

BISHOP'S PROPERTY (β), SVEP AND DUNFORD PROPERTY (C) *

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Abstract. If $S, T \in B(H)$ have Bishop's property (β) , does S + T have Bishop's property (β) ? In this paper, a special case of this question is studied. Also given are a necessary and sufficient condition for a 2 × 2 operator matrix to have Bishop's property (β) . Finally, the Helton class of an operator which has Bishop's property (β) is studied.

Key words. Hyponormal operators, Bishop's property (β), SVEP, Dunford property (C).

AMS subject classifications. 47B47, 47A30, 47B20, 47B10.

1. Introduction. Let B(H) be the algebra of all bounded linear operators acting on infinite dimensional separable complex Hilbert space H. An operator $T \in B(H)$ is said to have the single-valued extension property (or SVEP) if for every open subset G of \mathbb{C} and any analytic function $f: G \to H$ such that $(T-z)f(z) \equiv 0$ on G, we have $f(z) \equiv 0$ on G. For $T \in B(H)$ and $x \in H$, the set $\rho_T(x)$ is defined to consist of elements $z_0 \in \mathbb{C}$ such that there exists an analytic function f(z) defined in a neighborhood of z_0 , with values in H, which verifies (T-z)f(z) = x, and it is called the *local resolvent set of* T at x. We denote the complement of $\rho_T(x)$ by $\sigma_T(x)$, called the local spectrum of T at x, and define the local spectral subspace of $T, H_T(F) = \{x \in H : \sigma_T(x) \subset F\}$ for each subset F of C. Bishop [1] introduced Bishop's property (β). The study of operators satisfying Bishops property (β) is of significant interest and is currently being done by a number of mathematicians around the world [12, 13]. An operator $T \in B(H)$ is said to have Bishop's property (β) if for every open subset G of \mathbb{C} and every sequence $f_n: G \to H$ of H-valued analytic functions such that $(T-z)f_n(z)$ converges uniformly to 0 in norm on compact subsets of G, $f_n(z)$ converges uniformly to 0 in norm on compact subsets of G. An operator $T \in B(H)$ is said to have Dunford's property (C) if $H_T(F)$ is closed for each closed subset F of \mathbb{C} . It is well known that

Bishop's property(β) \Rightarrow Dunford's property(C) \Rightarrow SVEP.

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In particular, the single valued extension property of operators was first introduced by N. Dunford to investigate the class of spectral operators which is another important generalization of normal operators (see [3]). In the local spectral theory, given an operator T on a complex Banach space \mathcal{X} and a vector $x \in \mathcal{X}$, one is often interested in the existence and the uniqueness of analytic solution $f(.) : U \to \mathcal{X}$ of the local resolvent equation

$$(T - \lambda)f(\lambda) = x$$

on a suitable open subset U of \mathbb{C} . Obviously, if T has SVEP, then the existence of analytic solution to any local resolvent equation (related to T) implies the uniqueness of its analytic solution. The SVEP is possessed by many important classes of operators such as hyponormal operators and decomposable operators [2, 11]. To emphasize the significance of Bishop's property (β), we mention the important connections to sheaf theory and the spectral theory of several commuting operators from the monograph by Eschmeier and Putinar [4]. There are also interesting applications to invariant subspaces [4], harmonic analysis [5], and the theory of automatic continuity [10]. Unfortunately, but perhaps not surprisingly, the direct verification of property (β) in concrete cases tends to be a difficult task. It is therefore desirable to have sufficient conditions for property (β) which are easier to handle.

In [7], Helton initiated the study of operators T satisfying

$$T^{*m} - \begin{pmatrix} m \\ 1 \end{pmatrix} T^{*m-1}T + \dots + (-1)^m T^m = 0.$$

Let $R, S \in B(H)$. In [8], the authors studied the operator $C(R, S) : B(H) \to B(H)$ defined by C(R, S)(A) = RA - AS. Then

$$C(R,S)^{k}(I) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} R^{j} S^{k-j}.$$

DEFINITION 1.1. [8] Let $R \in B(H)$. If there is an integer $k \ge 1$ such that an operator S satisfies $C(R, S)^k(I) = 0$, we say that S belongs to the Helton class of R. We denote this by $S \in \text{Helton}_k(R)$.

We remark that $C(R, S)^k(I) = 0$ does not imply $C(S, R)^k(I) = 0$ in general [8].

EXAMPLE 1.2. [8] Let $S, R \in B(H \oplus H \oplus H)$ be defined by the following matrix operators:

$$S = \begin{bmatrix} 0 & A & B \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 0 & 0 & C \\ 0 & 0 & D \\ 0 & 0 & 0 \end{bmatrix}$$



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where A, B, C and D are bounded linear operators defined on H. Then it is easy to calculate that $C(R, S)^2(I) = 0$, but $C(S, R)^2(I) \neq 0$.

If $S, T \in B(H)$ have Bishop's property (β) , does S + T have Bishop's property (β) ? So far we do not know the answer to this question. In this paper, we study a special case of this question. We also give a necessary and sufficient condition for 2×2 operator matrix to have Bishop's property (β) . Finally, we study the Helton class of an operator which has Bishop's property (β) .

2. Main results. First we consider the case of nilpotent perturbation.

THEOREM 2.1. Let T = S + N be in B(H), where SN = NS and $N^k = 0$ for some nonnegative integer k. Then S has Bishop's property (β) if and only if T does.

Proof. Let $f_n: U \to H$ be any sequence of analytic functions on an arbitrary open set U such that $(\lambda I - T)f_n(\lambda) \to 0$ as $n \to \infty$ uniformly on all compact subsets of U. Then $(\lambda I - S)f_n(\lambda) - Nf_n(\lambda) \to 0$ as $n \to \infty$ uniformly on all compact subsets of U. Since $N^k = 0$ and SN = NS, $(\lambda I - S)N^{k-1}f_n(\lambda) \to 0$ as $n \to \infty$ uniformly on all compact subsets of U. Since S has Bishop's property (β) , $N^{k-1}f_n(\lambda) \to 0$ as $n \to \infty$ uniformly on all compact subsets of U. Since $(\lambda I - S)N^{k-2}f_n(\lambda) \to 0$ as $n \to \infty$ uniformly on all compact subsets of U and S has Bishop's property (β) , $N^{k-2}f_n(\lambda) \to 0$ as $n \to \infty$ uniformly on all compact subsets of U. By induction, we can show that $f_n(\lambda) \to 0$ as $n \to \infty$ uniformly on all compact subsets of U. Hence, T has Bishop's property (β) , SVEP and Dunford property (C).

The converse implication is similar. \Box

REMARK 2.2. Recall that if T = S + N are in B(H), where S is similar to a hyponormal operator (i.e, $S^*S \ge SS^*$), SN = NS and $N^k = 0$, then T is called a hypo-Jordan operator of order k [9]. Since every hyponormal operator has Bishop's property (β), SVEP and Dunford property (C), from Theorem 2.1 we get the following corollary.

COROLLARY 2.3. Every hypo-Jordan operator of order k has Bishop's property (β) , SVEP and Dunford property (C).

Next we consider another special case of our question.

THEOREM 2.4. Let R, S in B(H) have Bishop's property (β). If T = R + S, where SR = 0, then T has Bishop's property (β).

Proof. Let $f_n : U \to H$ be any sequence of analytic functions on the open set U such that $(\lambda I - T)f_n(\lambda) \to 0$ as $n \to \infty$ uniformly on all compact subsets of U. Then $(\lambda I - R - S)f_n(\lambda) \to 0$ as $n \to \infty$ uniformly on all compact subsets of U. Since SR = 0, we get $(\lambda I - S)Sf_n(\lambda) \to 0$ as $n \to \infty$ uniformly on all compact subsets of U.

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Since S has Bishop's property (β) , $Sf_n(\lambda) \to 0$ as $n \to \infty$ uniformly on all compact subsets of U. Since $(\lambda I - R - S)f_n(\lambda) \to 0$ as $n \to \infty$ uniformly on all compact subsets of U, $(\lambda I - R)f_n(\lambda) \to 0$ as $n \to \infty$ uniformly on all compact subsets of U. Since R has Bishop's property (β) , we have $f_n(\lambda) \to 0$ as $n \to \infty$ uniformly on all compact subsets of U. Hence, T has Bishop's property (β) . \square

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THEOREM 2.5. Let H and K be infinite complex Hilbert spaces and let $T \in B(H \oplus K)$ be the operator matrix of the form

$$T = \left[\begin{array}{cc} T_1 & T_2 \\ 0 & T_3 \end{array} \right].$$

Assume T_3 has Bishop's property β . Then the following assertions are equivalent:

- (i) T has Bishop's property β .
- (ii) T_1 has Bishop's property β .

Proof. (i) \Rightarrow (ii): Assume that T has Bishop's property β . Let $f_n : U \to H$ be any sequence of analytic functions on the open set U such that $(T_1 - \lambda I)f_n(\lambda) \to 0$ as $n \to \infty$ uniformly on all compact subsets of U. Define $h_n : U \to H \oplus K$ by

$$h_n(\lambda) = \left[\begin{array}{c} f_n(\lambda) \\ 0 \end{array} \right]$$

for all $\lambda \in \mathbb{C}$ and for all $n \in \mathbb{N}$. Then (h_n) is a sequence of analytic functions on U. Hence,

$$(T - \lambda I)h_n(\lambda) = \begin{bmatrix} T_1 - \lambda I & T_2 \\ 0 & T_3 - \lambda I \end{bmatrix} \begin{bmatrix} f_n(\lambda) \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} (T_1 - \lambda I)f_n(\lambda) \\ 0 \end{bmatrix} \to 0$$

as $n \to \infty$ uniformly on all compact subsets of U. Since T has Bishop's property β , $f_n(\lambda) \to 0$ as $n \to \infty$ uniformly on all compact subsets of U. Hence, T_1 has Bishop's property β .

(ii) \Rightarrow (i): Assume T_1 has Bishop's property β . Let U be any open set and let $g_n : U \to H \oplus K$ be any sequence of analytic functions on U such that $(T - \lambda I)g_n(\lambda) \to 0$ as $n \to \infty$ uniformly on all compact subsets of U. Put

$$g_n(\lambda) = \left[\begin{array}{c} g_{1n}(\lambda) \\ g_{2n}(\lambda) \end{array}\right]$$

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for every $\lambda \in U$, where g_{1n} and g_{2n} are analytic functions on U for all $n \in \mathbb{N}$. Assume $(T - \lambda I)g_n(\lambda) \to 0$ as $n \to \infty$ uniformly on all compact subsets of U. Then

$$(T - \lambda I)g_n(\lambda) = \begin{bmatrix} T_1 - \lambda I & T_2 \\ 0 & T_3 - \lambda I \end{bmatrix} \begin{bmatrix} g_{1n}(\lambda) \\ g_{2n}(\lambda) \end{bmatrix}$$
$$= \begin{bmatrix} (T_1 - \lambda I)g_{1n}(\lambda) + T_2g_{2n}(\lambda) \\ (T_3 - \lambda)g_{2n}(\lambda) \end{bmatrix} \rightarrow 0$$

as $n \to \infty$ uniformly on all compact subsets of U. Since T_3 has Bishop's property β , $T_2g_{2n}(\lambda) \to 0$ as $n \to \infty$ uniformly on all compact subsets of U. Therefore $(T_1 - \lambda I)g_{1n} \to 0$ as $n \to \infty$ uniformly on all compact subsets of U. Since T_1 has Bishop's property β , $g_n(\lambda) \to 0$ as $n \to \infty$ uniformly on all compact subsets of U. Therefore T has Bishop's property β . \Box

Now we consider the Helton class of an operator which has Bishop's property (β).

THEOREM 2.6. Let $R \in B(H)$ has Bishop's property (β). If $S \in Helton_k(R)$, then S has Bishop's property (β).

Proof. Let $f_n: U \to H$ be any sequence of analytic functions on U (any open set) such that $(\lambda I - S)f_n(\lambda) \to 0$ as $n \to \infty$ uniformly on all compact subsets of U. Since the terms of the below equation equal to zero when $j + s \neq r$, it suffices to consider only the case of j + s = r. Then we have

$$\begin{split} \sum_{j=0}^k (-1)^{k-j} \begin{pmatrix} k \\ j \end{pmatrix} (R-\lambda)^j (\lambda-S)^{k-j} \\ &= \sum_{j=0}^k \sum_{s=0}^j \sum_{s=0}^{k-j} (-1)^{k-(s+r)} \begin{pmatrix} k \\ j \end{pmatrix} \begin{pmatrix} j \\ r \end{pmatrix} \begin{pmatrix} k-j \\ s \end{pmatrix} R^j S^{k-j} \\ &= \sum_{j=0}^k (-1)^{k-j} \begin{pmatrix} k \\ j \end{pmatrix} R^j S^{k-j}. \end{split}$$

Thus, we have

$$\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} R^{j} S^{k-j} - (R-\lambda)^{k} f_{n}(\lambda)$$
$$= \sum_{j=0}^{k} \binom{k}{j} (R-\lambda)^{j} (\lambda-S)^{k-j} - (R-\lambda)^{k} f_{n}(\lambda)$$

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$$= \sum_{j=0}^{k-1} \binom{k}{j} (R-\lambda)^j (\lambda-S)^{k-j} f_n(\lambda)$$
$$= \left[\sum_{j=0}^{k-1} \binom{k}{j} (R-\lambda)^j (\lambda-S)^{k-j-1} \right] (\lambda-S) f_n(\lambda) \to 0$$

as $n \to \infty$ uniformly on all compact subsets of U. Since

$$\sum_{j=0}^{k} (-1)^{k-j} \begin{pmatrix} k \\ j \end{pmatrix} R^{j} S^{k-j} = 0,$$

we have $(R-\lambda)^k f_n(\lambda) \to 0$ as $n \to \infty$ uniformly on all compact subsets of U. Since R has Bishop's property (β) , $(R-\lambda)^{k-1} f_n(\lambda) \to 0$ as $n \to \infty$ uniformly on all compact subsets of U. By induction we get that $f_n(\lambda) \to 0$ as $n \to \infty$ uniformly on all compact subsets of U. Hence, S has Bishop's property (β) . \square

In the following theorem, we will study a special case of our question for the Helton class of operators.

THEOREM 2.7. If R has Bishop's property (β) , $S \in Helton_k(R)$ and RS = SR, then T = R + S has Bishop's property (β) .

Proof. It is easy to see that $C(2R, S)^k(I) = C(R, S)^k(I) = 0$. Hence, $T = R + S \in$ Helton_k(2R). Since 2R has Bishop's property (β), it follows from Theorem 2.1 that T has Bishop's property (β). \square

DEFINITION 2.8. We say that a certain property (P) (of operators on a Hilbert space H) is a bad property [6] if the following three conditions are fulfilled:

1. If A has property (P), then $\alpha + \beta A$ has the property (P) for all $\alpha \in \mathbb{C}$ and $\beta \neq 0$.

2. If A has property (P) and T is similar to A, then T has the property (P), and

3. If A has property (P) and $\sigma(A) \cap \sigma(B) = \emptyset$, then $A \oplus B$ has the property (P), where $A \oplus B$ denote the orthogonal direct sum of A and B.

Define an operator T has the property (P) by "T does not have Bishop's property (β) ". Let

 $\mathcal{P} = \{T \in B(H) : T \text{ does not have Bishop's property } \beta \}$

LEMMA 2.9. [6, Theorem 3.51] If (P) is a bad property and there exists some operator A with the property (P), then

$$\{T \in B(H) : T \text{ satisfies}(P)\}$$



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is dense in B(H).

LEMMA 2.10. The set

 $\mathcal{P} = \{T \in B(H) : T \text{ does not have Bishop's property}(\beta)\}$

is dense in B(H).

Proof. It is not difficult to verify that \mathcal{P} is a bad property. Hence, it follows from Lemma 2.9 that \mathcal{P} is dense in B(H). \square

By using Lemma 2.10 the following result is clear.

THEOREM 2.11. Given $T \in B(H)$ and $\epsilon > 0$, there exists $S \in B(H)$ with $||S|| < \epsilon$ such that T + S does not have Bishop's property (β).

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