A NOTE ON THE LEAST EIGENVALUE OF A GRAPH WITH GIVEN MAXIMUM DEGREE

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Abstract. This note investigates the least eigenvalues of connected graphs with \( n \) vertices and maximum degree \( \Delta \), and characterizes the unique graph whose least eigenvalue achieves the minimum among all the connected graphs with \( n \) vertices and maximum vertex degree \( \Delta > \frac{n}{2} \).

Key words. Maximum vertex degree, Least eigenvalue, Spectral radius.

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1. Introduction. Throughout this paper all graphs are finite and simple. Graph theoretical terms used but not defined follow [3].

Let \( G = (V(G), E(G)) \) be a graph with \( n \) vertices. By \( G - U \) we mean the induced subgraph \( G[V - U] \) if \( U \subset V(G) \). Denote by \( N_G(v) \) (or \( N(v) \) for short) the set of all neighbors of \( v \) in \( G \). The adjacency matrix of \( G \) is

\[
A(G) = (a_{ij})_{n \times n}, \quad a_{ij} = 1 \quad \text{if two vertices } v_i \text{ and } v_j \text{ are adjacent in } G \quad \text{and } a_{ij} = 0 \text{ otherwise.}
\]

All eigenvalues of \( A(G) \) are real and can be arranged in order as

\[
\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)
\]

since \( A(G) \) is a real symmetric matrix. The largest eigenvalue \( \lambda_1(G) \) of \( A(G) \) is called the spectral radius of \( G \), denoted by \( \rho(G) \). The least eigenvalue \( \lambda_n(G) \) is also denoted by \( \lambda_{\min}(G) \). Assume that \( x = (x_v, x_{v_2}, \ldots, x_{v_n})^T \in \mathbb{R}^n \) and \( x \) is a unit eigenvector of \( A(G) \) corresponding to \( \lambda_{\min}(G) \). Then by the Rayleigh-Ritz Theorem, we have

\[
\lambda_{\min}(G) = \min_{y \in \mathbb{R}^n, ||y||=1} y^T A(G) y = x^T A(G) x = 2 \sum_{v_i, v_j \in E(G)} x_{v_i} x_{v_j}
\]

and

\[
\lambda_{\min}(G) x_v = \sum_{u \in N_G(v)} x_u, \quad \text{for each } v \in V(G).
\]

The research for the least eigenvalue of graphs in some class is well-studied and interesting. For example, Bell et al. [1, 2] studied the extremal graphs with \( n \) vertices...
and $m$ edges having the minimal least eigenvalues. Constantine \cite{5} showed that

$$\lambda_{\text{min}}(G) \geq -\sqrt{\left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil}$$

for any graph of order $n$, where equality holds if and only if $G = K_{\left\lfloor \frac{n}{2} \right\rfloor, \left\lceil \frac{n}{2} \right\rceil}$. Cioaba, Gregory, and Nikiforov \cite{4} also proved that if $G$ is a graph of order $n$ with diameter $D$ and maximum vertex degree $\Delta$, then

$$\lambda_{\text{min}}(G) \geq -\Delta + 1 + \frac{1}{n(D+1)}.$$  \hspace{1cm} (1.3)

Fan et al. \cite{7} obtained unicyclic graphs having the minimal least eigenvalues. Liu et al. \cite{8} determined the graph with the minimal least eigenvalue among all unicyclic graphs with a given number of pendant vertices. Petrović et al. \cite{9} obtained the bicyclic graph minimizing the least eigenvalue. Wang and Fan \cite{10} gave the graph of order $n$ with $k$ cut vertices having the minimal least eigenvalue. Ye et al. \cite{11} discussed the least eigenvalues of graphs with given connectivity. Zhu \cite{12} also consider the least eigenvalues of graphs with cut vertices or edges.

In this note, we consider the following problem: what is the structure of the graph having the minimal least eigenvalue among all connected graphs with $n$ vertices and maximum vertex degree $\Delta$?

Let the complete bipartite graph $K_{p,q}$ have the vertex bipartition $(V_1, V_2)$, where $V_1 = \{v_1, v_2, \ldots, v_p\}$ and $V_2 = \{v_{p+1}, \ldots, v_{p+q}\}$. Let the graph $H_{p,q}^s,t$ be obtained from $K_{p,q}$ by adding a new vertex $u$ and joining $u$ to $s$ vertices of $V_1$ and $t$ vertices of $V_2$.

From the above result of Constantine, we know that

$$\lambda_{\text{min}}(G) \geq -\sqrt{\left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil}$$

for any connected graph $G$ with $n$ vertices and maximum degree $\Delta = \left\lceil \frac{n}{2} \right\rceil$, where equality holds if and only if $G = K_{\left\lfloor \frac{n}{2} \right\rfloor, \left\lceil \frac{n}{2} \right\rceil}$. The main result of this note is the following:

**Theorem 1.1.** Of all the connected graphs of order $n$ with given maximum degree $\Delta > \left\lfloor \frac{n}{2} \right\rfloor$, (i) the graph with the minimal least eigenvalue is isomorphic to $H_{\left\lfloor \frac{n}{2} \right\rfloor + 2, \left\lfloor \frac{n}{2} \right\rceil - 2}$ for $\Delta = \left\lfloor \frac{n}{2} \right\rfloor + 1$ and $n \geq 5$; (ii) the graph with the minimal least eigenvalue is isomorphic to either $H_{\left\lfloor \frac{n}{2} \right\rfloor + 1, \left\lfloor \frac{n}{2} \right\rceil - 1}$ or $H_{\left\lfloor \frac{n}{2} \right\rfloor + 1, \left\lfloor \frac{n}{2} \right\rceil - 1}$ for $\Delta > \left\lfloor \frac{n}{2} \right\rfloor + 1$.

**Remark 1.2.** From the proof of Theorem 1.1, we know that $\lambda_{\text{min}}(G) > -\sqrt{\left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil}$ if $G$ is a connected graph of order $n$ with maximum degree $\Delta > \left\lfloor \frac{n}{2} \right\rfloor$. Thus,

$$\lambda_{\text{min}}(G) > -\sqrt{\left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil} > -\Delta + \frac{1}{n(D+1)}.$$
This improves the bound in Eq. (1.3).

By this theorem, we further obtain the next result.

**Corollary 1.3.** Let \( G \) be a connected graph of order \( n \) with given maximum degree \( \Delta > \lceil \frac{n}{2} \rceil + 1 \).

(i) If \( n \) is even for \( \lceil \frac{n}{2} \rceil + 1 < \Delta \leq \frac{3n-2-\sqrt{3n^2-12n-2}}{2} \) or \( n \) is odd for \( \lceil \frac{n}{2} \rceil + 1 < \Delta \leq \frac{3n-2-\sqrt{2n^2-4n+3}}{2} \), then we have

\[
\lambda_{\min}(G) \geq \lambda_{\min}\left(H\left[\frac{n-1}{2}\right]_{\Delta-\lceil \frac{n}{2} \rceil-1}\right)
\]

with equality if and only if \( G \equiv H\left[\frac{n-1}{2}\right]_{\Delta-\lceil \frac{n}{2} \rceil-1} \).

(ii) If \( n \) is even for \( \frac{3n-2-\sqrt{3n^2-12n-2}}{2} \leq \Delta \leq n-1 \) or \( n \) is odd for \( \frac{3n-2-\sqrt{2n^2-4n+3}}{2} \leq \Delta \leq n-1 \), then we have

\[
\lambda_{\min}(G) \geq \lambda_{\min}\left(H\left[\frac{n-1}{2}\right]_{\Delta-\lceil \frac{n}{2} \rceil-1}\right)
\]

with equality if and only if \( G \equiv H\left[\frac{n-1}{2}\right]_{\Delta-\lceil \frac{n}{2} \rceil-1} \).

**Remark 1.4.** In general, it is hard to characterize the corresponding extremal graphs having the minimal least eigenvalue over all the connected graphs of order \( n \) with given maximum degree \( \Delta < \frac{n}{2} \). It is well known that \( \rho(G) \leq \Delta \) with equality if and only if \( G \) is regular, and \( G \) is bipartite if and only if \( \rho(G) = -\lambda_{\min}(G) \) (see [6]). Thus, if \( G \) is a connected graph of order \( n \) with maximum degree \( \Delta < \frac{n}{2} \), then \( \lambda_{\min}(G) \geq -\Delta \) with equality if and only if \( G \) is a regular bipartite graph.

**2. Proof of Theorem 1.1.** Let \( G \) be a connected graph of order \( n \) with given maximum vertex degree \( \Delta > \lceil \frac{n}{2} \rceil \). Assume that \( \lambda_{\min}(G) \) is as small as possible and that \( x = (x_1, x_2, \ldots, x_n)^T \) is a unit eigenvector of \( A(G) \) corresponding to \( \lambda_{\min}(G) \). Denote by \( V^+ = \{ v \in V(G) : x_v > 0 \} \), \( V^- = \{ v \in V(G) : x_v < 0 \} \) and \( V^0 = \{ v \in V(G) : x_v = 0 \} \). We will divide the following proof into two cases.

**Case 1.** If \( |V^0| = 0 \), then we have \( n - \Delta \leq |V^-|, |V^+| \leq \Delta \). Otherwise, we have

\[
\lambda_{\min}(G) = 2 \sum_{v_i,v_j \in E(G)} x_{v_i}x_{v_j} \geq 2 \sum_{v_i \in V^-, v_j \in V^+} x_{v_i}x_{v_j} \geq \lambda_{\min}(K_{|V^-|,|V^+|}) = -\sqrt{|V^-||V^+|} > -\sqrt{\Delta(n-\Delta)} = \lambda_{\min}(K_{\Delta, n-\Delta}),
\]
a contradiction. On the other hand, if \( \{|V^-|,|V^+|\} = \{n-\Delta,\Delta\} \), then every vertex in \( V^+ \) must be adjacent to all vertices in \( V^- \) by Eq. (1.1) since \( \lambda_{\min}(G) \) is as small as possible, which implies that \( G = K_{n-\Delta,\Delta} \). Thus, in what follows we only need to consider the case for \( n-\Delta+1 \leq |V^-|,|V^+| \leq \Delta-1 \).

Let \( u \) be a vertex with maximum degree. Without loss of generality, we can assume that \( x_u > 0 \). Consequently, we obtain that every vertex in \( V^+ \) must be adjacent to all vertices in \( V^- \); otherwise there exists two disjunct vertices \( u_1 \in V^+ \) and \( v_1 \in V^- \), and then by Eq. (1.1), we have a graph \( G + u_1v_1 \) with maximum degree \( \Delta \) and

\[
\lambda_{\min}(G) = 2 \sum_{v_i,v_j \in E(G)} x_{v_i}x_{v_j} > 2 \sum_{v_i,v_j \in E(G)} x_{v_i}x_{v_j} + 2x_u{x}_{v_1} \geq \lambda_{\min}(G + u_1v_1)
\]

since \( G \) is connected, which contradicts that \( \lambda_{\min}(G) \) is as small as possible. Thus, we can write \( G \) as \( H_{s,n-s-1}^{\Delta-s} \) if we assume that \( |V^-| = s \) and \( |V^+| = n-s \). By Eq. (1.2), \( x \) has a constant value on the vertices of \( V^- \) (respectively, \( V^+ \cap N(u) \) and \( V^+ \setminus N(u) \)), denoted by \( x_1 \) (respectively, \( x_2 \) and \( x_3 \)), e.g., \( \lambda_{\min}(G)x_{v_1} = \sum_{t \in V^+} x_t = \lambda_{\min}(G)x_{u_2} \) for any two vertices \( v_1, v_2 \in V^- \), which implies that \( x_{v_1} = x_{v_2} \) since \( \lambda_{\min}(G) < 0 \). Hence, by Eq. (1.2), we have the following equations

\[
\begin{align*}
\lambda_{\min}(G)x_u &= sx_1 + (\Delta - s)x_2, \\
\lambda_{\min}(G)x_1 &= x_u + (\Delta - s)x_2 + (n - \Delta - 1)x_3, \\
\lambda_{\min}(G)x_2 &= sx_1 + x_u, \\
\lambda_{\min}(G)x_3 &= sx_1.
\end{align*}
\]

This implies that \( \lambda_{\min}(G) \) is the least root of the following equation in \( \lambda \):

\[
(2.1) \quad \lambda^4 - [s(n - 1 - s) + \Delta]\lambda^2 + (\Delta - s)s(n - 1 - \Delta) - 2(\Delta - s)s\lambda = 0.
\]

From Eq. (2.1), we know that \( \lambda_{\min}\left(\begin{bmatrix} \frac{n-1}{2} \\ \frac{n-1}{2} \\ -\Delta \end{bmatrix}\right) \) is the least root of the following equation:

\[
\lambda^4 - \left(\begin{bmatrix} \frac{n-1}{2} \\ \frac{n-1}{2} \\ -\Delta \end{bmatrix} + \Delta \right)\lambda^2 + \left(\Delta - \begin{bmatrix} \frac{n-1}{2} \\ \frac{n-1}{2} \end{bmatrix}\right) \left(\begin{bmatrix} \frac{n-1}{2} \\ \frac{n-1}{2} \end{bmatrix} \right) (n - 1 - \Delta - 2\lambda) = 0.
\]

Note that

\[
\lambda^4 - \left(\begin{bmatrix} \frac{n-1}{2} \\ \frac{n-1}{2} \\ -\Delta \end{bmatrix} + \Delta \right)\lambda^2 + \left(\Delta - \begin{bmatrix} \frac{n-1}{2} \\ \frac{n-1}{2} \end{bmatrix}\right) \left(\begin{bmatrix} \frac{n-1}{2} \\ \frac{n-1}{2} \end{bmatrix} \right) (n - 1 - \Delta - 2\lambda) \leq 0
\]
for \( \lambda = -\sqrt{\left\lfloor \frac{n-1}{2} \right\rfloor \left\lceil \frac{n-1}{2} \right\rceil} \), where equality holds if and only if \( \Delta = n - 1 \) is even, which implies that

\[
\lambda_{\text{min}}(G) \leq \lambda_{\text{min}} \left( H \left( \left\lfloor \frac{n-1}{2} \right\rfloor, \left\lceil \frac{n-1}{2} \right\rceil \right) \right) \leq -\sqrt{\left\lfloor \frac{n-1}{2} \right\rfloor \left\lceil \frac{n-1}{2} \right\rceil},
\]

where equality in the second inequality holds if and only if \( \Delta = n - 1 \) is even.

Hence, we have \(-\sqrt{\left\lfloor \frac{n-1}{2} \right\rfloor \left\lceil \frac{n-1}{2} \right\rceil} < \lambda_{\text{min}}(G) \leq -\sqrt{\left\lfloor \frac{n-1}{2} \right\rfloor \left\lceil \frac{n-1}{2} \right\rceil} \), which can be seen from the result of Constantine and inequalities in (2.2). Note that \(-\sqrt{\Delta(n-\Delta)} \) is also the least root of the above Eq. (2.1) for \( s = \Delta \). Thus, in the following we will consider the least root of above Eq. (2.1) for \( n - \Delta + 1 \leq s \leq \Delta \) since \( \lambda_{\text{min}}(G) = -\sqrt{\Delta(n-\Delta)} \) for \( \{ |V^-|, |V^+| \} = \{ n - \Delta, \Delta \} \).

Assume that \( f(s) = \lambda^4 - [s(n-1-s) + \Delta]\lambda^2 + (\Delta-s)s(n-1+\Delta) - 2(\Delta-s)s\lambda \). Then \( f(s) = [(\lambda+1)^2 - n + \Delta]s^2 - [(n-1)\lambda^2 + 2\lambda\Delta - \Delta(n-1-\Delta)]s + \lambda^4 - \Delta \lambda^2 \) is a quadratic function in \( s \), whose axis of symmetry is

\[
s_0 = \frac{(n-1)\lambda^2 + 2\lambda \Delta - \Delta(n-1-\Delta)}{2[(\lambda+1)^2 - n + \Delta]} = \frac{n-1}{2} + \frac{(n-1-\Delta)(n-1-\Delta-2\lambda)}{2[(\lambda+1)^2 - n + \Delta]}.
\]

To consider the minimal least root of Eq. (2.1) for \( n - \Delta + 1 \leq s \leq \Delta \), we only need to know when \( f(s) \) attains its minimum. We need the following two claims.

**Claim 2.1.** Let \(-\sqrt{\left\lfloor \frac{n-1}{2} \right\rfloor \left\lceil \frac{n-1}{2} \right\rceil} < \lambda \leq -\sqrt{\left\lfloor \frac{n-1}{2} \right\rfloor \left\lceil \frac{n-1}{2} \right\rceil} \) and \( \left\lfloor \frac{n}{2} \right\rfloor + 1 < \frac{n+1}{2} \leq \Delta \). Then we have \( (\lambda+1)^2 - n + \Delta > 0 \) and

\[
0 \leq \frac{(n-1-\Delta)(n-1-\Delta-2\lambda)}{2[(\lambda+1)^2 - n + \Delta]} < \frac{3}{2}
\]

where \( (n-1-\Delta)(n-1-\Delta-2\lambda) = 0 \) if and only if \( \Delta = n - 1 \).

**Proof.** In fact, it is clear for the lower bound, and the upper bound is equivalent to

\[-\Delta^2 + (2n+1-2\lambda)\Delta - n^2 - n + 2n\lambda + 3\lambda^2 + 4\lambda + 2 > 0.
\]

It suffices to show that

\[
-\frac{(n+4)^2}{4} + \frac{(2n+1-2\lambda)(n+4)}{2} - n^2 - n + 2n\lambda + 3\lambda^2 + 4\lambda + 2 \\
= \frac{3n}{2} - \frac{n^2}{4} + n\lambda + 3\lambda^2 \\
> 0,
\]

where equality holds if and only if \( \Delta = n - 1 \) is even.
which is trivial for \( -\sqrt{\frac{1}{2}} \leq \lambda \leq -\sqrt{\frac{n-1}{2}} \).

Similarly, we can easily prove the next claim.

**Claim 2.2.** Let \( -\sqrt{\frac{1}{2}} \leq \lambda \leq -\sqrt{\frac{n-1}{2}} \), \( \Delta = \lceil \frac{n}{2} \rceil + 1 \) and \( n \geq 5 \). Then we have

\[
\left\lceil \frac{n-1}{2} \right\rceil + 2 < \frac{n-1}{2} + \frac{(n-1-\Delta)(n-1-\Delta-2\lambda)}{2(\lambda+1)^2 - n + \Delta} < \left\lceil \frac{n-1}{2} \right\rceil + \frac{5}{2}.
\]

Thus, by virtue of Claim 2.1, \( f(s) \) can attain the minimum value only at \( s = \lfloor \frac{n-1}{2} \rfloor + 1 \) or \( \lfloor \frac{n-1}{2} \rfloor = 1 \) for \( \lfloor \frac{n-1}{2} \rfloor + 2 \leq \Delta < n - 1 \), and \( f(s) \) can attain the minimum \( f(\lfloor \frac{n-1}{2} \rfloor) = f(\lfloor \frac{n-1}{2} \rfloor) \) for \( \Delta = n - 1 \) if and only if \( s = \lfloor \frac{n-1}{2} \rfloor \) or \( s = \lfloor \frac{n-1}{2} \rfloor + 1 \). It follows that if \( \Delta > \lfloor \frac{n}{2} \rfloor + 1 \) then the graph \( G \) with the minimal least eigenvalue is isomorphic to \( H[\frac{n-1}{2}], \frac{n-1}{2} \] or \( H[\frac{n-1}{2} + 1, \frac{n-1}{2} - 1] \). In addition, in view of Claim 2.2, \( f(s) \) can attain the minimum value only at \( s = \lfloor \frac{n-1}{2} \rfloor + 2 \) for \( \Delta = \lfloor \frac{n}{2} \rfloor + 1 \) and \( n \geq 5 \). This completes the proof of the Case 1.

**Case 2.** If \( |V^0| \neq 0 \), then let \( y \) be a subvector of \( x \) by deleting the entries corresponding to \( v \in V^0 \). By Eq. (1.1), we have

\[
\lambda_{\min}(G) = 2 \sum_{v_i v_j \in E(G)} x_{v_i} x_{v_j} = y^T A(G - V^0) y = \lambda_{\min}(G - V^0) \geq -\sqrt{\left\lceil \frac{n-1}{2} \right\rceil \left\lfloor \frac{n-1}{2} \right\rfloor},
\]

where the equality in the first inequality holds if and only if \( y \) is a least vector of \( G - V^0 \), and the equality in the second inequality holds if and only if \( G - V^0 \cong K[\frac{n-1}{2}, \frac{n-1}{2}] \). Thus, if \( |V^0| \neq 0 \), then we have

\[
\lambda_{\min}(G) \geq -\sqrt{\left\lceil \frac{n-1}{2} \right\rceil \left\lfloor \frac{n-1}{2} \right\rfloor}
\]

with equality only when \( V^0 = \{u\} \) and \( G = H_{\left\lfloor \frac{n-1}{2} \right\rfloor + 1, \left\lfloor \frac{n-1}{2} \right\rfloor} \), for some positive integer \( b \). This completes the proof of the Case 2.

Finally, if the extremal graph \( G \) belongs to Case 2, then

\[
\lambda_{\min}(G) \geq -\sqrt{\left\lceil \frac{n-1}{2} \right\rceil \left\lfloor \frac{n-1}{2} \right\rfloor}
\]
with equality only when $V^0 = \{ u \}$ and $G = H^{b, \Delta - b}$, for some positive integer $b$; if the extremal graph $G$ belongs to Case 1, then by inequalities (2.2) we have

$$\lambda_{\min}(G) \leq -\sqrt[n-1\over 2]{n-1\over 2},$$

where equality holds if and only if $\Delta = n - 1$ is even. Thus, the extremal graph $G$ belongs to Case 1 and the desired results (i) and (ii) follow from the Case 1.

3. Proof of Corollary 1.3. (i) If $n$ is even, then let $n = 2k$. From the proof of (i) and Eq. (2.1), it suffices to show

$$1 < \frac{(n-1-\Delta)(n-1-\Delta-2\lambda)}{2(\lambda+1)^2-n+\Delta} \leq \frac{3}{2}$$

for $-\sqrt{n-1\over 2}\leq \lambda \leq -\sqrt{n-1\over 2}\left[\frac{n-1}{2}\right]$ and $\left[\frac{n}{2}\right] + 1 < \Delta \leq \frac{3n-2-\sqrt{3n^2-12n}}{2}$. In view of Claim 2.1 we only need to show that the lower bound holds, which is equivalent to

$$2\lambda^2 + 2\lambda(n+1-\Delta) + 2(1+\Delta - n) - (n-1-\Delta)^2 < 0.$$

Thus, it follows from

$$2\lambda^2 + 2\lambda(n+1-\Delta) + 2(1+\Delta - n) - (n-1-\Delta)^2$$

$$\leq 2k(k-1) - 2\sqrt{k(k-1)(2k-\Delta + 1) + 2(1+\Delta - 2k) - (2k-1-\Delta)^2}$$

$$< 2k(k-1) - 2\sqrt{k(k-1)(2k-\Delta + 1) + 2(1+\Delta - 2k) - (2k-1-\Delta)^2}$$

$$= -\Delta^2 + 2(3k-1)\Delta - 6k^2 - 1$$

$$\leq 0,$$

for $\Delta \leq 3k - 1 - \sqrt{3k^2 - 6k} = \frac{3n-2-\sqrt{3n^2-12n}}{2}$.

If $n$ is odd, then let $n = 2k + 1$. In view of the proof of (i) and Eq. (2.1), we know that it suffices to show

$$\frac{1}{2} < \frac{(n-1-\Delta)(n-1-\Delta-2\lambda)}{2(\lambda+1)^2-n+\Delta} < \frac{3}{2}$$

for $-\sqrt{n-1\over 2}\left[\frac{n}{2}\right] \leq \lambda \leq -\sqrt{n-1\over 2}\left[\frac{n-1}{2}\right]$ and $\left[\frac{n}{2}\right] + 1 < \Delta \leq \frac{3n-1-\sqrt{2n^2-2n+1}}{2}$. Thus, we only need to show that the lower bound holds, that is

$$\lambda^2 + 2(2k - \Delta + 1)\lambda - 2k + \Delta - (2k - \Delta)^2 < 0.$$
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Hence, it follows from
\[ \lambda^2 + 2(2k - \Delta + 1)\lambda - 2k + \Delta - (2k - \Delta)^2 \]
\[ < k(k + 1) - 2(2k - \Delta + 1)k - 2k + \Delta - (2k - \Delta)^2 \]
\[ = -\Delta^2 + (1 + 6k)\Delta - 7k^2 - 3k \]
\[ \leq 0 \]
for \( \Delta \leq \frac{6k + 1 - \sqrt{8k^2 + 1}}{2} = \frac{6n - 2\sqrt{2n^2 - 4n + 1}}{2} \).

(ii) Note that it follows from the proof of (i) for \( \Delta = n - 1 \). Thus, in what follows
we only need to consider the case for \( \Delta \leq n - 2 \).

If \( n \) is even, then let \( n = 2k \). From the proof of (i) and Eq. (2.1), it suffices to show
\[ 0 < \frac{(n - 1 - \Delta)(n - 1 - \Delta - 2\lambda)}{2(\lambda + 1)^2 - n + \Delta} < 1 \]
for \( -\sqrt{\frac{2}{3}} \leq \lambda \leq -\sqrt{\left\lfloor \frac{n - 1}{2} \right\rfloor - \left\lfloor \frac{n - 1}{2} \right\rfloor} \) and \( \left\lceil \frac{3n - 2\sqrt{3n^2 - 8n + 2}}{2} \right\rceil \leq \Delta \leq n - 2 \). Consequently, we only need to show that the upper bound holds, which follows from
\[ 2\lambda^2 + 2\lambda(n + 1 - \Delta) + 2(1 + \Delta - n) - (n - 1 - \Delta)^2 \]
\[ > 2k(k - 1) - 2k(2k - \Delta + 1) + 2(1 + \Delta - 2k) - (2k - 1 - \Delta)^2 \]
\[ = -\Delta^2 + 6k\Delta - 6k^2 - 4k + 1 \]
\[ \geq 0 \]
for \( \frac{3n - 2\sqrt{3n^2 - 8n + 2}}{2} = 3k - \sqrt{3k^2 - 4k + 1} \leq \Delta \leq n - 2 \).

If \( n \) is odd, then let \( n = 2k + 1 \). From the proof of (i) and Eq. (2.1), it suffices to show
\[ 0 \leq \frac{(n - 1 - \Delta)(n - 1 - \Delta - 2\lambda)}{2(\lambda + 1)^2 - n + \Delta} < \frac{1}{2} \]
for \( -\sqrt{\frac{2}{3}} \leq \lambda \leq -\sqrt{\left\lfloor \frac{n - 1}{2} \right\rfloor - \left\lfloor \frac{n - 1}{2} \right\rfloor} \) and \( \frac{3n - 2\sqrt{2n^2 - 6n + 6}}{2} \leq \Delta \leq n - 2 \). Consequently, we only need to show that the upper bound holds, which follows from
\[ \lambda^2 + 2(2k - \Delta + 1)\lambda - 2k + \Delta - (2k - \Delta)^2 \]
\[ > k^2 - 2(2k - \Delta + 1)k - 2k + \Delta - (2k - \Delta)^2 \]
\[ = -\Delta^2 + 3(1 + 2k)\Delta - 7k^2 - 4k \]
\[ \geq 0 \]
for \( \frac{3n - 2\sqrt{2n^2 - 6n + 6}}{2} = \frac{6k + 1 - \sqrt{8k^2 + 4k + 1}}{2} \leq \Delta \leq n - 2 \).
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