

ON EP ELEMENTS, NORMAL ELEMENTS AND PARTIAL ISOMETRIES IN RINGS WITH INVOLUTION*

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Abstract. This is a continuation to the study of EP elements, normal elements and partial isometries in rings with involution. The aim of this paper is to give the negative solution to three conjectures on this subject. Moreover, some new characterizations of EP elements in rings with involution are presented.

Key words. EP elements, Moore-Penrose inverses, Group inverses, Partial isometries, Rings with involution.

AMS subject classifications. 16W10, 16U99.

1. Introduction. EP matrices and normal matrices, as well as EP linear operators and normal linear operators on Banach or Hilbert spaces have been investigated by many authors (see, for example, [1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 13, 15, 17, 18, 20]). D. Mosić et al. in [21, 24] use the setting of rings with involution to investigate EP elements and normal elements, giving new characterizations to them and providing simpler and more transparent proofs to already existing ones. D. Mosić and D.S. Djordjević also characterize partial isometries in terms of the pure theory of rings, generalizing known results for complex matrices (see [22, 23]).

Let R be an associative ring with the unit 1, and let $a \in R$. Then a is group invertible if there is $a^\# \in R$ such that $aa^\#a = a$, $a^\#aa^\# = a^\#$ and $aa^\# = a^\#a$. The element $a^\#$ is called the group inverse of a and it is uniquely determined by previous equations. If a is invertible, then $a^\#$ coincides with the ordinary inverse a^{-1} of a . It is proved in [21, p. 733] that the group inverse $a^\#$ double commutes with a , that is, $ax = xa$ implies $a^\#x = xa^\#$, where $x \in R$. It turns out that $ax = xa$ if and only if $a^\#x = xa^\#$ since $(a^\#)^\# = a$. Moreover, because $aa^\#$ is an idempotent and $aa^\# = a^\#a$, $a^n(a^\#)^n = aa^\#$, $a^{n+1}(a^\#)^n = a$, and $(a^\#)^{n+1}a^n = a^\#$ for any positive integer $n \geq 1$. We use the symbol $R^\#$ to denote the set of all group invertible elements in R .

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An involution $*$ in a ring R is an anti-isomorphism of degree 2, that is, $(a^*)^* = a$, $(a + b)^* = a^* + b^*$, and $(ab)^* = b^*a^*$ for all $a, b \in R$. An element $a \in R$ satisfying $a^* = a$ is called symmetric, and a is called normal if $aa^* = a^*a$.

We say that $b = a^\dagger$ is the Moore-Penrose inverse (or MP-inverse) of a , if the following conditions hold: $aba = a$, $bab = b$, $(ab)^* = ab$, and $(ba)^* = ba$. There is at most one b such that the above conditions hold (see [6, 12, 14, 16]). If a is invertible, then $a^\dagger$ coincides with the ordinary inverse a^{-1} of a . The set of all Moore-Penrose inverse elements of R will be denoted by $R^\dagger$. An element $a \in R^\dagger$ satisfying $a^* = a^\dagger$ is called a partial isometry.

An element a of a ring R with involution is said to be EP if $a \in R^\# \cap R^\dagger$ and $a^\# = a^\dagger$ [19]. It is easy to see that an element $a \in R$ is an EP element if and only if $a \in R^\dagger$ and $aa^\dagger = a^\dagger a$ (see [24, Lemma 1.1]). It is known by [19, Theorem 7.3] that $a \in R$ is EP if and only if a is group invertible and $aa^\#$ is symmetric. It should be noted that the Moore-Penrose inverse and the group inverse are useful in solving overdetermined systems of linear equations, and the importance of EP elements lies in the fact that they are characterized by the commutativity with their Moore-Penrose inverses. There are close connections among EP elements, partial isometries and normal elements in rings with involution (see [22, 23]).

The following result is well known for complex matrices and for linear operators on Hilbert spaces (see [1, 5, 8, 9]). But it is unknown for rings with involution. So the authors of [24] stated it as a conjecture.

CONJECTURE 1.1. [24, Conjecture 1] *Let R be a ring with involution. Then $a \in R$ is an EP element if and only if $a \in R^\# \cap R^\dagger$ and one of the following equivalent conditions hold:*

- (i) $(a^\dagger)^2 a^\# = a^\# (a^\dagger)^2$;
- (ii) $aa^\dagger = a^2 (a^\dagger)^2$;
- (iii) $a^\dagger a = (a^\dagger)^2 a^2$.

It is known that the next result holds for linear bounded operators on Hilbert spaces. However it is not known for rings with involution.

CONJECTURE 1.2. [21, Conjecture 2.1] *Let R be a ring with involution and $a \in R^\dagger$. Then a is normal if and only if one of the following conditions hold:*

- (i) $a(a^* + a^\dagger) = (a^* + a^\dagger)a$;
- (ii) $a \in R^\#$ and $a^*a(aa^*)^\dagger a^*a = aa^*$;
- (iii) $a \in R^\#$ and $aa^*(a^*a)^\dagger aa^* = a^*a$;
- (iv) there exists some $x \in R$ such that $aa^*x = a^*a$ and $a^*ax = aa^*$;
- (v) $aa^\dagger a^*aaa^\dagger = aa^*$.

Although the following result is well known for complex matrices, it is unknown for rings with involution.

CONJECTURE 1.3. [22, p. 9, Conjecture] *Let R be a ring with involution and $a \in R^\dagger$. Then a is a partial isometry if and only if one of the following equivalent conditions holds: (i) $a^*aa^* = a^\dagger$; (ii) $aa^*aa^*a = a$.*

The main purpose of this note is to give the negative solution to the above conjectures and present some new characterizations of EP elements in rings with involution for the further study.

For a ring R , we use the symbol $M_n(R)$ to denote the $n \times n$ matrix ring over R . The ring of integers modulo a positive integer n is denoted by $\mathbb{Z}_n$.

2. On the EP elements conjecture. We start this section with the following counterexample.

EXAMPLE 2.1. There exists a ring R with involution such that R contains an element $a \in R^\# \cap R^\dagger$ satisfying the conditions (ii) and (iii) in Conjecture 1.1, but a is not an EP element.

Proof. Let S be any commutative ring with characteristic 2, for example, $S = \mathbb{Z}_2$. Then $R = M_3(S)$ is a ring with involution $*$, the transposition of a matrix in R . Take

$$a = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then both a and b are idempotents in R . Hence, a is group invertible with $a^\# = a$. It is easy to check that

$$ab = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad ba = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Hence, both ab and ba are symmetric. Moreover, $aba = a$ and $bab = b$ hold. This implies that a is Moore-Penrose invertible with $a^\dagger = a^* = b$. It is easy to see that $aa^\dagger = a^2(a^\dagger)^2$ since a and $a^\dagger$ are idempotents. But $a^\# \neq a^\dagger$, and so a is not an EP element. Clearly, the condition $a^\dagger a = (a^\dagger)^2 a^2$ also holds. Note that in this example, $a^\dagger = b \in R^\#$. $\square$

More generally, one may choose S to be a commutative ring with characteristic p , where p is any prime. Let $R = M_{p+1}(S)$, then R is a ring with involution $*$, the transposition of a matrix. Take $a = e_{11} + e_{12} + \cdots + e_{1,p+1}$, where e_{1i} are the matrix units, and $b = a^*$. Similar to the case of $p = 2$, one would get more counterexamples for (ii) and (iii) in Conjecture 1.1.

Next we show that the conditions (ii) and (iii) in Conjecture 1.1 are in fact equivalent.

PROPOSITION 2.2. *Let R be a ring with involution. Then the following conditions are equivalent for an element $a \in R^\# \cap R^\dagger$:*

- (ii) $aa^\dagger = a^2(a^\dagger)^2$;
- (iii) $a^\dagger a = (a^\dagger)^2 a^2$.

Proof. Assume that (ii) holds. Then $aa^\dagger = a^2(a^\dagger)^2$. Multiplying $a^\#$ from left sides of this equation, $a^\#aa^\dagger = a(a^\dagger)^2$. Multiplying $a^\dagger$ from left sides of this equation, $a^\dagger a^\#aa^\dagger = (a^\dagger)^2$. Multiplying a from right sides of this equation, $a^\dagger a^\#a = (a^\dagger)^2 a$. It follows that $a^\dagger a = (a^\dagger)^2 a^2$ by multiplying a from right sides of the last equation. Hence, (ii) implies (iii). Similar to the above argument, we may get that (iii) implies (ii). The proof is complete. $\square$

PROPOSITION 2.3. *Let R be a ring with involution. Then $a \in R$ is EP if and only if $a \in R^\# \cap R^\dagger$ and satisfies the following two conditions:*

- (i) $(a^\dagger)^2 a^\# = a^\#(a^\dagger)^2$;
- (ii) $aa^\dagger = a^2(a^\dagger)^2$.

Proof. Assume that (i) and (ii) hold. Then $a(a^\dagger)^2 = a^\#a^2(a^\dagger)^2 = a^2a^\#(a^\dagger)^2 = a^2(a^\dagger)^2a^\# = aa^\dagger a^\# = aa^\dagger a(a^\#)^2 = a(a^\#)^2 = a^\#$. It yields that $a^2(a^\dagger)^2 = aa^\#$ and so $a^\#a = aa^\dagger$. This means that a is an EP element by [19, Theorem 7.3]. The necessity is obvious. $\square$

If a is an EP element, then the conditions (i), (ii) and (iii) in Conjecture 1.1 are clearly satisfied. Hence, Example 2.1, Proposition 2.2 and Proposition 2.3 imply that the conditions (i), (ii) and (iii) in Conjecture 1.1 are not equivalent.

THEOREM 2.4. *Let R be a ring with involution. Then $a \in R$ is EP if and only if $a \in R^\# \cap R^\dagger$ and satisfies one of the following conditions:*

- (i) $a^n a^\dagger = a^\dagger a^n$, for some $n \geq 1$;
- (ii) $(a^\#)^n a^\dagger = a^\dagger (a^\#)^n$, for some $n \geq 1$;
- (iii) $(a^\dagger)^n = (a^\#)^n$, for some $n \geq 1$.

Proof. If a is EP, then it commutes with its Moore-Penrose inverse and $a^\# = a^\dagger$. It is easy to verify that the conditions (i) – (iii) hold.

Conversely, we assume that $a \in R^\# \cap R^\dagger$. To conclude a is EP, we show that $aa^\dagger = a^\dagger a$ or $aa^\# = aa^\dagger$ holds under each assumption.

- (i) Assume that $a^n a^\dagger = a^\dagger a^n$. In the case of $n = 1$, then $aa^\dagger = a^\dagger a$. This means

$a^\dagger = a^\#$ by [24, Lemma 1.1], or by the uniqueness of $a^\#$. If $n > 1$, then we have

$$(a^\#)^{n-1} a^n a^\dagger = (a^\#)^{n-1} a^\dagger a^n,$$

that is,

$$aa^\dagger = (a^\#)^{n-1} a^\dagger a^n.$$

Multiplying $(a^\#)^{n-1}$ from right sides of this equation,

$$aa^\dagger (a^\#)^{n-1} = (a^\#)^{n-1} a^\dagger a^n (a^\#)^{n-1} = (a^\#)^{n-1} a^\dagger a.$$

From this equality, we get

$$aa^\dagger (a^\#)^{n-1} a^{n-1} = (a^\#)^{n-1} a^\dagger aa^{n-1} = (a^\#)^{n-1} a^\dagger a^n = (a^\#)^{n-1} a^n a^\dagger = aa^\dagger.$$

On the other hand,

$$aa^\dagger (a^\#)^{n-1} a^{n-1} = aa^\dagger aa^\# = aa^\#.$$

It follows that $aa^\# = aa^\dagger$.

(ii) Using the assumption $(a^\#)^n a^\dagger = a^\dagger (a^\#)^n$, we have

$$aa^\dagger = a^{n+1} (a^\#)^n a^\dagger = a^{n+1} a^\dagger (a^\#)^n.$$

This means that

$$aa^\dagger a^{n+1} = a^{n+1} a^\dagger (a^\#)^n a^{n+1} = a^{n+1} a^\dagger a.$$

Multiplying $(a^\#)^{n+1}$ from right sides of this equality, we get

$$aa^\dagger aa^\# = a^{n+1} a^\dagger (a^\#)^n = a^{n+1} (a^\#)^n a^\dagger = aa^\dagger.$$

Since $aa^\dagger aa^\# = aa^\#$, it yields that $aa^\# = aa^\dagger$.

(iii) Assume that $(a^\dagger)^n = (a^\#)^n$. Then

$$aa^\# = a^n (a^\#)^n = a^n (a^\dagger)^n = a^n (a^\dagger)^n aa^\dagger = a^n (a^\#)^n aa^\dagger = aa^\# aa^\dagger = aa^\dagger. \quad \square$$

COROLLARY 2.5. *Let R be a ring with involution. Then $a \in R$ is EP if and only if $a \in R^\# \cap R^\dagger$ and satisfies the following two conditions:*

- (i) $a^\dagger \in R^\#$;
- (ii) $(a^\dagger)^2 a^\# = a^\# (a^\dagger)^2$.

Proof. Since the group inverse $a^\#$ double commutes with a , the condition (ii) is equivalent to $(a^\dagger)^2 a = a^2 (a^\dagger)^2$. Let $b = a^\dagger$. Then $b \in R^\# \cap R^\dagger$, and $b^2 b^\dagger = b^\dagger b^2$. This means that $b^\dagger = b^\#$ by Theorem 2.4 (i). Hence, $a = (a^\dagger)^\#$, and so $a^\dagger = a^\#$. $\square$

We point that the condition (ii) in the above corollary is not superfluous by Example 2.1, but we do not know whether the condition (i) in Corollary 2.5 is superfluous or not.

3. On the normal elements and isometries conjectures. In this section, we present several counterexamples for Conjecture 1.2 and Conjecture 1.3.

EXAMPLE 3.1. There exists an involution ring R in which the condition (i) $a(a^* + a^\dagger) = (a^* + a^\dagger)a$ in Conjecture 1.2 does not imply that a is normal.

Proof. Let S be any commutative ring with characteristic 2, and let $R = M_2(S)$. Then R is a ring with involution $*$, the transposition of a matrix. Take

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then it is easy to check that

$$ab = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad ba = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad aba = a, \quad \text{and} \quad bab = b.$$

Since both ab and ba are symmetric, $b = a^\dagger = a^*$. Hence, $a^* + a^\dagger = 0$ in R , and so $a(a^* + a^\dagger) = (a^* + a^\dagger)a$ holds. But a is not normal since $aa^* = ab \neq ba = a^*a$. $\square$

More counterexamples may be obtained if one substitutes $R = M_2(S)$ for $R = M_n(S)$ ($n > 2$), and takes $a = e_{1n}$, $b = e_{n1}$ in $R = M_n(S)$ where e_{1n} and e_{n1} are the matrix units.

EXAMPLE 3.2. There exists an involution ring R in which the conditions (ii), (iii) and (iv) in Conjecture 1.2 hold, but a is not a normal element.

Proof. Let $R = M_2(\mathbb{Z}_8)$ and $*$ be the transposition of a matrix. Then R is a ring with involution $*$. Take $a = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, then $a^* = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. It is easy to check that

$$aa^* = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad a^*a = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}.$$

Hence,

$$\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 6 & 1 \end{pmatrix} \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix},$$

that is, $aa^*(a^*a)^\dagger aa^* = a^*a$. Similarly,

$$\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} 1 & 6 \\ 6 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}.$$

This means that $a^*a(aa^*)^\dagger a^*a = aa^*$ holds.

Finally, let $x = (aa^*)^{-1}a^*a$. Then $aa^*x = a^*a$ holds. On the other hand,

$$\begin{aligned} a^*ax &= \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} 1 & 6 \\ 6 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} = aa^*. \end{aligned}$$

But a is clearly not normal. $\square$

EXAMPLE 3.3. There exists an involution ring R in which the condition (v) $aa^\dagger a^*aaa^\dagger = aa^*$ in Conjecture 1.2 does not imply that a is normal.

Proof. We use Example 2.1. Let S be any commutative ring with characteristic 2, for example, $S = \mathbb{Z}_2$. Then $R = M_3(S)$ is a ring with involution $*$, the transposition of a matrix in R . Take

$$a = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then both a and b are idempotents in R . It has been checked in Example 2.1 that

$$ab = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad ba = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Hence, both ab and ba are symmetric. Moreover, $aba = a$ and $bab = b$ hold. This implies that a is Moore-Penrose invertible with $a^\dagger = a^* = b$. We claim that the condition $aa^\dagger a^*aaa^\dagger = aa^*$ holds. In fact,

$$aa^\dagger a^*aaa^\dagger = abbaab = abaab = a^2b = ab = aa^*. \quad \square$$

More generally, one may choose S to be a commutative ring with characteristic p where p is any prime. Let $R = M_{p+1}(S)$, then R is a ring with involution $*$, the transposition of a matrix. Take $a = e_{11} + e_{12} + \cdots + e_{1,p+1}$, where e_{1i} are the matrix units, and $b = a^*$. Similar to the above argument, it is easy to show that a is not normal although it satisfies the condition $aa^\dagger a^*aaa^\dagger = aa^*$. We conclude this note with the following counterexample.

EXAMPLE 3.4. There exists an involution ring R in which the equivalent conditions (i) $a^*aa^* = a^\dagger$ and (ii) $aa^*aa^*a = a$ in Conjecture 1.3 do not imply that a is a partial isometry.

Proof. Let S be a ring with characteristic 5, for example, $S = \mathbb{Z}_5$. Consider

$R = M_2(S)$ with the transposition involution $*$. Take $a = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Then

$$a^2 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}, \quad a^3 = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}, \quad a^4 = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}, \quad \text{and} \quad a^5 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = a.$$

This means that $aa^3a = a$, and $a^3aa^3 = a^5a^2 = a^3$, and so a is group invertible with $a^\# = a^3$. Since aa^3 is a symmetric element, a is Moore-Penrose invertible with $a^\dagger = a^3$. On the other hand, since a is symmetric, $a = a^*$. It follows that $a^\dagger \neq a^*$. But the conditions (i) $a^*aa^* = a^\dagger$; and (ii) $aa^*aa^*a = a$ in Conjecture 1.3 are obviously satisfied. Next we show that the conditions (i) and (ii) in Conjecture 1.3 are equivalent. It well known that $a^* = a^*aa^\dagger = a^\dagger aa^*$ (see [21]). Multiplying a from two sides of the equation (i), we get the equation(ii). Multiplying $a^\dagger$ from two sides of equation(ii), we obtain the equation (i). Also note that if a is a partial isometry, then the two conditions are clearly satisfied. $\square$

For the simplest counterexample to Conjecture 1.3, one may consider the commutative ring $R = \mathbb{Z}_5$ for the trivial involution $x^* = x$. Take $a = 2$. Then $a^4 = 1$. This implies that $a^\dagger = a^{-1} = a^3$. Note that $a^* = a \neq a^\dagger$. So a is not an isometry, but it satisfies the two conditions in Conjecture 1.3. This means that even for a finite field R and $a \in R$ is invertible, Conjecture 1.3 may not hold yet.

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REFERENCES

- [1] O.M. Baksalary and G. Trenkler. Characterizations of EP, normal, and Hermitian matrices. *Linear Multilinear Algebra*, 56:299–304, 2008.
- [2] A. Ben-Israel and T.N.E. Greville. *Generalized Inverses: Theory and Applications*, second edition. Springer, New York, 2003.
- [3] E. Boasso. On the Moore-Penrose inverse, EP Banach space operators, and EP Banach algebra elements. *J. Math. Anal. Appl.*, 339:1003–1014, 2008.
- [4] S.L. Campbell and C.D. Meyer Jr. EP operators and generalized inverses. *Canad. Math. Bull.*, 18:327–333, 1975.
- [5] S. Cheng and Y. Tian. Two sets of new characterizations for normal and EP matrices. *Linear Algebra Appl.*, 375:181–195, 2003.
- [6] D. Cvetković, D.S. Djordjević, and J.J. Koliha. Moore-Penrose inverse in rings with involution. *Linear Algebra Appl.*, 426:371–381, 2007.
- [7] D.S. Djordjević. Products of EP operators on Hilbert spaces. *Proc. Amer. Math. Soc.*, 129(6):1727–1731, 2000.
- [8] D.S. Djordjević. Characterization of normal, hyponormal and EP operators. *J. Math. Anal. Appl.*, 329(2):1181–1190, 2007.
- [9] D.S. Djordjević and J.J. Koliha. Characterizing Hermitian, normal and EP operators. *Filomat*, 21(1):39–54, 2007.

- [10] D.S. Djordjević, J.J. Koliha, and I. Strakšraba. Factorization of EP elements in C^* -algebras. *Linear Multilinear Algebra*, 57(6):587–594, 2009.
- [11] D.S. Djordjević and V. Rakočević. Lectures on generalized inverses, Faculty of Sciences and Mathematics, University of Niš, 2008.
- [12] M.P. Drazin. Pseudo-inverses in associative rings and semigroups. *Amer. Math. Monthly*, 65:506–514, 1958.
- [13] D. Drivaliaris, S. Karanasios and D. Pappas. Factorizations of EP operators. *Linear Algebra Appl.* 429(7):1555–1567, 2008.
- [14] R.E. Harte and M. Mbekhta. On generalized inverses in C^* -algebras. *Studia Math.*, 103:71–77, 1992.
- [15] R.E. Hartwig and I.J. Katz. On products of EP matrices. *Linear Algebra Appl.*, 252:339–345, 1997.
- [16] J.J. Koliha. The Drazin and Moore-Penrose inverse in C^* -algebras. *Math. Proc. R. Ir. Acad.*, 99A:17–27, 1999.
- [17] J.J. Koliha. A simple proof of the product theorem for EP matrices. *Linear Algebra Appl.*, 294:213–215, 1999.
- [18] J.J. Koliha. Elements of C^* -algebras commuting with their Moore-Penrose inverse. *Studia Math.*, 139:81–90, 2000.
- [19] J.J. Koliha and P. Patrício. Elements of rings with equal spectral idempotents. *J. Aust. Math. Soc.*, 72:137–152, 2002.
- [20] G. Lešnjak. Semigroups of EP linear transformations. *Linear Algebra Appl.*, 304:109–118, 2000.
- [21] D. Mosić and D.S. Djordjević. Moore-Penrose-invertible normal and Hermitian elements in rings. *Linear Algebra Appl.*, 431: 732–745, 2009.
- [22] D. Mosić and D.S. Djordjević. Partial isometries and EP elements in rings with involution. *Electron. J. Linear Algebra*, 18:761–772, 2009.
- [23] D. Mosić and D.S. Djordjević. Further results on partial isometries and EP elements in rings with involution. *Math. Comput. Modelling*, 54:460–465, 2011.
- [24] D. Mosić, D.S. Djordjević, and J.J. Koliha EP elements in rings. *Linear Algebra Appl.*, 431:527–535, 2009.