A NOTE ON THE LEAST EIGENVALUE OF A GRAPH WITH GIVEN MAXIMUM DEGREE*

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Abstract. This note investigates the least eigenvalues of connected graphs with n vertices and maximum degree Δ , and characterizes the unique graph whose least eigenvalue achieves the minimum among all the connected graphs with n vertices and maximum vertex degree $\Delta > \frac{n}{2}$.

Key words. Maximum vertex degree, Least eigenvalue, Spectral radius.

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1. Introduction. Throughout this paper all graphs are finite and simple. Graph theoretical terms used but not defined follow [3].

Let $G = (V(G), E(G))$ be a graph with n vertices. By $G - U$ we mean the induced subgraph $G[V - U]$ if $U \subset V(G)$. Denote by $N_G(v)$ (or $N(v)$ for short) the set of all neighbors of v in G . The *adjacency matrix* of G is $A(G) = (a_{ij})_{n \times n}$, where $a_{ij} = 1$ if two vertices v_i and v_j are adjacent in G and $a_{ij} = 0$ otherwise. All eigenvalues of $A(G)$ are real and can be arranged in order as $\lambda_1(G) \geq \lambda_2(G) \geq \dots \geq \lambda_n(G)$ since $A(G)$ is a real symmetric matrix. The largest eigenvalue $\lambda_1(G)$ of $A(G)$ is called the *spectral radius* of G , denoted by $\rho(G)$. The least eigenvalue $\lambda_n(G)$ is also denoted by $\lambda_{\min}(G)$. Assume that $x = (x_{v_1}, x_{v_2}, \dots, x_{v_n})^T \in \mathbb{R}^n$ and x is a unit eigenvector of $A(G)$ corresponding to $\lambda_{\min}(G)$. Then by the Rayleigh-Ritz Theorem, we have

$$(1.1) \quad \lambda_{\min}(G) = \min_{\substack{y \in \mathbb{R}^n \\ \|y\|=1}} y^T A(G) y = x^T A(G) x = 2 \sum_{v_i v_j \in E(G)} x_{v_i} x_{v_j}$$

and

$$(1.2) \quad \lambda_{\min}(G) x_v = \sum_{u \in N_G(v)} x_u, \text{ for each } v \in V(G).$$

The research for the least eigenvalue of graphs in some class is well-studied and interesting. For example, Bell et al. [1, 2] studied the extremal graphs with n vertices

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and m edges having the minimal least eigenvalues. Constantine [5] showed that

$$\lambda_{\min}(G) \geq -\sqrt{\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil}$$

for any graph of order n , where equality holds if and only if $G = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$. Cioaba, Gregory, and Nikiforov [4] also proved that if G is a graph of order n with diameter D and maximum vertex degree Δ , then

$$(1.3) \quad \lambda_{\min}(G) \geq -\Delta + \frac{1}{n(D+1)}.$$

Fan et al. [7] obtained unicyclic graphs having the minimal least eigenvalues. Liu et al. [8] determined the graph with the minimal least eigenvalue among all unicyclic graphs with a given number of pendant vertices. Petrović et al. [9] obtained the bicyclic graph minimizing the least eigenvalue. Wang and Fan [10] gave the graph of order n with k cut vertices having the minimal least eigenvalue. Ye et al. [11] discussed the least eigenvalues of graphs with given connectivity. Zhu [12] also consider the least eigenvalues of graphs with cut vertices or edges.

In this note, we consider the following problem: what is the structure of the graph having the minimal least eigenvalue among all connected graphs with n vertices and maximum vertex degree Δ ?

Let the complete bipartite graph $K_{p,q}$ have the vertex bipartition (V_1, V_2) , where $V_1 = \{v_1, v_2, \dots, v_p\}$ and $V_2 = \{v_{p+1}, \dots, v_{p+q}\}$. Let the graph $H_{p,q}^{s,t}$ be obtained from $K_{p,q}$ by adding a new vertex u and joining u to s vertices of V_1 and t vertices of V_2 .

From the above result of Constantine, we know that $\lambda_{\min}(G) \geq -\sqrt{\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil}$ for any connected graph G with n vertices and maximum vertex degree $\Delta = \lceil \frac{n}{2} \rceil$, where equality holds if and only if $G = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$. The main result of this note is the following:

THEOREM 1.1. *Of all the connected graphs of order n with given maximum degree $\Delta > \lceil \frac{n}{2} \rceil$, (i) the graph with the minimal least eigenvalue is isomorphic to $H_{\lceil \frac{n-1}{2} \rceil+2, \Delta-\lceil \frac{n-1}{2} \rceil-2}^{\lceil \frac{n-1}{2} \rceil+2, \Delta-\lceil \frac{n-1}{2} \rceil-2}$ for $\Delta = \lceil \frac{n}{2} \rceil + 1$ and $n \geq 5$; (ii) the graph with the minimal least eigenvalue is isomorphic to either $H_{\lceil \frac{n-1}{2} \rceil, \Delta-\lceil \frac{n-1}{2} \rceil}^{\lceil \frac{n-1}{2} \rceil, \Delta-\lceil \frac{n-1}{2} \rceil}$ or $H_{\lceil \frac{n-1}{2} \rceil+1, \Delta-\lceil \frac{n-1}{2} \rceil-1}^{\lceil \frac{n-1}{2} \rceil+1, \Delta-\lceil \frac{n-1}{2} \rceil-1}$ for $\Delta > \lceil \frac{n}{2} \rceil + 1$.*

REMARK 1.2. From the proof of Theorem 1.1, we know that $\lambda_{\min}(G) > -\sqrt{\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil}$ if G is a connected graph of order n with maximum degree $\Delta > \lceil \frac{n}{2} \rceil$. Thus,

$$\lambda_{\min}(G) > -\sqrt{\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil} > -\Delta + \frac{1}{n(D+1)}.$$

This improves the bound in Eq. (1.3).

By this theorem, we further obtain the next result.

COROLLARY 1.3. *Let G be a connected graph of order n with given maximum degree $\Delta > \lceil \frac{n}{2} \rceil + 1$.*

- (i) *If n is even for $\lceil \frac{n}{2} \rceil + 1 < \Delta \leq \frac{3n-2-\sqrt{3n^2-12n}}{2}$ or n is odd for $\lceil \frac{n}{2} \rceil + 1 < \Delta \leq \frac{3n-2-\sqrt{2n^2-4n+3}}{2}$, then we have*

$$\lambda_{\min}(G) \geq \lambda_{\min} \left(H_{\lceil \frac{n-1}{2} \rceil + 1, \lfloor \frac{n-1}{2} \rfloor - 1}^{\lceil \frac{n-1}{2} \rceil + 1, \Delta - \lceil \frac{n-1}{2} \rceil - 1} \right)$$

with equality if and only if $G \cong H_{\lceil \frac{n-1}{2} \rceil + 1, \lfloor \frac{n-1}{2} \rfloor - 1}^{\lceil \frac{n-1}{2} \rceil + 1, \Delta - \lceil \frac{n-1}{2} \rceil - 1}$.

- (ii) *If n is even for $\frac{3n-\sqrt{3n^2-8n+4}}{2} \leq \Delta \leq n-1$ or n is odd for $\frac{3n-2-\sqrt{2n^2-6n+5}}{2} \leq \Delta \leq n-1$, then we have*

$$\lambda_{\min}(G) \geq \lambda_{\min} \left(H_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor}^{\lceil \frac{n-1}{2} \rceil, \Delta - \lceil \frac{n-1}{2} \rceil} \right)$$

with equality if and only if $G \cong H_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor}^{\lceil \frac{n-1}{2} \rceil, \Delta - \lceil \frac{n-1}{2} \rceil}$.

REMARK 1.4. In general, it is hard to characterize the corresponding extremal graphs having the minimal least eigenvalue over all the connected graphs of order n with given maximum degree $\Delta < \frac{n}{2}$. It is well known that $\rho(G) \leq \Delta$ with equality if and only if G is regular, and G is bipartite if and only if $\rho(G) = -\lambda_{\min}(G)$ (see [6]). Thus, if G is a connected graph of order n with maximum degree $\Delta < \frac{n}{2}$, then $\lambda_{\min}(G) \geq -\Delta$ with equality if and only if G is a regular bipartite graph.

2. Proof of Theorem 1.1. Let G be a connected graph of order n with given maximum vertex degree $\Delta > \lceil \frac{n}{2} \rceil$. Assume that $\lambda_{\min}(G)$ is as small as possible and that $x = (x_1, x_2, \dots, x_n)^T$ is a unit eigenvector of $A(G)$ corresponding to $\lambda_{\min}(G)$. Denote by $V^+ = \{v \in V(G) : x_v > 0\}$, $V^- = \{v \in V(G) : x_v < 0\}$ and $V^0 = \{v \in V(G) : x_v = 0\}$. We will divide the following proof into two cases.

Case 1. If $|V^0| = 0$, then we have $n - \Delta \leq |V^-|, |V^+| \leq \Delta$. Otherwise, we have

$$\begin{aligned} \lambda_{\min}(G) &= 2 \sum_{v_i v_j \in E(G)} x_{v_i} x_{v_j} \\ &\geq 2 \sum_{v_i \in V^-, v_j \in V^+} x_{v_i} x_{v_j} \\ &\geq \lambda_{\min}(K_{|V^-|, |V^+|}) \\ &= -\sqrt{|V^-| |V^+|} \\ &> -\sqrt{\Delta(n - \Delta)} \\ &= \lambda_{\min}(K_{\Delta, n-\Delta}), \end{aligned}$$

a contradiction. On the other hand, if $\{|V^-|, |V^+|\} = \{n - \Delta, \Delta\}$, then every vertex in V^+ must be adjacent to all vertices in V^- by Eq. (1.1) since $\lambda_{\min}(G)$ is as small as possible, which implies that $G = K_{n-\Delta, \Delta}$. Thus, in what follows we only need to consider the case for $n - \Delta + 1 \leq |V^-|, |V^+| \leq \Delta - 1$.

Let u be a vertex with maximum degree. Without loss of generality, we can assume that $x_u > 0$. Consequently, we obtain that every vertex in V^+ must be adjacent to all vertices in V^- ; otherwise there exists two disjunct vertices $u_1 \in V^+$ and $v_1 \in V^-$, and then by Eq. (1.1), we have a graph $G + u_1v_1$ with maximum degree Δ and

$$\begin{aligned}\lambda_{\min}(G) &= 2 \sum_{v_i v_j \in E(G)} x_{v_i} x_{v_j} \\ &> 2 \sum_{v_i v_j \in E(G)} x_{v_i} x_{v_j} + 2x_{u_1} x_{v_1} \\ &\geq \lambda_{\min}(G + u_1v_1)\end{aligned}$$

since G is connected, which contradicts that $\lambda_{\min}(G)$ is as small as possible. Thus, we can write G as $H_{s, n-s-1}^{s, \Delta-s}$ if we assume that $|V^-| = s$ and $|V^+| = n - s$. By Eq. (1.2), x has a constant value on the vertices of V^- (respectively, $V^+ \cap N(u)$ and $V^+ \setminus N(u)$), denoted by x_1 (respectively, x_2 and x_3), e.g., $\lambda_{\min}(G)x_{v_1} = \sum_{t \in V^+} x_t = \lambda_{\min}(G)x_{v_2}$

for any two vertices $v_1, v_2 \in V^-$, which implies that $x_{v_1} = x_{v_2}$ since $\lambda_{\min}(G) < 0$. Hence, by Eq. (1.2), we have the following equations

$$\begin{aligned}\lambda_{\min}(G)x_u &= sx_1 + (\Delta - s)x_2, \\ \lambda_{\min}(G)x_1 &= x_u + (\Delta - s)x_2 + (n - \Delta - 1)x_3, \\ \lambda_{\min}(G)x_2 &= sx_1 + x_u, \\ \lambda_{\min}(G)x_3 &= sx_1.\end{aligned}$$

This implies that $\lambda_{\min}(G)$ is the least root of the following equation in λ :

$$(2.1) \quad \lambda^4 - [s(n - 1 - s) + \Delta]\lambda^2 + (\Delta - s)s(n - 1 - \Delta) - 2(\Delta - s)s\lambda = 0.$$

From Eq. (2.1), we know that $\lambda_{\min}\left(H_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor}^{\lceil \frac{n-1}{2} \rceil, \Delta - \lceil \frac{n-1}{2} \rceil}\right)$ is the least root of the following equation:

$$\lambda^4 - \left(\left\lceil \frac{n-1}{2} \right\rceil \left\lfloor \frac{n-1}{2} \right\rfloor + \Delta\right)\lambda^2 + \left(\Delta - \left\lceil \frac{n-1}{2} \right\rceil\right)\left\lceil \frac{n-1}{2} \right\rceil(n - 1 - \Delta - 2\lambda) = 0.$$

Note that

$$\lambda^4 - \left(\left\lceil \frac{n-1}{2} \right\rceil \left\lfloor \frac{n-1}{2} \right\rfloor + \Delta\right)\lambda^2 + \left(\Delta - \left\lceil \frac{n-1}{2} \right\rceil\right)\left\lceil \frac{n-1}{2} \right\rceil(n - 1 - \Delta - 2\lambda) \leq 0$$

for $\lambda = -\sqrt{\lceil \frac{n-1}{2} \rceil \lfloor \frac{n-1}{2} \rfloor}$, where equality holds if and only if $\Delta = n - 1$ is even, which implies that

$$(2.2) \quad \lambda_{\min}(G) \leq \lambda_{\min}\left(H_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor}^{\lceil \frac{n-1}{2} \rceil, \Delta - \lceil \frac{n-1}{2} \rceil}\right) \leq -\sqrt{\left\lceil \frac{n-1}{2} \right\rceil \left\lfloor \frac{n-1}{2} \right\rfloor},$$

where equality in the second inequality holds if and only if $\Delta = n - 1$ is even.

Hence, we have $-\sqrt{\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor} < \lambda_{\min}(G) \leq -\sqrt{\lceil \frac{n-1}{2} \rceil \lfloor \frac{n-1}{2} \rfloor}$, which can be seen from the result of Constantine and inequalities in (2.2). Note that $-\sqrt{\Delta(n-\Delta)}$ is also the least root of the above Eq. (2.1) for $s = \Delta$. Thus, in the following we will consider the least root of above Eq. (2.1) for $n - \Delta + 1 \leq s \leq \Delta$ since $\lambda_{\min}(G) = -\sqrt{\Delta(n-\Delta)}$ for $\{|V^-|, |V^+|\} = \{n - \Delta, \Delta\}$.

Assume that $f(s) = \lambda^4 - [s(n-1-s) + \Delta]\lambda^2 + (\Delta-s)s(n-1-\Delta) - 2(\Delta-s)s\lambda$. Then $f(s) = [(\lambda+1)^2 - n + \Delta]s^2 - [(n-1)\lambda^2 + 2\lambda\Delta - \Delta(n-1-\Delta)]s + \lambda^4 - \Delta\lambda^2$ is a quadratic function in s , whose axis of symmetry is

$$\begin{aligned} s_0 &= \frac{(n-1)\lambda^2 + 2\lambda\Delta - \Delta(n-1-\Delta)}{2[(\lambda+1)^2 - n + \Delta]} \\ &= \frac{n-1}{2} + \frac{(n-1-\Delta)(n-1-\Delta-2\lambda)}{2[(\lambda+1)^2 - n + \Delta]}. \end{aligned}$$

To consider the minimal least root of Eq. (2.1) for $n - \Delta + 1 \leq s \leq \Delta$, we only need to know when $f(s)$ attains its minimum. We need the following two claims.

CLAIM 2.1. Let $-\sqrt{\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor} < \lambda \leq -\sqrt{\lceil \frac{n-1}{2} \rceil \lfloor \frac{n-1}{2} \rfloor}$ and $\lceil \frac{n}{2} \rceil + 1 < \frac{n+4}{2} \leq \Delta$. Then we have $(\lambda+1)^2 - n + \Delta > 0$ and

$$0 \leq \frac{(n-1-\Delta)(n-1-\Delta-2\lambda)}{2[(\lambda+1)^2 - n + \Delta]} < \frac{3}{2},$$

where $(n-1-\Delta)(n-1-\Delta-2\lambda) = 0$ if and only if $\Delta = n - 1$.

Proof. In fact, it is clear for the lower bound, and the upper bound is equivalent to

$$-\Delta^2 + (2n+1-2\lambda)\Delta - n^2 - n + 2n\lambda + 3\lambda^2 + 4\lambda + 2 > 0.$$

It suffices to show that

$$\begin{aligned} & -\frac{(n+4)^2}{4} + \frac{(2n+1-2\lambda)(n+4)}{2} - n^2 - n + 2n\lambda + 3\lambda^2 + 4\lambda + 2 \\ &= \frac{3n}{2} - \frac{n^2}{4} + n\lambda + 3\lambda^2 \\ &> 0, \end{aligned}$$

which is trivial for $-\sqrt{\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor} \leq \lambda \leq -\sqrt{\lceil \frac{n-1}{2} \rceil \lfloor \frac{n-1}{2} \rfloor}$. $\square$

Similarly, we can easily prove the next claim.

CLAIM 2.2. Let $-\sqrt{\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor} < \lambda \leq -\sqrt{\lceil \frac{n-1}{2} \rceil \lfloor \frac{n-1}{2} \rfloor}$, $\Delta = \lceil \frac{n}{2} \rceil + 1$ and $n \geq 5$. Then we have

$$\left\lceil \frac{n-1}{2} \right\rceil + 2 < \frac{n-1}{2} + \frac{(n-1-\Delta)(n-1-\Delta-2\lambda)}{2[(\lambda+1)^2 - n + \Delta]} < \left\lceil \frac{n-1}{2} \right\rceil + \frac{5}{2}.$$

Thus, by virtue of Claim 2.1, $f(s)$ can attain the minimum value only at $s = \lceil \frac{n-1}{2} \rceil$ or $\lceil \frac{n-1}{2} \rceil + 1$ for $\lceil \frac{n}{2} \rceil + 2 \leq \Delta < n-1$, and $f(s)$ can attain the minimum $f(\lceil \frac{n-1}{2} \rceil) = f(\lfloor \frac{n-1}{2} \rfloor)$ for $\Delta = n-1$ if and only if $s = \lceil \frac{n-1}{2} \rceil$ or $s = \lfloor \frac{n-1}{2} \rfloor$. It follows that if $\Delta > \lceil \frac{n}{2} \rceil + 1$ then the graph G with the minimal least eigenvalue is isomorphic to $H_{\lceil \frac{n-1}{2} \rceil, \Delta - \lceil \frac{n-1}{2} \rceil}^{\lceil \frac{n-1}{2} \rceil, \Delta - \lceil \frac{n-1}{2} \rceil}$ or $H_{\lceil \frac{n-1}{2} \rceil + 1, \Delta - \lceil \frac{n-1}{2} \rceil - 1}^{\lceil \frac{n-1}{2} \rceil + 1, \Delta - \lceil \frac{n-1}{2} \rceil - 1}$. In addition, in view of Claim 2.2, $f(s)$ can attain the minimum value only at $s = \lceil \frac{n-1}{2} \rceil + 2$ for $\Delta = \lceil \frac{n}{2} \rceil + 1$ and $n \geq 5$. This completes the proof of the Case 1.

Case 2. If $|V^0| \neq 0$, then let y be a subvector of x by deleting the entries corresponding to $v \in V^0$. By Eq. (1.1), we have

$$\begin{aligned} \lambda_{\min}(G) &= 2 \sum_{v_i v_j \in E(G)} x_{v_i} x_{v_j} \\ &= y^T A(G - V^0) y \\ &\geq \lambda_{\min}(G - V^0) \\ &\geq -\sqrt{\left\lfloor \frac{n-1}{2} \right\rfloor \left\lceil \frac{n-1}{2} \right\rceil}, \end{aligned}$$

where the equality in the first inequality holds if and only if y is a least vector of $G - V^0$, and the equality in the second inequality holds if and only if $G - V^0 \cong K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor}$. Thus, if $|V^0| \neq 0$, then we have

$$\lambda_{\min}(G) \geq -\sqrt{\left\lfloor \frac{n-1}{2} \right\rfloor \left\lceil \frac{n-1}{2} \right\rceil}$$

with equality only when $V^0 = \{u\}$ and $G = H_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor}^{b, \Delta-b}$ for some positive integer b . This completes the proof of the Case 2.

Finally, if the extremal graph G belongs to Case 2, then

$$\lambda_{\min}(G) \geq -\sqrt{\left\lfloor \frac{n-1}{2} \right\rfloor \left\lceil \frac{n-1}{2} \right\rceil}$$

with equality only when $V^0 = \{u\}$ and $G = H_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor}^{b, \Delta-b}$ for some positive integer b ; if the extremal graph G belongs to Case 1, then by inequalities (2.2) we have

$$\lambda_{\min}(G) \leq -\sqrt{\left\lceil \frac{n-1}{2} \right\rceil \left\lfloor \frac{n-1}{2} \right\rfloor},$$

where equality holds if and only if $\Delta = n - 1$ is even. Thus, the extremal graph G belongs to Case 1 and the desired results (i) and (ii) follow from the Case 1. $\square$

3. Proof of Corollary 1.3. (i) If n is even, then let $n = 2k$. From the proof of (i) and Eq. (2.1), it suffices to show

$$1 < \frac{(n-1-\Delta)(n-1-\Delta-2\lambda)}{2[(\lambda+1)^2-n+\Delta]} \leq \frac{3}{2}$$

for $-\sqrt{\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor} \leq \lambda \leq -\sqrt{\lceil \frac{n-1}{2} \rceil \lfloor \frac{n-1}{2} \rfloor}$ and $\lceil \frac{n}{2} \rceil + 1 < \Delta \leq \frac{3n-2-\sqrt{3n^2-12n}}{2}$. In view of Claim 2.1, we only need to show that the lower bound holds, which is equivalent to

$$2\lambda^2 + 2\lambda(n+1-\Delta) + 2(1+\Delta-n) - (n-1-\Delta)^2 < 0.$$

Thus, it follows from

$$\begin{aligned} & 2\lambda^2 + 2\lambda(n+1-\Delta) + 2(1+\Delta-n) - (n-1-\Delta)^2 \\ & \leq 2k(k-1) - 2\sqrt{k(k-1)}(2k-\Delta+1) + 2(1+\Delta-2k) - (2k-1-\Delta)^2 \\ & < 2k(k-1) - 2(k-1)(2k-\Delta+1) + 2(1+\Delta-2k) - (2k-1-\Delta)^2 \\ & = -\Delta^2 + 2(3k-1)\Delta - 6k^2 - 1 \\ & \leq 0, \end{aligned}$$

for $\Delta \leq 3k-1 - \sqrt{3k^2-6k} = \frac{3n-2-\sqrt{3n^2-12n}}{2}$.

If n is odd, then let $n = 2k+1$. In view of the proof of (i) and Eq. (2.1), we know that it suffices to show

$$\frac{1}{2} < \frac{(n-1-\Delta)(n-1-\Delta-2\lambda)}{2[(\lambda+1)^2-n+\Delta]} < \frac{3}{2}$$

for $-\sqrt{\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor} \leq \lambda \leq -\sqrt{\lceil \frac{n-1}{2} \rceil \lfloor \frac{n-1}{2} \rfloor}$ and $\lceil \frac{n}{2} \rceil + 1 < \Delta \leq \frac{3n-1-\sqrt{2n^2-2n+1}}{2}$. Thus, we only need to show that the lower bound holds, that is

$$\lambda^2 + 2(2k-\Delta+1)\lambda - 2k + \Delta - (2k-\Delta)^2 < 0.$$

Hence, it follows from

$$\begin{aligned} & \lambda^2 + 2(2k - \Delta + 1)\lambda - 2k + \Delta - (2k - \Delta)^2 \\ & < k(k + 1) - 2(2k - \Delta + 1)k - 2k + \Delta - (2k - \Delta)^2 \\ & = -\Delta^2 + (1 + 6k)\Delta - 7k^2 - 3k \\ & \leq 0 \end{aligned}$$

for $\Delta \leq \frac{6k+1-\sqrt{8k^2+1}}{2} = \frac{3n-2-\sqrt{2n^2-4n+3}}{2}$.

(ii) Note that it follows from the proof of (i) for $\Delta = n - 1$. Thus, in what follows we only need to consider the case for $\Delta \leq n - 2$.

If n is even, then let $n = 2k$. From the proof of (i) and Eq. (2.1), it suffices to show

$$0 < \frac{(n-1-\Delta)(n-1-\Delta-2\lambda)}{2[(\lambda+1)^2-n+\Delta]} < 1$$

for $-\sqrt{\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor} \leq \lambda \leq -\sqrt{\lceil \frac{n-1}{2} \rceil \lfloor \frac{n-1}{2} \rfloor}$ and $\lceil \frac{3n-\sqrt{3n^2-8n+4}}{2} \rceil \leq \Delta \leq n-2$. Consequently, we only need to show that the upper bound holds, which follows from

$$\begin{aligned} & 2\lambda^2 + 2\lambda(n+1-\Delta) + 2(1+\Delta-n) - (n-1-\Delta)^2 \\ & > 2k(k-1) - 2k(2k-\Delta+1) + 2(1+\Delta-2k) - (2k-1-\Delta)^2 \\ & = -\Delta^2 + 6k\Delta - 6k^2 - 4k + 1 \\ & \geq 0 \end{aligned}$$

for $\frac{3n-\sqrt{3n^2-8n+4}}{2} = 3k - \sqrt{3k^2-4k+1} \leq \Delta \leq n-2$.

If n is odd, then let $n = 2k + 1$. From the proof of (i) and Eq. (2.1), it suffices to show

$$0 \leq \frac{(n-1-\Delta)(n-1-\Delta-2\lambda)}{2[(\lambda+1)^2-n+\Delta]} < \frac{1}{2}$$

for $-\sqrt{\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor} \leq \lambda \leq -\sqrt{\lceil \frac{n-1}{2} \rceil \lfloor \frac{n-1}{2} \rfloor}$ and $\frac{3n-2-\sqrt{2n^2-6n+5}}{2} \leq \Delta \leq n-2$. Consequently, we only need to show that the upper bound holds, which follows from

$$\begin{aligned} & \lambda^2 + 2(2k - \Delta + 1)\lambda - 2k + \Delta - (2k - \Delta)^2 \\ & > k^2 - 2(2k - \Delta + 1)k - 2k + \Delta - (2k - \Delta)^2 \\ & = -\Delta^2 + 3(1+2k)\Delta - 7k^2 - 4k \\ & \geq 0 \end{aligned}$$

for $\frac{3n-2-\sqrt{2n^2-6n+5}}{2} = \frac{6k+1-\sqrt{8k^2-4k+1}}{2} \leq \Delta \leq n-2$. $\square$

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