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SPECTRAL PROPERTIES FOR A NEW COMPOSITION OF A MATRIX AND A COMPLEX REPRESENTATION*

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Abstract. A way to compose a matrix P and a finite dimensional representation ρ of $\mathbb{C}$ via a map h into a new matrix $P *_h \rho$ is defined. Several results about the spectrum, eigenvectors, kernel and rank of $P *_h \rho$ are proved.

Key words. Spectrum, Complex representation of $\mathbb{C}$.

AMS subject classifications. 15A18, 15A99.

1. Introduction. In two recent papers [4, 5], it has been pointed out the interest of the symmetric matrix

$$P(\omega) := \begin{bmatrix} P_1(\omega) & P_2(\omega) \\ P_2^{\mathsf{T}}(\omega) & P_1(\omega) \end{bmatrix}$$

for the design of some signal filters, where $P_1(\omega)$ and $P_2(\omega)$ are the square matrices of order N whose entries are

$$\big(P_1(\omega)\big)_{i,j=1}^{N} := (i+j-2)\cos((i-j)\omega), \quad \big(P_2(\omega)\big)_{i,j=1}^{N} := (i+j-2)\sin((i-j)\omega).$$

In particular, it has been conjectured that the spectrum of $P(\omega)$, i.e., its eigenvalues and their multiplicities, is actually independent of ω. In this paper we prove this fact as a consequence of a more general result (Theorem 2.5 and Proposition 2.8 here below). Indeed, we introduce a procedure which generalizes the construction of $P(\omega)$ and we prove the conjecture for each matrix we obtain in this way. Several other results exploring the connection of the new operation with other ways to combine matrices into a new matrix are given.

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2. Results. Let ρ be an r dimensional linear representation of $\mathbb{C}$, i.e., a map $\mathbb{C} \to \mathrm{GL}(r, \mathbb{C})$ satisfying the condition

$$\rho(x+y) = \rho(x) \cdot \rho(y) \quad \forall x, y \in \mathbb{C}, \tag{2.1}$$

let $h\colon \mathbb{N} \times \mathbb{N} \to \mathbb{C}$ be any function for which

$$h(i,k) + h(k,j) = h(i,j) \quad \forall i,j,k, \tag{2.2}$$

and finally let P be any complex $n \times m$ matrix. With these ingredients we define a new matrix $P *_h \rho$ in $\mathcal{M}(nr \times mr, \mathbb{C})$ as

$$P *_h \rho := \begin{bmatrix} (P *_h \rho)_{(1,1;\cdot,\cdot)} & \cdots & (P *_h \rho)_{(1,r;\cdot,\cdot)} \\ \dots\dots & \dots\dots & \dots\dots \\ (P *_h \rho)_{(r,1;\cdot,\cdot)} & \cdots & (P *_h \rho)_{(r,r;\cdot,\cdot)} \end{bmatrix},$$

where each block $(P *_h \rho)_{(I,J;\cdot,\cdot)}$ is itself an $n \times m$ matrix and is defined as

$$(P *_h \rho)_{(I,J;i,j)} := P_{i,j}\rho_{I,J}(h(i,j)), \qquad i = 1, \dots, n, \;\; j = 1, \dots, m.$$

Remark 2.1. There is a canonic way to build ρ: let T be an arbitrary square matrix in $\mathcal{M}(r, \mathbb{C})$, and take

$$\rho(x) := \exp(xT) := \sum_{k=0}^{\infty} \frac{x^k}{k!}\, T^k, \quad \forall x \in \mathbb{C}.$$

Every regular (analytic) representation is of this form (see [1, Ch. 6 Appendix A] and [3, Ch. 8]).

Remark 2.2. It is immediate to realize that (2.2) is satisfied if and only if $h(i,j) = g(i) - g(j)$ for some map $g\colon \mathbb{N} \to \mathbb{C}$.

Remark 2.3. The matrix $P(\omega)$ is of the form $P *_h \rho$ with

$$\begin{aligned} &P \in \mathcal{M}(N, \mathbb{C}): \quad (P)_{i,j} := i + j - 2, \\ &\rho : \mathbb{C} \to \mathrm{GL}(2, \mathbb{C}): \quad \rho(x) := \left[\begin{smallmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{smallmatrix}\right], \\ &h(i,j) := (i-j)\omega. \end{aligned}$$

The following theorem gives a first set of properties for $P *_h \rho$ in terms of analogous properties for P.

Theorem 2.4.

1. *$(\mu P + \nu Q) *_h \rho = \mu P *_h \rho + \nu Q *_h \rho$ for every $P, Q \in \mathcal{M}(n \times m, \mathbb{C})$ and for every $\mu, \nu \in \mathbb{C}$. Moreover, $P *_h \rho = \mathbf{0}$ if and only if $P = \mathbf{0}$.*

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2. *Let P be diagonal, then $P *_h \rho$ is the direct sum of r copies of P and therefore it is diagonal too. In particular, $\mathbb{I}_n *_h \rho = \mathbb{I}_{nr}$.*

3. *$(P *_h \rho)^* = P^* *_{-h} \rho^*$. In particular, if P is a square matrix and the restriction of ρ on the range of h is unitary (i.e., if $\rho^*(x) = \rho(-x)$ for every x in the range of h), then $P *_h \rho$ is self-adjoint if and only if P is self-adjoint.*

4. *Let $P \in \mathcal{M}(n \times l, \mathbb{C})$ and $Q \in \mathcal{M}(l \times m, \mathbb{C})$, then*

$$(P *_h \rho) \cdot (Q *_h \rho) = (P \cdot Q) *_h \rho.$$

5. *Let P be a square matrix. The minimal polynomials of P and $P *_h \rho$ are equal.*

6. *$P *_h \rho$ is diagonalizable if and only if P is diagonalizable, and a complex number λ is an eigenvalue for $P *_h \rho$ if and only if it is an eigenvalue for P.*

7. *$P \in \mathrm{GL}(n, \mathbb{C})$ if and only if $P *_h \rho \in \mathrm{GL}(nr, \mathbb{C})$, with $(P *_h \rho)^{-1} = P^{-1} *_h \rho$.*

Proof.

1. The linearity of $P *_h \rho$ as a function of P is evident; it implies that $\mathbf{0} *_h \rho = \mathbf{0}$. Suppose that $P_{i,j}\rho_{I,J}(h(i,j)) = 0$ for every $I, J = 1, \ldots, r$ and every $i = 1, \ldots, n$, $j = 1, \ldots, m$, and that by absurd $P_{\bar{\imath},\bar{\jmath}} \neq 0$ for a couple of indexes $\bar{\imath}, \bar{\jmath}$. Then $\rho_{I,J}(h(\bar{\imath}, \bar{\jmath})) = 0$ for every I and J, which is impossible because $\rho(h(\bar{\imath}, \bar{\jmath})) \in \mathrm{GL}(r, \mathbb{C})$.

2. Let P be diagonal, so that $P_{i,j} = a_i \delta_{i,j}$, then:

$$(P *_h \rho)_{(I,J;i,j)} = a_i \delta_{i,j} \rho_{I,J}(h(i,j)) = a_i \delta_{i,j} \rho_{I,J}(h(i,i)) = a_i \delta_{i,j} \rho_{I,J}(0)$$

because h is an odd map, and this is $a_i \delta_{i,j} \delta_{I,J}$, because $\rho(0) = \mathbb{I}_r$.

3. The equality $(P *_h \rho)^* = P^* *_{-h} \rho^*$ is an immediate consequence of the definition of the $*_h$-product. Now suppose that $\rho^*(-h(i,j)) = \rho(h(i,j))$, then $(P *_h \rho)^* = P^* *_h \rho$ so that this is equal to $P *_h \rho$ if and only if $(P^* - P) *_h \rho = \mathbf{0}$, i.e., if and only if $P^* = P$, by Item 1.

4. The proof is a direct consequence of Relations (2.1–2.2). In fact, for every couple of indexes $I, J = 1, \ldots, r$ and $i = 1, \ldots, n$, $j = 1, \ldots, m$ we have

$$\begin{aligned} ((P *_h \rho) \cdot (Q *_h \rho))_{(I,J;i,j)} &= \sum_{k,K} P_{i,k} \rho_{I,K}(h(i,k)) Q_{k,j} \rho_{K,J}(h(k,j)) \\ &= \sum_{k} P_{i,k} Q_{k,j} \sum_{K} \rho_{I,K}(h(i,k)) \rho_{K,J}(h(k,j)). \end{aligned}$$

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By (2.1) the inner sum is $\rho_{I,J}(h(i,k)+h(k,j))$, i.e., $\rho_{I,J}(h(i,j))$, by (2.2). Thus

$$((P *_h \rho)\cdot(Q *_h \rho))_{(I,J;i,j)} = \sum_k P_{i,k}Q_{k,j}\rho_{I,J}(h(i,j)),$$

which is the claim.

5. The formula in Item 4 implies that $f(P *_h \rho) = f(P) *_h \rho$ for every polynomial $f \in \mathbb{C}[x]$, so that $f(P *_h \rho)$ is null if and only if $f(P)$ is null as well, by Item 1. The claim follows by the definition of the minimal polynomial of a matrix A as the monic generator of the ideal of complex polynomials f for which $f(A) = \mathbf{0}$.

6. A matrix is diagonalizable if and only if its minimal polynomial $f \in \mathbb{C}[x]$ factorizes in $\mathbb{C}[x]$ as product of distinct linear polynomials. Therefore the first claim follows by Item 5. Moreover, the eigenvalues coincide with the roots of the minimal polynomial, therefore Item 5 implies also the second part of this claim.

7. A matrix is invertible if and only if 0 is not an eigenvalue. Therefore the first part of the claim follows by Item 6. The formula for $(P *_h \rho)^{-1}$ is an immediate consequence of Items 2 and 4. $\square$

Item 6 of the previous theorem already shows that the spectrum of $P *_h \rho$ and that one of P contain the same points, but their spectral structures are even more strictly related. In fact, the next two theorems prove that also the eigenvectors of $P *_h \rho$ can be easily deduced by those ones of P. We start with a general result which is of some independent interest.

THEOREM 2.5. *Let* $P \in \mathcal{M}(n \times m, \mathbb{C})$. *Then* $\dim\ker(P *_h \rho) = r \dim\ker(P)$ *and* $\operatorname{rank}(P *_h \rho) = r \operatorname{rank}(P)$.

Proof. Let s denote the rank of P. The definition of rank implies the existence of a permutation $\mathcal{P} \in \mathrm{GL}(n,\mathbb{C})$ and a permutation $\mathcal{P}' \in \mathrm{GL}(m,\mathbb{C})$ such that

$$P' := \mathcal{P}P\mathcal{P}' = \begin{bmatrix} P'' & * \\ * & * \end{bmatrix}$$

with P'' in $\mathrm{GL}(s,\mathbb{C})$. By Theorem 2.4 Item 7 the matrices $\mathcal{P} *_h \rho$ and $\mathcal{P}' *_h \rho$ are invertible, so that the rank of $P' *_h \rho = (\mathcal{P} *_h \rho)(P *_h \rho)(\mathcal{P}' *_h \rho)$ (by Item 4) is equal to that one of $P *_h \rho$. Moreover, the matrix $P'' *_h \rho$ is a submatrix in $P' *_h \rho$, therefore $\operatorname{rank}(P' *_h \rho) \geq \operatorname{rank}(P'' *_h \rho)$ and the rank of $P'' *_h \rho$ is rs because it is in $\mathrm{GL}(sr,\mathbb{C})$. As a consequence we have proved that

$$\operatorname{rank}(P *_h \rho) \geq r \operatorname{rank}(P). \tag{2.3}$$

Let $\boldsymbol{v}_1,\ldots,\boldsymbol{v}_k$ be a basis for the kernel of P. Let $V := [\boldsymbol{v}_1 \mid \ldots \mid \boldsymbol{v}_k]$ be the matrix having the vectors $\boldsymbol{v}_j$ for $j = 1,\ldots,k$ as columns. By (2.3) applied to V we get that

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$\operatorname{rank}(V *_h \rho) \geq kr$. The columns in $V *_h \rho$ belong to the kernel of $P *_h \rho$, by Item 4 in Theorem 2.4; this proves that

$$\dim\ker(P *_h \rho) \geq r \dim\ker(P). \tag{2.4}$$

Adding (2.3) and (2.4) and recalling the rank-nullity theorem we conclude that

$$mr = \operatorname{rank}(P *_h \rho) + \dim\ker(P *_h \rho) \geq r \operatorname{rank}(P) + r \dim\ker(P) = mr$$

which proves that the equality holds in (2.3) and (2.4). ▯

Let P be a square matrix. For each $\lambda \in \mathbb{C}$ let E_λ denote the kernel of $P - \lambda\mathbb{I}_n$ (i.e., the λ-eigenspace of P when λ belongs to the spectrum of P), and analogously let $E_{\lambda,\rho}$ denote the kernel of $P *_h \rho - \mathbb{I}_{nr}$.

Proposition 2.6. *For every $\lambda \in \mathbb{C}$, $\dim E_{\lambda,\rho} = r \dim E_\lambda$. In particular, P and $P *_h \rho$ have the same eigenvalues, and the multiplicity of every λ as eigenvalue for $P *_h \rho$ is r times its multiplicity as eigenvalue for P. Moreover, if the columns in $V \in \mathcal{M}(m \times \dim E_\lambda, \mathbb{C})$ are a basis for E_λ, then the columns of $V *_h \rho$ are a basis for $E_{\lambda,\rho}$.*

Proof. In fact, $E_{\lambda,\rho} = \ker(P *_h \rho - \lambda\mathbb{I}_{nr}) = \ker((P - \lambda\mathbb{I}_n) *_h \rho)$ so that the claims follow by the previous theorem. ▯

Remark 2.7. We can rephrase the claims of Proposition 2.6 by saying that the spectrum (i.e., the eigenvalues and the dimension of each eigenspace) of $P *_h \rho$ is independent of h and depends on ρ only via its dimension; this claim already suffices to completely determine the spectrum of $P *_h \rho$ since one sees immediately that $P *_h \rho$ collapses to the direct sum of r copies of P when h is taken equal to 0 identically. The conjectured independence of the spectrum of $P(\omega)$ of ω in [5], therefore, is evidently only a special case of the independence of the spectrum of $P *_h \rho$ on h claimed in Proposition 2.6 when it is restated in this way.

Let V and W be two matrices, respectively in $\mathcal{M}(n \times v, \mathbb{C})$ and $\mathcal{M}(n \times w, \mathbb{C})$. Then we can form the new matrix $[V \,|\, W]$ in $\mathcal{M}(n \times (v+w), \mathbb{C})$ by joining the columns of W to those ones of V. In general, the matrices $[V *_h \rho \,|\, W *_h \rho]$ and $[V \,|\, W] *_h \rho$ are distinct, but they are quite strictly related. We begin with a simple computation, which is useful in applications. Suppose that the columns of V and W be eigenvectors for a matrix P, so that $PV = VD_V$ and $PW = WD_W$ with D_V and D_W diagonal matrices. Then, using two times the multiplicativity property for the $*_h$-product (Item 4 of Theorem 2.4), we get

$$\begin{aligned}(P *_h \rho) \cdot [V \,|\, W] *_h \rho &= (P \cdot [V \,|\, W]) *_h \rho = ([PV \,|\, PW]) *_h \rho \\ &= ([VD_V \,|\, WD_W]) *_h \rho = ([V \,|\, W] \cdot (D_V \oplus D_W)) *_h \rho \\ &= ([V \,|\, W] *_h \rho) \cdot ((D_V \oplus D_W) *_h \rho).\end{aligned}$$

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The matrix $(D_V \oplus D_W) *_h \rho$ is diagonal and in particular is the direct sum of r copies of $D_V \oplus D_W$ (by Item 2 of Theorem 2.4, because $D_V \oplus D_W$ is diagonal by the assumption), hence we have proved the following claim.

PROPOSITION 2.8. *With the previous notations, the columns of $[V \mid W] *_h \rho$ are eigenvectors for $P *_h \rho$, and if $\boldsymbol{\lambda}_V$ and $\boldsymbol{\lambda}_W$ denote the eigenvalues for the columns of V and W (i.e., the main diagonals of D_V and D_W), then the eigenvalues corresponding to the columns of $[V \mid W] *_h \rho$ are the sequence $\boldsymbol{\lambda}_V, \boldsymbol{\lambda}_W, \boldsymbol{\lambda}_V, \boldsymbol{\lambda}_W, \ldots, \boldsymbol{\lambda}_V, \boldsymbol{\lambda}_W$ (r couples).*

Probably, the typical use of this computation will be in 'tandem' with Proposition 2.6, to produce a set of eigenvectors for $P *_h \rho$. We illustrate this through a simple example as follows. Consider P with $N = 2$, ρ, h and $P(\omega)$ as given in Remark 2.3, thus

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, \qquad P(\omega) = \begin{bmatrix} 0 & \cos\omega & 0 & -\sin\omega \\ \cos\omega & 2 & \sin\omega & 0 \\ 0 & \sin\omega & 0 & \cos\omega \\ -\sin\omega & 0 & \cos\omega & 2 \end{bmatrix}.$$

The eigenvalues of P (and hence also $P(\omega)$, with multiplicity 2) are $\lambda_\pm := 1 \pm \sqrt{2}$ with $v_\pm := \begin{bmatrix} 1 \\ 1\pm\sqrt{2} \end{bmatrix}$ as corresponding eigenvectors. By extending each $v_\pm$ through $v_\pm *_h \rho$ we form the matrix

$$\begin{aligned} Q(\omega) :=& [v_+ *_h \rho \mid v_- *_h \rho] \\ =& \begin{bmatrix} 1 & 0 & 1 & 0 \\ (1+\sqrt{2})\cos\omega & (1+\sqrt{2})\sin\omega & (1-\sqrt{2})\cos\omega & (1-\sqrt{2})\sin\omega \\ 0 & 1 & 0 & 1 \\ -(1+\sqrt{2})\sin\omega & (1+\sqrt{2})\cos\omega & -(1-\sqrt{2})\sin\omega & (1-\sqrt{2})\cos\omega \end{bmatrix} \end{aligned}$$

for which $P(\omega)Q(\omega) = Q(\omega)\,\mathrm{diag}\{\lambda_+, \lambda_+, \lambda_-, \lambda_-\}$.
Analogously, we can form the other matrix

$$Q'(\omega) := [v_+ \mid v_-] *_h \rho = \begin{bmatrix} 1 & \cos\omega & 0 & -\sin\omega \\ (1+\sqrt{2})\cos\omega & 1-\sqrt{2} & (1+\sqrt{2})\sin\omega & 0 \\ 0 & \sin\omega & 1 & \cos\omega \\ -(1+\sqrt{2})\sin\omega & 0 & (1+\sqrt{2})\cos\omega & 1-\sqrt{2} \end{bmatrix},$$

for which $P(\omega)Q'(\omega) = Q'(\omega)\,\mathrm{diag}\{\lambda_+, \lambda_-, \lambda_+, \lambda_-\}$. Both the procedures give a basis of eigenvectors for $P(\omega)$ from a basis of eigenvectors for P: for $Q(\omega)$ this is a consequence of Proposition 2.6, for $Q'(\omega)$ it is a consequence of Theorem 2.5 (or even by Item 7 in Theorem 2.4). These approaches to the construction of eigenvectors for $P(\omega)$ are in general convenient for applications: to obtain the eigenvectors directly

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from $\ker(P *_h \rho - \lambda \mathbb{I}_4)$ one would need to solve a system of equations involving polynomials in $\sin\omega$ and $\cos\omega$ and this could be a computationally quite difficult task. In fact, although there are computational algebraic methods for solving a system of equations in a polynomial ring with several independent variables $z_1, \ldots, z_k$, e.g., Gröbner bases [2], there is not a constructive method for solving a system of equations in the polynomial ring in $\sin\omega$ and $\cos\omega$ because now the variables $z_1 := \sin\omega$ and $z_2 := \cos\omega$ are algebraically dependent.

In the previous example it is easy to check that $\det Q(\omega) = -8$, $\det Q'(\omega) = 8$ and $\det([v_+ \,|\, v_-]) = \sqrt{8}$: the simple relation between these determinants is not casual, but it is a consequence of a general relation which we explore now.

Let V, W be generic $n \times v$ and $n \times w$ matrices. Then,

$$([V \,|\, W] *_h \rho)_{(I,J;i,j)} = \begin{cases} V_{i,j}\rho_{I,J}(h(i,j)) & \text{if } j \leq v, \\ W_{i,j-v}\rho_{I,J}(h(i,j)) & \text{if } j > v. \end{cases}$$

This proves that there exists a permutation $\mathcal{P}$ such that

$$[V \,|\, W] *_h \rho \cdot \mathcal{P}^{-1} = [V *_h \rho \,|\, S], \tag{2.5}$$

where

$$S \in \mathcal{M}(nr \times wr, \mathbb{C}), \qquad \text{with} \qquad S_{(I,J;i,j)} := W_{i,j}\rho_{I,J}(h(i, j+v)).$$

Noting that $h(i, j+v)$ can be written as $h(i,j) + h(j, j+v)$ and using the multiplicativity of the representation ρ, we get

$$\begin{aligned} S_{(I,J;i,j)} &= W_{i,j}\rho_{I,J}(h(i,j) + h(j,j+v)) \\ &= \sum_K W_{i,j}\rho_{I,K}(h(i,j))\rho_{K,J}(h(j,j+v)) \\ &= \sum_K (W *_h \rho)_{(I,K;i,j)}\rho_{K,J}(h(j,j+v)). \end{aligned}$$

In matricial form this equality can be written as

$$S = (W *_h \rho) \cdot \mathcal{B},$$

where

$$\mathcal{B} = \begin{bmatrix} \mathcal{B}_{1,1} & \ldots & \mathcal{B}_{1,r} \\ \ldots & \ldots & \ldots \\ \mathcal{B}_{r,1} & \ldots & \mathcal{B}_{r,r} \end{bmatrix}, \quad \mathcal{B}_{I,J} := \operatorname{diag}\{\rho_{I,J}(h(1, v+1)), \ldots, \rho_{I,J}(h(w, v+w))\}.$$

This structure proves the existence of two permutations $\mathcal{Q}$, $\mathcal{Q}'$ in $\mathrm{GL}(rw, \mathbb{C})$ such that

$$\mathcal{B} = \mathcal{Q} \bigoplus_{j=1}^{w} \rho(h(j, j+v))\mathcal{Q}'.$$

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Their definition makes evident that $\mathcal{Q}$ and $\mathcal{Q}'$ depend on ρ only via its order; when ρ is the trivial representation both $\mathcal{B}$ and $\bigoplus_{j=1}^{w} \rho(h(j, j+v))$ collapse to the identity, thus proving that $\mathcal{Q}' = \mathcal{Q}^{-1}$. As a consequence we have proved that

$$S = W *_h \rho \cdot \mathcal{Q} \bigoplus_{j=1}^{w} \rho(h(j, j+v)) \mathcal{Q}^{-1}.$$

With (2.5), this equality proves the following formula.

Theorem 2.9. *With the previous notations, we have*

$$[V \,|\, W] *_h \rho = [V *_h \rho \,|\, W *_h \rho] \cdot \big(\mathbb{I}_{rv} \oplus \mathcal{Q} \bigoplus_{j=1}^{w} \rho(h(j, j+v)) \mathcal{Q}^{-1}\big) \cdot \mathcal{P}. \tag{2.6}$$

In particular, the ranks of $[V \,|\, W] *_h \rho$ and of $[V *_h \rho \,|\, W *_h \rho]$ are equal and when $v + w = n$, i.e., when $[V \,|\, W]$ is a square matrix, we have

$$\det([V *_h \rho \,|\, W *_h \rho]) = (-1)^{vw\binom{r}{2}} \det([V \,|\, W] *_h \rho) \det\Big(\rho\big(\sum_{j=1}^{w} h(j+v, j)\big)\Big).$$

Proof. The formula for $\det([V *_h \rho \,|\, W *_h \rho])$ is a direct consequence of (2.6), apart the computation of the determinant of $\mathcal{P}$, for which we need the following explicit description coming directly from its definition in (2.5). Split the integers $\{1, \ldots, vr\}$ in r consecutive blocks denoted as $n_1, \ldots, n_r$ having v integers each one, and analogously split the integers $\{vr+1, \ldots, vr+wr\}$ in r consecutive blocks denoted as $m_1, \ldots, m_r$, having w integers each one. Then $\mathcal{P}$ is the 'shuffle' permutation which moves the blocks according to the following rule:

$$\mathcal{P} : \ (n_1, n_2, \ldots, n_r, m_1, m_2, \ldots, m_r) \to (n_1, m_1, n_2, m_2, \ldots, n_r, m_r).$$

It is now easy to verify that $\det(\mathcal{P}) = (-1)^{vw\binom{r}{2}}$. ◻

Remark 2.10. The permutation $\mathcal{Q}$ in Theorem 2.9 can be concretely described as follows. Each integer n in $\{0, \ldots, wr-1\}$ can be uniquely written both as $a+bw$ and as $a' + b'r$, with $0 \le a, b' < w$ and $0 \le a', b < r$. The map $a + bw \to b + ar$ is therefore a well defined bijection of $\{0, \ldots, wr-1\}$ in itself: $\mathcal{Q}$ is the matrix representing this permutation.

Theorem 2.9 here above explains the equality $\det Q(\omega) = -\det Q'(\omega)$ in our previous example. As we will see now, the other equality $\det Q'(\omega) = (\det([v_+ \,|\, v_-]))^2$ is a consequence of a general formula relating the characteristic polynomial of $P *_h \rho$ to that one of P (see next Theorem 2.13). We will deduce this formula via the Jordan decomposition of P and using the following proposition describing the behavior of the $*_h$-product with respect to a direct sum in its first argument.

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Proposition 2.11. *Let $P \in \mathcal{M}(p,\mathbb{C})$ and $Q \in \mathcal{M}(q,\mathbb{C})$. Then, there is a permutation $\mathcal{P} \in \mathrm{GL}((p+q)r,\mathbb{C})$ such that*

$$\mathcal{P} \cdot \big((P \oplus Q) *_h \rho\big) \cdot \mathcal{P}^{-1} = (P *_h \rho) \oplus (Q *_{h_P} \rho),$$

where $h_P(i,j) := h(i+p, j+p)$.

Proof. We have

$$(P \oplus Q)_{i,j}\rho_{I,J}(h(i,j)) = \begin{cases} P_{i,j}\rho_{I,J}(h(i,j)) & \text{if } i,j \le p, \\ Q_{i-p,j-p}\rho_{I,J}(h_P(i-p,j-p)) & \text{if } i,j > p, \\ 0 & \text{otherwise,} \end{cases}$$

for every I and J. Thus, according to the definition of the $*_h$-product, we see that $(P \oplus Q) *_h \rho$ can be obtained by permuting columns and rows of $(P *_h \rho) \oplus (Q *_{h_P} \rho)$, i.e.,

$$\mathcal{P}((P \oplus Q) *_h \rho)\mathcal{P}' = (P *_h \rho) \oplus (Q *_{h_P} \rho) \tag{2.7}$$

for two suitable permutations $\mathcal{P}$ and $\mathcal{P}'$. The formula also shows that these permutations depend on P and Q only via their orders, thus substituting P and Q with the identities of the same order and using the conclusion in Item 2 of Theorem 2.4, we get that

$$\begin{aligned}\mathcal{P}\mathcal{P}' &= \mathcal{P}((\mathbb{I}_{p+q}) *_h \rho)\mathcal{P}' = \mathcal{P}((\mathbb{I}_p \oplus \mathbb{I}_q) *_h \rho)\mathcal{P}' \\ &= (\mathbb{I}_p *_h \rho) \oplus (\mathbb{I}_q *_{h_P} \rho) = \mathbb{I}_{pr} \oplus \mathbb{I}_{qr} = \mathbb{I}_{(p+q)r},\end{aligned}$$

thus proving that $\mathcal{P}' = \mathcal{P}^{-1}$ in (2.7). ∎

Remark 2.12. The argument in the proof of Proposition 2.11 also shows that $\mathcal{P}$ coincides with the permutation having the same name already described in the proof of Theorem 2.9:

$$\mathcal{P}: \ (n_1, n_2, \ldots, n_r, m_1, m_2, \ldots, m_r) \to (n_1, m_1, n_2, m_2, \ldots, n_r, m_r),$$

where $n_1, \ldots, n_r$ are a partition of $\{1, \ldots, pr\}$ in blocks of consecutive integers having p integers each one, and $m_1, \ldots, m_r$ a partition of $\{pr+1, \ldots, pr+qr\}$ in blocks of consecutive integers having q integers each one.

Theorem 2.13. *The characteristic polynomial of $P *_h \rho$ is the rth power of that one of P.*

Proof. Let $\oplus_l \oplus_m (\lambda_l \mathbb{I}_{n_{l,m}} + \mathbb{J}_{n_{l,m}})$ be the Jordan canonical decomposition of P, where λ_l are the distinct eigenvalues of P and $\{\mathbb{J}_{n_{l,m}}\}_m$ are the Jordan blocks corresponding to the eigenvalue λ_l. Then,

$$P \quad \text{is similar to} \quad \oplus_l \oplus_m (\lambda_l \mathbb{I}_{n_{l,m}} + \mathbb{J}_{n_{l,m}})$$

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and by Items 4 and 7 of Theorem 2.4

$$P *_h \rho \quad \text{is similar to} \quad \big(\oplus_l \oplus_m (\lambda_l \mathbb{I}_{n_{l,m}} + \mathbb{J}_{n_{l,m}})\big) *_h \rho.$$

By Proposition 2.11

$$P *_h \rho \quad \text{is similar to} \quad \oplus_l \oplus_m \big((\lambda_l \mathbb{I}_{n_{l,m}} + \mathbb{J}_{n_{l,m}}) *_{h_{l,m}} \rho\big),$$

where each $h_{l,m}$ is a suitable map satisfying (2.2). Since

$$\mathbb{I}_{rp} = \oplus_l \oplus_m (\mathbb{I}_{n_{l,m}} *_{h_{l,m}} \rho),$$

(by Theorem 2.4, Item 2) we get that

$$x\mathbb{I}_{rp} - P *_h \rho \quad \text{is similar to} \quad \oplus_l \oplus_m (((x-\lambda_l)\mathbb{I}_{n_{l,m}} - \mathbb{J}_{n_{l,m}}) *_{h_{l,m}} \rho). \tag{2.8}$$

Consider a matrix of the form $(\lambda \mathbb{I}_n + \mathbb{J}_n) *_h \rho$. By Item 5 of Theorem 2.4 its minimal polynomial is $(x-\lambda)^n$. Thus, its characteristic polynomial must be a power of $(x-\lambda)$, and hence is $(x-\lambda)^{nr}$, i.e., the rth power of the characteristic polynomial of $\lambda\mathbb{I}_n + \mathbb{J}_n$. The claim now follows by (2.8), by multiplicativity. $\square$

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