l(\mathbf{l}'_q,\mathbf{q}))=(t_w+1:h,\ldots,t_i+1:t_{i+1},t_{i-1}+1:t_i,\ldots,0:t_1),$$

where $0\le t_1<t_2<\cdots<t_w<h$. Then, taking into account (3.7), we get

$$rev(z_l(\mathbf{l}_q,\mathbf{q}))=(t_w+1:h,\ldots,t_i+2:t_{i+1},\ t_{i-1}+1:t_i+1,\ldots,0:t_1).$$

Note that the consecutions and inversions that occur in $rev(z_l(\mathbf{l}'_q,\mathbf{q}))$ are the same as those that occur in $rev(z_l(\mathbf{l}_q,\mathbf{q}))$, except at t_i, where $rev(z_l(\mathbf{l}'_q,\mathbf{q}))$ has an inversion and $rev(z_l(\mathbf{l}_q,\mathbf{q}))$ has a consecution, and at t_i+1, where $rev(z_l(\mathbf{l}'_q,\mathbf{q}))$ has a consecution and $rev(z_l(\mathbf{l}_q,\mathbf{q}))$ has an inversion. Also, from (3.6), the parameters of the quasi-identity matrices S and S' coincide, except those in positions t_i+1 and t_i+2. Since the (t_i+1)th and the (t_i+2)th parameters of S' are 1 and -1, respectively, we just need to note that $z_l(\mathbf{l}_q,\mathbf{q})$ has an inversion at t_i and a consecution at t_i+1. Then, it follows that $M^T_{k-t_i-1}$ and $S'S_{t_i+2}$ commute, which implies that S has the desired form.

Assume that $s_r=t_i+1=0$. Then, M_0 is invertible and from (3.5), $S=M^T_{-k}S'S_1M^T_kS_k$. Clearly, M^T_{-k} and $S'S_1$ commute. Thus, $S=S'S_1$. Using the notation above for $rev(z_l(\mathbf{l}'_q,\mathbf{q})$, we have

$$rev(z_l(\mathbf{l}_q,\mathbf{q}))=(t_w+1:h,t_{w-1}+1:t_w,\ldots,t_1+1:t_2,1:t_1,0).$$

Similar arguments show that S satisfies (3.4). □

Note that the matrix S in Theorem 3.3 is independent of the matrix polynomial $P(\lambda)$.

Next we consider the particular case when $\mathbf{r}_q=\emptyset$ as an immediate corollary of the previous result.

COROLLARY 3.4. *Let $P(\lambda)$ be a (regular or singular) T-palindromic matrix polynomial of odd degree $k\ge 3$. Let $L(\lambda)=F_{\mathbf{l}_q,\mathbf{q},\emptyset}(\lambda)$ be a reverse-FPR of type 1 associated with*

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$P(\lambda)$. Then, $SRL(\lambda)$ is a T-palindromic linearization of $P(\lambda)$, where S is the quasi-identity matrix defined as follows:

$$S(i,i) = -I \quad \textit{if and only if} \quad \begin{cases} z_l(\mathbf{l}_q, \mathbf{q}) \textit{ has an inversion at } i-1, \\ \mathbf{q} \textit{ has a consecution at } k-i. \end{cases}$$

EXAMPLE 3.5. Assume that $P(\lambda)$ is an $n \times n$ T-palindromic matrix polynomial of degree 5. The T-palindromic linearizations given by Theorem 3.3 can be obtained from Table 3.1.

TABLE 3.1
Example 3.5

$\mathbf{l}_q$	$\mathbf{q}$	$\mathbf{r}_q$	S
$\emptyset$	$(0:2)$	(0)	S_4
$\emptyset$	$(0:2)$	$(0:1)$	S_5
$\emptyset$	$(0:2)$	$(0:1,0)$	I_5
(0)	$(2,1,0)$	$\emptyset$	S_2
$(1,0)$	$(2,1,0)$	$\emptyset$	S_1
$(0:1,0)$	$(2,1,0)$	$\emptyset$	I_5
$\emptyset$	$(1:2,0)$	(1)	S_1S_5
$\emptyset$	$(1:2,0)$	$(1,0)$	S_1
(0)	$(1:2,0)$	$\emptyset$	S_4
$\emptyset$	$(2,0:1)$	(0)	S_2
(1)	$(2,0:1)$	$\emptyset$	S_1S_5
$(0:1)$	$(2,0:1)$	$\emptyset$	S_5
(0)	$(1:2,0)$	(1)	S_5
(0)	$(1:2,0)$	$(1,0)$	I_5
(1)	$(2,0:1)$	(0)	S_1
$(0:1)$	$(2,0:1)$	(0)	I_5

The next result is a corollary of Theorem 3.3.

COROLLARY 3.6. *Let $P(\lambda)$ be a (regular or singular) T-palindromic matrix polynomial of odd degree $k \geq 3$. Let $L(\lambda) = F_{\mathbf{l}_q,\mathbf{q},\mathbf{r}_q}(\lambda)$ be a reverse-FPR of type 1 associated with $P(\lambda)$. Then, $L(\lambda)RS$ is a T-palindromic linearization of $P(\lambda)$, where S is the quasi-identity given by*

$$S(i,i) = -I \quad \textit{if and only if} \quad \begin{cases} z_l(\mathbf{l}_q, \mathbf{q}) \textit{ has an inversion at } k-i, \\ z_r(\mathbf{q}, \mathbf{r}_q) \textit{ has a consecution at } i-1. \end{cases} \tag{3.8}$$

Proof. Notice that $P(\lambda)^T$ is T-palindromic. Also $\mathbf{r}_q$ (resp., $\mathbf{l}_q$) is a right (resp., left) index tuple of type 1 relative to $\mathbf{q}$ if and only if $rev(\mathbf{r}_q)$ (resp., $rev(\mathbf{l}_q)$) is a left (resp.,

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A publication of the International Linear Algebra Society
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right) index tuple of type 1 relative to $rev(\mathbf{q})$. In particular, this implies that $L(\lambda)^T$ is a reverse-FPR of type 1 which is a strong linearization of $P(\lambda)^T$. Therefore, by Theorem 3.3, $SRL(\lambda)^T = (L(\lambda)RS)^T$ is a T-palindromic strong linearization of $P(\lambda)^T$, where S is the quasi-identity matrix defined as follows:

$$S(i,i) = -I \quad \text{if and only if} \quad \begin{cases} z_l(rev(\mathbf{r}_q), rev(\mathbf{q})) \text{ has an inversion at } i-1, \\ z_r(rev(\mathbf{q}), rev(\mathbf{l}_q)) \text{ has a consecution at } k-i. \end{cases}$$

Since $(L(\lambda)RS)^T$ is a T-palindromic strong linearization of $P(\lambda)^T$, we deduce that $L(\lambda)RS$ is a T-palindromic strong linearization of $P(\lambda)$.

It can be easily seen that S is the matrix defined in (3.8). $\square$

The linearizations produced by the previous corollary are not the same as those produced by Theorem 3.3, as the following example shows. Moreover, note that, in this example, it is not possible to obtain one linearization from the other by permuting rows and columns.

EXAMPLE 3.7. Let $P(\lambda)$ be a T-palindromic matrix polynomial of degree $k = 7$. Let $\mathbf{l}_q = (2, 0)$, $\mathbf{q} = (3, 1 : 2, 0)$, $\mathbf{r}_q = (1)$ and consider the reverse-FPR of type 1 $F_{\mathbf{l}_q,\mathbf{q},\mathbf{r}_q}(\lambda)$.

Then, the T-palindromic linearization of $P(\lambda)$ produced by Theorem 3.3 is given by

$$\lambda \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -A_0 & 0 \\ 0 & 0 & 0 & 0 & -I & 0 & 0 \\ 0 & 0 & 0 & 0 & -A_2 & -A_1 & I \\ 0 & I & A_2^T & A_3^T & 0 & 0 & 0 \\ A_0^T & 0 & A_1^T & A_2^T & 0 & 0 & 0 \\ 0 & 0 & A_0^T & A_1^T & 0 & 0 & 0 \\ 0 & 0 & 0 & -I & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & -A_0 & 0 & 0 \\ 0 & 0 & 0 & -I & 0 & 0 & 0 \\ 0 & 0 & 0 & -A_2 & -A_1 & -A_0 & 0 \\ 0 & 0 & 0 & -A_3 & -A_2 & -A_1 & I \\ 0 & I & A_2^T & 0 & 0 & 0 & 0 \\ A_0^T & 0 & A_1^T & 0 & 0 & 0 & 0 \\ 0 & 0 & -I & 0 & 0 & 0 & 0 \end{bmatrix},$$

while the T-palindromic linearization of $P(\lambda)$ produced by Corollary 3.6 is given by

$$\lambda \begin{bmatrix} 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & A_1^T & A_0^T & 0 & 0 \\ 0 & 0 & 0 & A_2^T & A_1^T & 0 & A_0^T \\ 0 & 0 & 0 & A_3^T & A_2^T & -I & 0 \\ -I & -A_1 & -A_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & -A_0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & A_1^T & 0 & A_0^T \\ 0 & 0 & 0 & 0 & A_2^T & -I & 0 \\ -I & -A_1 & -A_2 & -A_3 & 0 & 0 & 0 \\ 0 & -A_0 & -A_1 & -A_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & -A_0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The reader may wonder if there could be any other T-palindromic linearizations from the FPR if we relax the condition of being of type 1. The next example shows that, in general, the construction we have used for the new family of T-palindromic linearizations does not produce T-palindromic pencils when $\mathbf{r}_q$ or $\mathbf{l}_q$ are not of type 1.

EXAMPLE 3.8. Let $P(\lambda)$ be a T-palindromic matrix polynomial of degree $k = 5$. Let $\mathbf{q} = (0 : 2)$, $\mathbf{l}_q = \emptyset$, and $\mathbf{r}_q = (1)$. Note that $\mathbf{r}_q$ is not a right index tuple of type 1 relative to

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A publication of the International Linear Algebra Society
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$\mathbf{q}$. Consider the pencil

$$L(\lambda) = \lambda M_{-4}M_{-3:-5}M_1 - M_{-4}M_{0:2}M_1.$$

A calculation shows that

$$RL(\lambda) = \lambda \begin{bmatrix} 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & A_1 & I \\ A_0^T & A_1^T & A_2^T & 0 & 0 \\ 0 & I & A_1^T & 0 & 0 \\ 0 & 0 & I & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & A_0 & 0 & 0 \\ 0 & 0 & A_1 & -I & 0 \\ 0 & 0 & A_2 & A_1 & -I \\ -I & -A_1^T & 0 & 0 & 0 \\ 0 & -I & 0 & 0 & 0 \end{bmatrix}.$$

It is easy to see that there is no quasi-identity matrix S such that SRL is T-palindromic, which happens because of the occurrence of the 2-by-2 blocks

$$\begin{bmatrix} I & 0 \\ A_1 & I \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -I & -A_1^T \\ 0 & -I \end{bmatrix}$$

in each of the coefficient matrices of $RL(\lambda)$.

4. T-Anti-palindromic linearizations from reverse-FPR of type 1. A polynomial $P(\lambda)$ of the form (1.1) is said to be *T-anti-palindromic* if $A_i = -A_{k-i}^T$, for $i = 1, \ldots, k$. In this section, we construct T-anti-palindromic linearizations for T-anti-palindromic matrix polynomials from reverse-FPR of type 1. For that purpose we consider the following result proven in [6].

LEMMA 4.1. [6] *Let $P(\lambda)$ be any T-anti-palindromic $n \times n$ matrix polynomial with odd degree. Define $Q(\lambda) := P(-\lambda)$. Then, $Q(\lambda)$ is T-palindromic. Moreover, if $\tilde{L}$ is any strong T-palindromic linearization of $Q(\lambda)$, then $\tilde{L}(\lambda) := L(-\lambda)$ is a strong T-anti-palindromic linearization of $P(\lambda)$.*

The next result is a corollary of Theorem 3.3 and Lemma 4.1.

COROLLARY 4.2. *Let $P(\lambda)$ be a (regular or singular) T-anti-palindromic matrix polynomial of odd degree $k \geq 3$. Let $L(\lambda) = F_{\mathbf{l}_q,\mathbf{q},\mathbf{r}_q}(\lambda)$ be a reverse-FPR of type 1 associated with $P(-\lambda)$. Then, $SRL(-\lambda)$ is a T-anti-palindromic strong linearization of $P(\lambda)$, where S is the quasi-identity given by*

$$S(i,i) = -I \quad \text{if and only if} \quad \begin{cases} z_l(\mathbf{l}_q, \mathbf{q}) \text{ has an inversion at } i-1, \\ z_r(\mathbf{q}, \mathbf{r}_q) \text{ has a consecution at } k-i. \end{cases}$$

5. Conclusions. In this paper, we characterize a new family of strong linearizations for matrix polynomials of odd degree $k \geq 3$, which are T-palindromic when the polynomial is. These linearizations are obtained from the Fiedler pencils with repetition [17], in which the tuples with allowed repetitions are of type 1 (in [3] this subfamily of the FPR is defined and is

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 562-577, June 2012

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called FPR of type 1). Our construction extends the T-palindromic linearizations constructed in [6]. It is an open question if there exist T-palindromic companion forms strictly equivalent to a FPR outside the type 1 subfamily. Also, when a T-palindromic matrix polynomial with even degree has a T-palindromic linearization, it is not known if such a linearization exists among the T-palindromic companion forms strictly equivalent to a FPR.

Acknowledgment. We thank the two anonymous referees for their thoughtful suggestions and comments that helped improve the presentation of this paper.

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