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A REAL COUNTEREXAMPLE TO THE DRURY PERMANENT CONJECTURE*

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Abstract. The Drury permanent conjecture, published by Zhang in 2016 and disproved over the complex numbers by Hutchinson in 2017, has remained open in the real case, where a proof would imply the real case of the Chollet conjecture. The real 5×5 counterexample reported by Paksoy in 2023 concerns a different inequality, whose principal submatrices have order $n-1$ rather than order $n-2$ as in the conjecture; it is verified that Paksoy's matrix satisfies the conjectured inequality strictly, under every simultaneous permutation of its rows and columns. The real case is then settled in the negative: an 8×8 real positive semidefinite matrix with integer entries and rank 3 is exhibited that violates the inequality, with an exact integer certificate of the violation. Direct sums with identity matrices give counterexamples of every order $n \geq 8$.

Key words. Permanent, Positive semidefinite matrix, Drury permanent conjecture, Chollet conjecture, Counterexample.

AMS subject classifications. 15A15, 15B48.

1. Introduction.

In 2016, Zhang published the following conjecture, attributing it to Stephen Drury and citing a private communication [4, Eq. (20)]. Zhang writes the submatrix in the second term as B_{kk} while summing over $k = 2, \ldots, n$. Following Hutchinson [2], we number the rows and columns of B from 1 to $n-1$ and write this submatrix as $B_{k-1,k-1}$.

CONJECTURE 1.1 (Drury permanent conjecture). *Let $A = (a_{ij})$ be an $n \times n$ positive semidefinite Hermitian matrix. Let B be obtained from A by deleting its first row and column, and, for $1 \leq j \leq n-1$, let B_{jj} be obtained from B by deleting its jth row and column. Then,*

$$(a_{11} \operatorname{per} B)^2 + \left(\sum_{k=2}^{n} |a_{1k}|^2 \operatorname{per} B_{k-1,k-1} \right)^2 \leq (\operatorname{per} A)^2. \tag{1.1}$$

Zhang observed that Conjecture 1.1 would imply the Chollet conjecture [1, 4]. Hutchinson subsequently disproved it using a 4×4 complex correlation matrix [2]. He concluded: "It is unknown whether the Drury permanent conjecture holds for all *real* positive semi-definite matrices. This is an open problem worth pursuing, as this would imply the real case of the Chollet conjecture" [2, p. 302].

This note has two purposes. The first is to clarify the status of the real case. Remark 1.1 of [3] reports a real 5×5 counterexample, and a natural reading of that remark suggests that the real case is already settled. The inequality disproved there is, however, a different one, in which the principal submatrices have order $n-1$; in the conjectured inequality (1.1) they have order $n-2$. In Section 3, we verify that Paksoy's matrix satisfies (1.1) strictly, under every simultaneous permutation of its rows and columns. To our knowledge, no real counterexample to (1.1) has previously appeared in the literature.

The second purpose is to settle the real case in the negative. The counterexample, found by a computer search over low-rank integer Gram factors $A = XX^{\mathsf{T}}$, is presented in Section 2.

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2. The counterexample.

THEOREM 2.1. *There exists an* 8×8 *real positive semidefinite matrix with integer entries and rank* 3 *that violates* (1.1). *Consequently, there are counterexamples of every order* $n \geq 8$.

Proof. Let

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 3 \\ -1 & 0 & 3 \\ 1 & -4 & -2 \\ -1 & 4 & -2 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \end{bmatrix}, \qquad A = XX^{\mathsf{T}}.$$

Explicitly,

$$A = \begin{bmatrix} 1 & 0 & 1 & -1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 3 & 3 & -2 & -2 & -2 & 2 \\ 1 & 3 & 10 & 8 & -5 & -7 & -5 & 7 \\ -1 & 3 & 8 & 10 & -7 & -5 & -7 & 5 \\ 1 & -2 & -5 & -7 & 21 & -13 & -11 & -19 \\ -1 & -2 & -7 & -5 & -13 & 21 & 19 & 11 \\ 1 & -2 & -5 & -7 & -11 & 19 & 21 & 13 \\ 1 & 2 & 7 & 5 & -19 & 11 & 13 & 21 \end{bmatrix}.$$

Since $A = XX^{\mathsf{T}}$, the matrix A is real and positive semidefinite. Moreover,

$$X^{\mathsf{T}}X = \operatorname{diag}(7, 64, 35),$$

so X has rank 3, and hence $\operatorname{rank} A = 3$.

Let B be obtained from A by deleting its first row and column. Exact integer computations give

$$\operatorname{per} A = 1\,204\,741\,248, \qquad \operatorname{per} B = 1\,193\,136\,480,$$

together with

$$\operatorname{per} B_{22} = \operatorname{per} B_{33} = 38\,965\,032$$

and

$$\operatorname{per} B_{44} = \operatorname{per} B_{55} = \operatorname{per} B_{66} = \operatorname{per} B_{77} = 23\,155\,580.$$

The first row of A is

$$(1, 0, 1, -1, 1, -1, 1, 1).$$

Hence, $a_{12} = 0$ and $a_{1k}^2 = 1$ for $3 \leq k \leq 8$, so

$$\sum_{k=2}^{8} a_{1k}^2 \operatorname{per} B_{k-1,k-1} = 2(38\,965\,032) + 4(23\,155\,580) = 170\,552\,384.$$

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Consequently,

$$(a_{11}\operatorname{per} B)^2+\left(\sum_{k=2}^{8}a_{1k}^2\operatorname{per} B_{k-1,k-1}\right)^2-(\operatorname{per} A)^2 \\ =1\,261\,300\,962\,276\,352>0.$$

Thus, A is a real counterexample of order 8.

The permanent values were verified using both Ryser's formula and direct summation over all permutations, with exact integer arithmetic.

For $m\geq 1$, let $\widetilde{A}=A\oplus I_m$. The matrix obtained from $\widetilde{A}$ by deleting its first row and column is $\widetilde{B}=B\oplus I_m$. Since permanents are multiplicative over direct sums (a permutation contributing a nonzero term must preserve the blocks),

$$\operatorname{per}\widetilde{A}=\operatorname{per} A,\qquad \operatorname{per}\widetilde{B}=\operatorname{per} B.$$

For $2\leq k\leq 8$, the corresponding principal submatrices satisfy

$$\operatorname{per}\widetilde{B}_{k-1,k-1}=\operatorname{per}\bigl(B_{k-1,k-1}\oplus I_m\bigr)=\operatorname{per} B_{k-1,k-1}.$$

For $k>8$, the new first-row entries satisfy $\widetilde{a}_{1k}=0$. Therefore, the difference between the two sides of (1.1) is unchanged, and $\widetilde{A}$ is a counterexample of order $8+m$. ∎

Remark 2.2. *The matrix X belongs to the three-parameter family*

$$X(a,b,c)=\begin{bmatrix}1&0&0\\0&0&1\\1&0&a\\-1&0&a\\1&-b&-c\\-1&b&-c\\1&b&-c\\1&b&c\end{bmatrix},\qquad X=X(3,4,2).$$

For every member of this family, the first row of $A(a,b,c)=X(a,b,c)X(a,b,c)^{\mathsf{T}}$ has entries in $\{0,\pm1\}$, so the weights a_{1k}^2 in (1.1) are all 0 or 1. Writing $\Delta(a,b,c)$ for the difference between the left- and right-hand sides of (1.1) for $A(a,b,c)$, exact computation shows that among the integer triples $(a,b,c)\in\{1,\dots,6\}^3$, precisely four give a violation, namely

$$(3,4,2),\quad (4,4,2),\quad (4,5,2),\quad (5,6,3).$$

Among these four triples, the relative violation $\Delta(a,b,c)/\bigl(\operatorname{per} A(a,b,c)\bigr)^2$ is largest at $(3,4,2)$. Moreover, since Δ is a polynomial in (a,b,c), every real triple sufficiently close to $(3,4,2)$ also gives a violation. The counterexample of Theorem 2.1 is therefore not an isolated example.

3. Relation to Paksoy's example. In the conjectured inequality (1.1), the submatrices in the second term are obtained by deleting the first row and column of A and then deleting a further row and column. They, therefore, have order $n-2$.

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The inequality studied in [3], printed there as Eq. (1.1), is

$$(a_{11}\operatorname{per} A_{11})^2 + \left(\sum_{k=2}^{n} |a_{1k}|^2 \operatorname{per} A_{kk}\right)^2 \le (\operatorname{per} A)^2, \tag{3.2}$$

where A_{kk} is obtained from A by deleting its kth row and column. These principal submatrices have order $n-1$.

Remark 1.1 of [3, p. 72] reports a 5×5 real correlation matrix C, given in [3, Eq. (1.3)], for which (3.2) fails. Exact rational computation confirms the values reported there.

Let D be obtained from C by deleting its first row and column. For the formulation (1.1), the same matrix gives

$$(c_{11}\operatorname{per} D)^2 + \left(\sum_{k=2}^{5} c_{1k}^2 \operatorname{per} D_{k-1,k-1}\right)^2 = \frac{60\,061\,039\,533\,025}{16\,000\,000\,000\,000},$$

and

$$(\operatorname{per} C)^2 = \frac{207\,315\,794\,356\,225}{16\,000\,000\,000\,000}.$$

Hence,

$$(c_{11}\operatorname{per} D)^2 + \left(\sum_{k=2}^{5} c_{1k}^2 \operatorname{per} D_{k-1,k-1}\right)^2 < (\operatorname{per} C)^2.$$

The left-hand side of (1.1) is invariant under simultaneous permutations of the rows and columns of C that fix the first index: the permanent is unchanged by simultaneous row and column permutations, and such a relabeling merely permutes the terms of the sum. The left-hand side, therefore, depends only on which index is placed first. Exact computation for each of the remaining four choices of leading index gives the common left-hand-side value

$$\frac{108\,322\,248\,528\,025}{16\,000\,000\,000\,000},$$

again strictly smaller than $(\operatorname{per} C)^2$. Paksoy's matrix therefore does not violate (1.1) under any simultaneous permutation of its rows and columns.

We note that the matrix A of Theorem 2.1 violates (3.2) as well: for that matrix, the left-hand side of (3.2) exceeds $(\operatorname{per} A)^2$ by $4\,276\,375\,678\,264\,320$.

4. Concluding remarks. A remaining question is the minimal order of a real counterexample. The complex example of Hutchinson has order 4; the matrix of Theorem 2.1 has order 8, and the family of Remark 2.2 does not shrink: exact evaluation of every Gram matrix of order 4 to 7 obtained by deleting rows of $X(a,b,c)$, for all $(a,b,c)\in\{1,\dots,6\}^3$ and every choice of leading index, produces no violation of (1.1). The minimal order is thus at most 8; its exact value is open.

Hutchinson called the real case of Conjecture 1.1 worth pursuing because its truth would have implied the real case of the Chollet conjecture [1, 2]. Theorem 2.1 closes that route — the real Chollet conjecture cannot be reached through (1.1). The Chollet conjecture itself remains open.

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