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BINARY RANKS AND BINARY FACTORIZATIONS OF NONNEGATIVE INTEGER MATRICES*

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Abstract. A matrix is binary if each of its entries is either 0 or 1. The binary rank of a nonnegative integer matrix A is the smallest integer b such that $A = BC$, where B and C are binary matrices, and B has b columns. In this paper, bounds for the binary rank are given, and nonnegative integer matrices that attain the lower bound are characterized. Moreover, binary ranks of nonnegative integer matrices with low ranks are determined, and binary ranks of nonnegative integer Jacobi matrices are estimated.

Key words. Nonnegative integer matrix, Rank, Binary rank, Binary factorization.

AMS subject classifications. 15A23, 15A36.

1. Introduction. A matrix is *binary* if each of its entries is either 0 or 1. Let $\mathscr{B}^{m\times n}$ be the set of all $m \times n$ binary matrices. We denote by $\mathbb{Z}_+^{m\times n}$ the set of all $m\times n$ nonnegative integer matrices. For $A \in \mathbb{Z}_+^{m\times n}$, if there exist two matrices $B \in \mathscr{B}^{m\times k}$ and $C \in \mathscr{B}^{k\times n}$ such that $A = BC$, then we say that $A = BC$ is a *binary factorization* of A, where B and C are called a left and a right binary factor of A, respectively. The smallest k that makes the factorization possible is called the *binary rank* of A and denoted by $b(A)$. It is easily seen from the definition of $b(A)$ that $\operatorname{rank}(A) \le b(A)$.

The problem of obtaining such decompositions of nonnegative integer matrices arises frequently in a variety of scientific contexts such as symmetric designs, combinatorial optimization, probability and statistics. De Caen [4] made several possible interpretations of binary factorizations of nonnegative integer matrices, which touched on symmetric designs, bipartite graphs, directed graphs, and combinatorial designs. For example, a binary factorization of a nonnegative integer matrix A corresponds to a partition of the edge-set of $G(A)$ into bicliques. Pullman and Stanford [8] studied the biclique partitions of a special class of graphs, the regular bipartite graphs.

The following two problems were posed by de Caen [4] (see also [6]):

Problem 1.1. Given a nonnegative integer matrix A, determine $b(A)$.

Problem 1.2. Find some restrictions on the structure of factors B and C,

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especially in the extreme case where the number of columns of B is equal to $b(A)$.

In a special symmetric version of Problem 1.1, Berman and Xu [3] defined a matrix to be $\{0,1\}$ completely positive ($\{0,1\}$-cp, for short) matrix if $A \in \mathbb{Z}_+^{n\times n}$ can be expressed as $A = BB^T$ with $B \in \mathscr{B}^{n\times k}$. They call the smallest possible number of columns of B in such a factorization the $\{0,1\}$*-rank* of A.

In this paper, we mainly deal with Problem 1.1. Determining the exact binary rank and computing the corresponding factorization of a given nonnegative integer matrix, however, are known to be NP-hard (see [7]). The aim of this paper is to give bounds for binary rank, characterize the nonnegative integer matrices that attain the lower bound, establish some necessary conditions for a nonnegative integer matrix to achieve the upper bound and compute the binary ranks of some special classes of nonnegative integer matrices.

Let $A \in \mathbb{Z}_+^{m\times n}$. We refer to the *binary column rank* of A, denoted by $bc(A)$, as the smallest nonnegative integer q for which there exist vectors $v_1, v_2, \ldots, v_q$ in $\mathscr{B}^{n\times 1}$ such that each column of A can be represented as a linear combination of $v_1, v_2, \ldots, v_q$ with coefficients from the set $\{0,1\}$. The *binary row rank* of A, denoted by $br(A)$, is defined as the binary column rank of A^T, the transpose of A.

The following result gives equivalent characterizations of binary rank; its proof is almost the same as that of [5] and we omit it.

Lemma 1.3. *Let $A \in \mathbb{Z}_+^{m\times n}$ and let q be a nonnegative integer. Then the following statements are equivalent:*
(i) $q = bc(A)$;
(ii) $q = br(A)$;
(iii) $q = b(A)$;
(iv) q is the smallest integer for which there exist vectors $b_1, b_2, \ldots, b_q$ and $c_1, c_2, \ldots, c_q$ in $\mathscr{B}^{n\times 1}$ such that $A = \sum_{i=1}^{q} b_i c_i^T$.

We refer to the representation of A in (iv) as a *binary rank-one decomposition* of A.

2. Bounds for the binary ranks of nonnegative integer matrices. In this section, we study upper and lower bounds for the binary ranks of nonnegative integer matrices. For $A = (a_{ij}) \in \mathbb{Z}_+^{m\times n}$, denote $\|A\|_\infty = \max_{i,j} a_{ij}$, $\|A\|_r = \sum_i \max_j a_{ij}$ and $\|A\|_c = \sum_j \max_i a_{ij}$. It is clear that $\|A\|_r = \|A^T\|_c$. Let $j_n \in \mathscr{B}^{n\times 1}$ denote the vector of all 1's.

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LEMMA 2.1. *Let* $A \in \mathbb{Z}_+^{m\times n}$. *Then,* $\|A\|_\infty \le b(A) \le \min\{\|A\|_r, \|A\|_c\}$.

Proof. The first inequality is clear. To prove the second, we will show that any nonnegative integer matrix A can be expressed as a product $A = BC$ for some binary matrices B and C, where B has $\min\{\|A\|_r, \|A\|_c\}$ columns. For $A = (a_{ij}) \in \mathbb{Z}_+^{m\times n}$, without loss of generality, assume $\|A\|_r \le \|A\|_c$. Let r_i be the largest component of the i-th row of A, $i = 1,\dots,m$. Let $B = \oplus_{i=1}^m j_{r_i}^T$ and let $C = [C_1^T, C_2^T, \dots, C_m^T]^T$, where $C_i \in \mathscr{B}^{r_i\times n}$ and the j-th column of C_i has exactly a_{ij} 1's, $j = 1,\dots,n$. Now, it follows that $A = BC$ and B has $\|A\|_r$ columns. Thus, $b(A) \le \|A\|_r$. ∎

COROLLARY 2.2. *Let* $A \in \mathscr{B}^{m\times n}$. *If* $\operatorname{rank}(A) = \min\{m,n\}$, *then* $b(A) = \operatorname{rank}(A)$.

Proof. Since A is binary, by Lemma 2.1, $b(A) \le \min\{\|A\|_r, \|A\|_c\} \le \min\{m,n\}$. On the other hand, it is obvious that $\operatorname{rank}(A) \le b(A)$. Hence, if $\operatorname{rank}(A) = \min\{m,n\}$, then $b(A) = \operatorname{rank}(A) = \min\{m,n\}$. ∎

Notice that the upper and lower bounds given in Lemma 2.1 are sharp. Corollary 2.2 implies that any full-rank binary matrix achieves the upper bound. The following theorem characterizes the nonnegative integer matrices that attain the lower bound.

THEOREM 2.3. *Let* $A = (a_{ij}) \in \mathbb{Z}_+^{m\times n}$ *and let* $a_{i_0j_0}$ *be a maximum entry of* A. *If* $\min\{m,n\} = 2$, *then* $b(A) = \|A\|_\infty$ *if and only if*

(i) $a_{kj_0} = \max\limits_{1\le j\le n} a_{kj}$ *for any* k, $1 \le k \le m$;

(ii) $a_{i_0h} = \max\limits_{1\le i\le m} a_{ih}$ *for any* h, $1 \le h \le n$;

(iii) $a_{i_0j_0} + a_{st} \ge a_{i_0t} + a_{sj_0}$ *for any* s,t, $1 \le s \le m$, $1 \le t \le n$, $s \ne i_0$, $t \ne j_0$.

If $\min\{m,n\} > 2$, *let* $A = [A_1^T, A_2^T, \dots, A_m^T]^T$ *be row partition of* A. *Then* $b(A) = \|A\|_\infty$ *if and only if* A *satisfies (i), (ii), (iii) and there exists a positive integer* $i_1 \ne i_0$ *such that* $\begin{bmatrix} A_{i_0} \\ A_{i_1} \end{bmatrix} = BC$ *for some matrices* $B \in \mathscr{B}^{m\times \|A\|_\infty}$ *and* $C \in \mathscr{B}^{\|A\|_\infty\times n}$ *and for every* $j \in \{1,\dots,m\}\backslash\{i_0,i_1\}$, $A_j = x_jC$ *for some* $x_j \in \mathscr{B}^{1\times\|A\|_\infty}$.

Proof. First, we consider the case $\min\{m,n\} = 2$. Since $b(A) = b(A^T)$, we may assume $m = 2$.

If $b(A) = \|A\|_\infty$ and $A = BC$ is a binary factorization of A, where $B \in \mathscr{B}^{2\times\|A\|_\infty}$, $C \in \mathscr{B}^{\|A\|_\infty\times n}$. Let us partition B and C as $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ and $C = [C_1, C_2, \dots, C_n]$. Since $a_{i_0j_0} = \|A\|_\infty$, it follows that $B_{i_0}^T = C_{j_0} = j_{\|A\|_\infty}$. For any positive integer k, $1 \le k \le 2$, $a_{kj} = B_kC_j$ for any j, $1 \le j \le n$. Since $C_{j_0} = j_{\|A\|_\infty}$, it follows that $a_{kj_0} = B_kC_{j_0} = \max\limits_{1\le j\le n} a_{kj}$. Similarly, for any positive integer h, $1 \le h \le n$, $a_{ih} = B_iC_h$ for

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any i, $1 \leq i \leq m$. Since $B_{i_0} = j^T_{\|A\|_\infty}$, it follows that $a_{i_0 h} = B_{i_0} C_h = \max\limits_{1 \leq i \leq 2} a_{ih}$.

Observe that $a_{i_0 j_0} = B_{i_0} C_{j_0}$ and for any nonnegative integers s and t, $s \neq i_0$, $t \neq j_0$, $a_{st} = B_s C_t$, $a_{i_0 t} = B_{i_0} C_t$, $a_{s j_0} = B_s C_{j_0}$. It is not difficult to check that $(B_{i_0} - B_s)(C_{j_0} - C_t) \geq 0$, which is equivalent to $B_{i_0} C_{j_0} + B_s C_t \geq B_{i_0} C_t + B_s C_{j_0}$, i.e., $a_{i_0 j_0} + a_{st} \geq a_{i_0 t} + a_{s j_0}$.

Conversely, we may assume $a_{11} = \|A\|_\infty$. Let us partition B and C as $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ and $C = [C_1, C_2, \ldots, C_n]$, respectively. Let $B_1^T = C_1 = j_{\|A\|_\infty}$ and let $B_2 = (j^T_{a_{21}}, \mathbf{0})$. Now, B is fixed. Since $a_{21} + a_{1j} - a_{11} \leq a_{2j} \leq \min\{a_{21}, a_{1j}\}$ for every j, $2 \leq j \leq n$, let $C_j = \begin{bmatrix} C_{j1} \\ C_{j2} \end{bmatrix}$, where $C_{j1} \in \mathscr{B}^{a_{21} \times 1}$ and $C_{j2} \in \mathscr{B}^{(a_{11} - a_{21}) \times 1}$, such that C_{j1} has a_{2j} 1's and C_{j2} has $a_{1j} - a_{2j}$ 1's, $j = 2, \ldots, n$. Now, it is readily seen that $A = BC$.

If $\min\{m, n\} > 2$, let us partition A as $A = [A_1^T, A_2^T, \ldots, A_m^T]^T$. Then $a_{i_0 j_0}$ is in A_{i_0}. If A satisfies conditions (i), (ii) and (iii), then for a positive integer $i_1 \neq i_0$, it follows from the case $\min\{m, n\} = 2$ that there exist two binary matrices $B' \in \mathscr{B}^{2 \times \|A\|_\infty}$ and $C \in \mathscr{B}^{\|A\|_\infty \times n}$ such that $\begin{bmatrix} A_{i_0} \\ A_{i_1} \end{bmatrix} = B'C$. For every $j \in \{1, \ldots, m\} \backslash \{i_0, i_1\}$, if $A_j = x_j C$ for some $x_j \in \mathscr{B}^{1 \times \|A\|_\infty}$, let B be the matrix whose i_0-th row and i_1-th row are the first row and the second row of B', respectively, and j-th row, $j \in \{1, \ldots, m\} \backslash \{i_0, i_1\}$, is x_j. Then $A = BC$, and thus, $b(A) = \|A\|_\infty$. Conversely, if $b(A) = \|A\|_\infty$, then by the above argument, A satisfies (i), (ii) and (iii). Moreover, it can be seen from $A = BC$ that C is a right binary factor of the submatrix $\begin{bmatrix} A_{i_0} \\ A_{i_1} \end{bmatrix}$ and for every $j \in \{1, \ldots, m\} \backslash \{i_0, i_1\}$, A_j can be expressed as $A_j = x_j C$ for some $x_j \in \mathscr{B}^{1 \times \|A\|_\infty}$. $\square$

Next, we will give some necessary conditions for $b(A) = \min\{\|A\|_r, \|A\|_c\}$. Since $b(A) = b(A^T)$, we assume $\|A\|_r \leq \|A\|_c$ in the sequel.

Lemma 2.4. *Let $A = (a_{ij}) \in \mathbb{Z}_+^{n \times n}$ be a triangular matrix. If $a_{ii} = \max\limits_{1 \leq j \leq n} a_{ij}$ for every $1 \leq i \leq n$, then $b(A) = \|A\|_r$.*

Proof. Since $b(A) = b(A^T)$, we assume that A is upper triangular. Suppose that $A = BC$ is a binary factorization of A, where $B \in \mathscr{B}^{n \times k}$ and $C \in \mathscr{B}^{k \times n}$. Notice that the first row of B and the first column of C have at least a_{11} 1's and without loss of generality we assume that the first row of B is of the form $(j^T_{a_{11}}, *)$ and the first column of C is of the form $(j^T_{a_{11}}, *)^T$. Exchanging some rows of B and the corresponding columns of C if necessary, it follows from $a_{21} = 0$ and $a_{22} = \max\limits_{1 \leq j \leq n} a_{2j}$ that the second row of B is of the form $(\mathbf{0}^T_{a_{11}}, j^T_{a_{22}}, *)$. Similarly, exchanging some rows of B

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and the corresponding columns of C if necessary, it follows from $a_{31} = a_{32} = 0$ and $a_{33} = \max\limits_{1\le j\le n} a_{3j}$ that the third row of B is of the form $(\mathbf{0}^T_{a_{11}+a_{22}}, j^T_{a_{33}}, *)$. Continuing in this way we conclude that the last row of B is of the form $(\mathbf{0}^T_{\sum_{i=1}^{n-1} a_{ii}}, j^T_{a_{nn}}, *)$. Thus, $b(A) \ge \|A\|_r$. On the other hand, by Lemma 2.1, $b(A) \le \|A\|_r$. Hence, $b(A) = \|A\|_r$. □

A matrix A is defined to be *nondegenerate* if A has no zero row or zero column. Let $\phi(A)$ be the number of zeros of A.

THEOREM 2.5. *Let $A \in \mathbb{Z}_+^{n\times n}$ be nondegenerate. If $b(A) = \|A\|_r$, then $n-1 \le \phi(A) \le n(n-1)$. Moreover, let k and n be positive integers such that $n-1 \le k \le n(n-1)$. Then there exists a matrix $A \in \mathbb{Z}_+^{n\times n}$ such that $\phi(A) = k$ and $b(A) = \|A\|_r$.*

Proof. If $b(A) = \|A\|_r$, we claim that there is at most one row of A all of whose components are nonzero. Suppose that all components of row i and row j of A are nonzero, $1 \le i, j \le n$, $i \ne j$. Assume that a_{it} and a_{js} are the maximum components of row i and row j of A, respectively. Let $A' = \begin{bmatrix} a_{i1} & \cdots & a_{in} \\ a_{j1} & \cdots & a_{jn} \end{bmatrix}$ and $B = \begin{bmatrix} j^T_{a_{it}-1} & 1 & \mathbf{0} \\ \mathbf{0} & 1 & j^T_{a_{js}-1} \end{bmatrix}$. Then, it is not difficult to see that there exists a binary matrix $C \in \mathscr{B}^{(a_{it}+a_{js}-1)\times n}$ such that $A' = BC$, i.e., $b(A') \le a_{it} + a_{js} - 1$. Thus, $b(A) \le \|A\|_r - 1$, contradicting the assumption that $b(A) = \|A\|_r$. Hence, there is at most one row of A all of whose components are nonzero, i.e., $\phi(A) \ge n-1$. Since A is nondegenerate, it follows that $\phi(A) \le n(n-1)$.

Let k and n be positive integers such that $n-1 \le k \le n(n-1)$. For an upper triangular matrix $A \in \mathbb{Z}_+^{n\times n}$, assigning the values of the diagonal entries of A such that $a_{ii} = \max\limits_{1\le j\le n} a_{ij}$ for every $1 \le i \le n$. Then, no matter what the entries in the strictly upper triangular part of A are, we have $b(A) = \|A\|_r$. Hence, if $\frac{n(n-1)}{2} \le k \le n(n-1)$, then we can find an upper triangular matrix $A \in \mathbb{Z}_+^{n\times n}$ with $k - \frac{n(n-1)}{2}$ zeros in its strictly upper triangular part such that $b(A) = \|A\|_r$. Next, we will show that if $n-1 \le k \le \frac{n(n-1)}{2} - 1$, then there exists a matrix $A \in \mathbb{Z}_+^{n\times n}$ such that $\phi(A) = k$ and $b(A) = \|A\|_r$. By Corollary 2.2, it suffices to show that for every k, $n-1 \le k \le \frac{n(n-1)}{2} - 1$, there exists a nonsingular binary matrix $A \in \mathscr{B}^{n\times n}$ such that $\phi(A) = k$. Let

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 0 & 1 \end{bmatrix} \in \mathscr{B}^{n\times n}.$$

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That is, $a_{i+1,i} = 0$ for every $1 \leq i \leq n-1$, and the remaining entries of A are 1. A direct computation shows that $det(A) = 1$, i.e., A is nonsingular. Moreover, observe that replacing any a_{ij}, $3 \leq i \leq n$, $1 \leq j \leq i-2$, by 0 will not change the determinant of A. Hence, if $k \in [n-1, \frac{n(n-1)}{2} - 1]$, then there exists a nonsingular binary matrix $A \in \mathscr{B}^{n\times n}$ such that $\phi(A) = k$. This completes the proof. $\square$

A natural question is: For a nondegenerate nonnegative integer matrix A, if $b(A) = \|A\|_r$, then how small can rank(A) be? The following theorem shows that for a nondegenerate nonnegative integer matrix A of order $n \geq 4$, if $b(A) = \|A\|_r$, then rank$(A) \geq 3$.

For a matrix $A \in \mathbb{Z}_+^{n\times n}$, let μ, ν be nonempty ordered subsets of $\{1, 2, \ldots, n\}$. Denote by $A[\mu|\nu]$ the submatrix of A with rows indexed by μ and columns indexed by ν. If $\mu = \nu$, then $A[\mu|\mu]$ is abbreviated to $A[\mu]$.

Theorem 2.6. *Let $A \in \mathbb{Z}_+^n$ be nondegenerate, $n \geq 4$. If $b(A) = \|A\|_r$, then* rank$(A) \geq 3$.

Proof. We will show in Theorem 3.1 that if rank$(A) = 1$, then $b(A) = \|A\|_\infty$. Hence, for a nondegenerate matrix A of order $n \geq 2$, if rank$(A) = 1$, then $b(A) < \|A\|_r$. Next, we will show that for a nondegenerate nonnegative integer matrix A of order $n \geq 4$, if rank$(A) = 2$, then $b(A) < \|A\|_r$.

We have shown in the proof of Theorem 2.5 that if $A \in \mathbb{Z}_+^{n\times n}$ with $b(A) = \|A\|_r$, then there is at most one row of A all of whose components are nonzero. Hence, there exists a submatrix of A, say $\tilde{A}$, which is permutation equivalent to one of the following three forms

$$\text{(I)} \begin{bmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ 0 & * & \cdots & * \\ 0 & * & \cdots & * \end{bmatrix}, \ \text{(II)} \begin{bmatrix} * & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ * & 0 & * & \cdots & * \end{bmatrix}, \ \text{(III)} \begin{bmatrix} * & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ * & 0 & * & \cdots & * \\ * & * & 0 & \cdots & * \end{bmatrix}.$$

If $\tilde{A}$ is of type (I) or type (II), then we only consider the first three rows of $\tilde{A}$. Without loss of generality, we assume

$$\tilde{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & a_{24} & \cdots & a_{2n} \\ 0 & a_{32} & a_{33} & a_{34} & \cdots & a_{3n} \\ * & * & * & * & \cdots & * \end{bmatrix}.$$

Since A is nondegenerate, the first column of $\tilde{A}$ has at least one nonzero component and we may assume $a_{11} \neq 0$. We consider the following three cases:

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Case 1: $a_{22} \neq 0$, $a_{32} \neq 0$. In this case, for any $3 \leq j \leq n$, if $\mathrm{rank}(A) = 2$, then $\tilde{A}[\{1,2,3\},\{1,2,j\}]$ is singular, which implies $a_{22}a_{3j} = a_{2j}a_{32}$. Hence, either a_{2j} and a_{3j} are positive integers or $a_{2j} = a_{3j} = 0$. It follows from the second and the third row of $\tilde{A}$ that $b(\tilde{A}) < \|\tilde{A}\|_r$, which implies $b(A) < \|A\|_r$, a contradiction.

Case 2: $a_{22} = 0$, $a_{32} \neq 0$ or $a_{22} \neq 0$, $a_{32} = 0$. Since A is nondegenerate, the second row of $\tilde{A}$ has at least one nonzero component, say, $a_{2j_0} \neq 0$. If $a_{22} = 0$ and $a_{32} \neq 0$, it follows that $\tilde{A}[\{1,2,3\},\{1,2,j_0\}]$ is nonsingular, contradicting the assumption that $\mathrm{rank}(A) = 2$. Similarly, if $a_{22} \neq 0$ and $a_{32} = 0$, then we can find a 3×3 nonsingular sbumatrix of $\tilde{A}$, also a contradiction.

Case 3: $a_{22} = a_{32} = 0$. If at least one of a_{23} and a_{33} is nonzero, then applying the argument of Case 1 and Case 2 to a_{23} and a_{33} we conclude that if $\mathrm{rank}(A) = 2$, then $b(A) < \|A\|_r$. If $a_{23} = a_{33} = 0$, then we consider a_{24} and a_{34}. Continuing in this way we conclude that if $\mathrm{rank}(A) = 2$, then $b(A) < \|A\|_r$.

If $\tilde{A}$ is of type (III), then without loss of generality, we assume

$$\tilde{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & a_{24} & \cdots & a_{2n} \\ a_{31} & 0 & a_{33} & a_{34} & \cdots & a_{3n} \\ a_{41} & a_{42} & 0 & a_{44} & \cdots & a_{4n} \end{bmatrix}.$$

If $\mathrm{rank}(A) = 2$, then $\det(\tilde{A}[\{2,3,4\},\{1,2,3\}]) = a_{22}a_{33}a_{41} + a_{23}a_{31}a_{42} = 0$. Since A is nonnegative, one must have that at least one of a_{22}, a_{33} and a_{41} is zero and at least one of a_{23}, a_{31} and a_{42} is zero. Thus, it can be reduced to type (I) or type (II). By the above argument, we conclude that if $\mathrm{rank}(A) = 2$, then $b(A) < \|A\|_r$. This completes the proof. ☐

We do not know if 3 is the best possible lower bound in Theorem 2.6. We will show in the next section that if $n = 4$, then there is a nondegenerate binary matrix A of order 4 satisfies $\mathrm{rank}(A) = 3$ and $b(A) = 4$. Now, we pose the following question: For any $n \geq 5$, there is a nondegenerate matrix $A \in \mathbb{Z}_+^{n\times n}$ such that $\mathrm{rank}(A) = 3$ and $b(A) = \|A\|_r$? Or there is a nondegenerate binary matrix $A \in \mathscr{B}^{n\times n}$ such that $\mathrm{rank}(A) = 3$ and $b(A) = n$?

3. Computing binary ranks of nonnegative integer matrices with low ranks. Let $\mathcal{M}_{m,n}(\mathcal{S})$ denote the set of all $m \times n$ matrices whose entries belong to the set $\mathcal{S}$. We refer to the *factor rank* of $A \in \mathcal{M}_{m,n}(\mathcal{S})$, denoted by $r_{\mathcal{S}}(A)$, as the minimum integer k such that $A = BC$ for some matrices $B \in \mathcal{M}_{m,k}(\mathcal{S})$ and $C \in \mathcal{M}_{k,n}(\mathcal{S})$. For example, if $\mathcal{S} = \mathcal{R}^+$, the set of nonnegative real numbers, then $r_{\mathcal{R}^+}(A)$ is the *nonnegative rank* of A (see [5]). Recently, comparing real rank with various "ranks" over various semirings has been object of interest. For example, in [5] and

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independently in [1], it was shown that for any $A \in \mathcal{M}_{m,n}(\mathcal{R}^+)$, if $\operatorname{rank}(A) \le 2$, then $r_{\mathcal{R}^+}(A) = \operatorname{rank}(A)$. In [2], it was proved that the nonnegative rank of a product of two nonnegative matrices is not greater than the product of ranks of these two matrices. In this section, we will compute the binary ranks of nonnegative integer matrices with low ranks.

Theorem 3.1. *Let $A \in \mathbb{Z}_+^{m\times n}$. If $\operatorname{rank}(A) = 1$, then $b(A) = \|A\|_\infty$.*

Proof. Exchanging some rows and some columns of A if necessary, we assume $a_{11} = \|A\|_\infty$ and $a_{11} \ge a_{12} \ge \cdots \ge a_{1n}$, $a_{11} \ge a_{21} \ge \cdots \ge a_{m1}$. For convenience, we assume $a_{1n} \ge 1$ and $a_{m1} \ge 1$. Then $a_{ij} \ge 1$ for any i, j, $1 \le i \le m$, $1 \le j \le n$. Since $\operatorname{rank}(A) = 1$, any two rows of A are linearly dependent. It follows from $a_{11}/a_{i1} = a_{1j}/a_{ij}$ that $a_{1j} \ge a_{ij}$ for any i, j, where $2 \le i \le m$ and $2 \le j \le n$. Similarly, any two columns of A are linearly dependent and it follows from $a_{11}/a_{1k} = a_{h1}/a_{hk}$ that $a_{h1} \ge a_{hk}$ for any h, k, where $2 \le h \le m$ and $2 \le k \le n$. Moreover, for any i, j, $1 \le i \le m-1$, $2 \le j \le n$, let $a_{i1}/a_{i+1,1} = a_{ij}/a_{i+1,j} = k$. Then $k \ge 1$ and

$$(a_{i1}-a_{i+1,1})-(a_{ij}-a_{i+1,j}) = a_{i+1,1}(k-1)-a_{i+1,j}(k-1) = (k-1)(a_{i+1,1}-a_{i+1,j}) \ge 0.$$

Let $t_{i1} = a_{i1} - a_{i+1,1}$, $1 \le i \le m-1$ and $t_{m1} = a_{m1}$. Partition A as $A = [A_1, A_2, \ldots, A_n]$. Then, A_1 can be expressed as

$$A_1 = \underbrace{\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix} + \cdots + \begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}}_{t_{m1}} + \underbrace{\begin{bmatrix}1\\\vdots\\1\\0\end{bmatrix} + \cdots + \begin{bmatrix}1\\\vdots\\1\\0\end{bmatrix}}_{t_{m-1,1}} + \cdots + \underbrace{\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix} + \cdots + \begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}}_{t_{11}};$$

that is, A_1 can be expressed as a linear combination of the above a_{11} vectors in $\mathscr{B}^{n\times 1}$ with coefficients all equal 1. Moreover, for every j, $2 \le j \le n$, A_j has a similar representation as that of A_1, we need only replace t_{i1} by t_{ij}, where $1 \le i \le m$, $t_{ij} = a_{ij} - a_{i+1,j}$ and $t_{mj} = a_{mj}$.

Notice that $a_{i1}-a_{i+1,1} \ge \max_{2\le j\le n}(a_{ij}-a_{i+1,j})$ for any $1 \le i \le m-1$, and $a_{m1} \ge a_{mj}$ for any $2 \le j \le n$, i.e., $t_{i1} \ge t_{ij}$ for any $1 \le i \le m$. Thus, for every $1 \le j \le n$, A_j can be expressed as a linear combination of a_{11} vectors in $\mathscr{B}^{n\times 1}$ with coefficients from $\{0, 1\}$. Hence, by Lemma 1.3, $b(A) = \|A\|_\infty$. □

It seems that it is not easy to determine the binary rank of a given nonnegative integer matrix of rank 2. However, for binary matrices with rank 2, we are able to determine their binary ranks.

Theorem 3.2. *Let $A \in \mathscr{B}^{m\times n}$. If $\operatorname{rank}(A) = 2$, then $b(A) = 2$.*

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Proof. Let $A = [A_1^T, A_2^T, \ldots, A_m^T]^T \in \mathscr{B}^{m\times n}$. If $\mathrm{rank}(A) = 2$ and $\min\{m,n\} = 2$, then it follows from Corollary 2.2 that $b(A) = 2$. If $\min\{m,n\} > 2$, then there exist two rows of A such that each row of A is a linear combination of these two rows. Exchanging some rows of A if necessary, assume that these two rows are A_1 and A_2. For $j = 3,\ldots,m$, let A_j has the representation $A_j = \alpha_j A_1 + \beta_j A_2$. Since $\mathrm{rank}(A) = 2$, there exists a positive integer i such that $(A_1)_i = 1$, $(A_2)_i = 0$ or $(A_1)_i = 0$, $(A_2)_i = 1$, where $(A_j)_i$ is the i-th component of A_j, $j = 1,2$. We only consider the case $(A_1)_i = 1$, $(A_2)_i = 0$ since the other case can be proved in a similar way. Without loss of generality, we assume $(A_1)_1 = 1$ and $(A_2)_1 = 0$. It follows that for any $3 \le j \le m$, $\alpha_j = \alpha_j(A_1)_1 + \beta_j(A_2)_1 \in \{0,1\}$. Notice that for any positive integer k, $2 \le k \le n$, $((A_1)_k,(A_2)_k) \in \{(1,0),(0,1),(1,1),(0,0)\}$. We consider the following two cases.

Case 1: There exists at least one k_0, $2 \le k_0 \le n$, such that $(A_1)_{k_0} = 0$ and $(A_2)_{k_0} = 1$. In this case, for any $3 \le j \le m$, it follows from $\alpha_j(A_1)_{k_0} + \beta_j(A_2)_{k_0} \in \{0,1\}$ that $\beta_j \in \{0,1\}$. Thus, any row of A can be expressed as a linear combination of A_1 and A_2 with coefficients from the set $\{0,1\}$. Hence, A has the following binary factorization:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \alpha_3 & \beta_3 \\ \vdots & \vdots \\ \alpha_m & \beta_m \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix},$$

which implies $b(A) = 2$.

Case 2: For any positive integer l, $((A_1)_l,(A_2)_l) \in \{(1,0),(1,1),(0,0)\}$, $2 \le l \le n$. We exclude the case $((A_1)_l,(A_2)_l) = (0,0)$ since otherwise A will have a zero column. Let $\tilde{A} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$. Then $\tilde{A}$ is of the form

$$\tilde{A} = \begin{bmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 \\ 0 & \cdots & 0 & 1 & \cdots & 1 \end{bmatrix} \tag{3.1}$$

or

$$\tilde{A} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \end{bmatrix}. \tag{3.2}$$

Observe that there exists a positive integer l_0 such that $(A_1)_{l_0} = (A_2)_{l_0} = 1$ in both forms above. It follows from $\alpha_j(A_1)_{l_0} + \beta_j(A_2)_{l_0} \in \{0,1\}$ that $\alpha_j + \beta_j \in \{0,1\}$. Thus, if $\alpha_j = 0$, then $\beta_j \in \{0,1\}$. If $\alpha_j = 1$, then $\beta_j \in \{-1,0\}$. Observe that if $\alpha_j = 1$ and $\beta_j = -1$, then

$$A_j = \begin{bmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \end{bmatrix} \tag{3.3}$$

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or

$$A_j = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}. \tag{3.4}$$

Thus, if $\tilde{A}$ is of the form (3.1), then any row of A is equal to one of the four types: (1) A_1; (2) A_2; (3) $\mathbf{0}$; (4) the vector in (3.3). If $\tilde{A}$ is of the form (3.2), then any row of A is equal to one of the four types: (1) A_1; (2) A_2; (3) $\mathbf{0}$; (4) the vector in (3.4). First, we consider the case that $\tilde{A}$ is of the form (3.2). Without loss of generality, suppose that A_h's, $3 \le h \le t \le m-1$, are of the form (3.3) and the rest rows of A equal A_1, A_2 or $\mathbf{0}$. Then

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ \gamma_{t+1} & \delta_{t+1} \\ \vdots & \vdots \\ \gamma_m & \delta_m \end{bmatrix} \begin{bmatrix} 1 & \cdots & 1 & 0 & \cdots 0 \\ 0 & \cdots & 0 & 1 & \cdots 1 \end{bmatrix}, \tag{3.5}$$

where

$$(\gamma_j, \delta_j) = \begin{cases} (1,1), & \text{if } (\alpha_j, \beta_j) = (1,0), \\ (0,1), & \text{if } (\alpha_j, \beta_j) = (0,1), \\ (0,0), & \text{if } (\alpha_j, \beta_j) = (0,0). \end{cases}$$

Now, it is readily seen from (3.5) that $b(A) = 2$.

If $\tilde{A}$ is of the form (3.2), replace the right factor of (3.5) by $\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 1 \end{bmatrix}$ continuing in a similar way to above, we have $b(A) = 2$. This completes the proof. $\square$

For a binary matrix A, if $\operatorname{rank}(A) \ge 3$, one may wonder if $\operatorname{rank}(A) = b(A)$ is still true? The answer is negative. A simple example is provided by

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

It is easy to see that $\operatorname{rank}(A) = 3$. Cohen and Rothblum [5] show that $\operatorname{rank}_{\mathcal{R}^+}(A) = 4$. Hence, $4 = b(A) = \operatorname{rank}_{\mathcal{R}^+}(A) > \operatorname{rank}(A) = 3$.

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4. Binary ranks of nonnegative integer Jacobi matrices. A matrix A is called a Jacobi matrix if it is of the form

$$A = \begin{bmatrix} a_1 & b_1 & 0 & \cdots & \cdots & 0 \\ b_1 & a_2 & b_2 & \ddots & & \vdots \\ 0 & b_2 & a_3 & b_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & a_{n-1} & b_{n-1} \\ 0 & \cdots & \cdots & 0 & b_{n-1} & a_n \end{bmatrix}, \tag{4.1}$$

where a_i are real and b_i are positive.

THEOREM 4.1. *Let $A = (a_{ij})$ be a nonnegative integer Jacobi matrix of the form (4.1). If $a_{ii} = \max\limits_{1\leq j\leq n} a_{ij}$ for every $1 \leq i \leq n$, then*

$$b(A) \geq a_1 + \sum_{i=2}^{n-1} \max\{a_i - b_{i-1}, b_i\} + a_n - b_{n-1}.$$

Proof. Suppose that $A = BC$ is a binary factorization of A, where $B \in \mathscr{B}^{n\times k}$, $C \in \mathscr{B}^{k\times n}$. Let us partition B and C as $B = [B_1^T, B_2^T, \ldots, B_n^T]^T$ and $C = [C_1, C_2, \ldots, C_n]$, respectively. Without loss of generality, assume that B_1 is of the form $(j_{a_1}^T, *)$. Then C_1 is of the form $(j_{a_1}^T, *)^T$, C_i's, $i = 3, \ldots, n$, are of the form $(\mathbf{0}_{a_1}^T, *)^T$ and B_i's, $i = 3, \ldots, n$, are of the form $(\mathbf{0}_{a_1}^T, *)$. Hence, to guarantee $B_2C_3 = b_2$, B_2 should have at least $a_1 + b_2$ components and without loss of generality, assume that B_2 is of the form $(\underbrace{*, \ldots, *}_{a_1}, j_{b_2}^T, *)$. Then C_3 is of the form $(\mathbf{0}_{a_1}^T, j_{b_2}^T, *)^T$ and C_i's, $i = 4, \ldots, n$, are of the form $(\mathbf{0}_{a_1}^T, \mathbf{0}_{a_1+b_2}^T, *)^T$. If $a_2 - b_1 \leq b_2$, then it is possible to assign the values of the first a_1 components of B_2 and the first $a_1 + b_2$ components of C_2 such that $B_1C_2 = B_2C_1 = b_1$ and $B_2C_2 = a_2$. If $a_2 - b_1 \geq b_2$, since the first a_1 components of C_2 have at most b_1 1's, to guarantee $B_2C_2 = a_2$, B_2 should have at least $a_1 + a_2 - b_1$ components. Similarly, it follows from $a_{24} = 0$ and $a_{34} = b_3$ that B_3 should have at least $a_1 + \max\{a_2 - b_1, b_2\} + b_3$ components. By comparing $a_3 - b_2$ with b_3 and using a similar argument as above we conclude that B_3 should have at least $a_1 + \max\{a_2 - b_1, b_2\} + \max\{a_3 - b_2, b_3\}$ components. Continuing in this way we conclude that B_{n-1} should have at least $a_1 + \sum\limits_{i=2}^{n-1} \max\{a_i - b_{i-1}, b_i\}$ components. Moreover, let $t = a_1 + \sum\limits_{i=2}^{n-1} \max\{a_i - b_{i-1}, b_i\}$. Then it follows from the last row of A that B_n is of the form $(\mathbf{0}_t^T, j_{b_{n-1}}^T, *)$. Hence, to guarantee $B_nC_n = a_n$,

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B_n should have at least $a_1 + \sum_{i=2}^{n-1} \max\{a_i - b_{i-1}, b_i\} + a_n - b_{n-1}$ components, i.e., $b(A) \geq a_1 + \sum_{i=2}^{n-1} \max\{a_i - b_{i-1}, b_i\} + a_n - b_{n-1}$. □

COROLLARY 4.2. [3] *Let A be a nonnegative integer Jacobi matrix of the form (4.1). If A is diagonally dominant, then*

$$b(A) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n-1} b_i.$$

Proof. Let

$$B = \begin{bmatrix} j_{b_1}^T & j_{a_1-b_1}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ j_{b_1}^T & \mathbf{0} & j_{b_2}^T & j_{a_2-b_1-b_2}^T & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & j_{b_2}^T & \mathbf{0} & j_{a_3-b_2}^T & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & j_{b_{n-1}}^T & j_{a_{n-1}-b_{n-2}}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & j_{b_{n-1}}^T & \mathbf{0} & j_{a_n-b_{n-1}}^T \end{bmatrix}.$$

Then $A = BB^T$ and B has $\sum_{i=1}^{n} a_i - \sum_{i=1}^{n-1} b_i$ columns. Thus, $b(A) \leq \sum_{i=1}^{n} a_i - \sum_{i=1}^{n-1} b_i$. On the other hand, by Theorem 4.1, $b(A) \geq \sum_{i=1}^{n} a_i - \sum_{i=1}^{n-1} b_i$. Hence, $b(A) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n-1} b_i$. □

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