Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 508-513, May 2012

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GENERAL POLYNOMIALS OVER DIVISION ALGEBRAS AND LEFT EIGENVALUES*

ADAM CHAPMAN†

Abstract. In this paper, we present an isomorphism between the ring of general polynomials over a division algebra D with center F and the group ring of the free monoid with $[D : F]$ variables over D. Using this isomorphism, we define the characteristic polynomial of any matrix over any division algebra, i.e., a general polynomial with one variable over the algebra whose roots are precisely the left eigenvalues. Furthermore, we show how the left eigenvalues of a 4×4 quaternion matrices can be obtained by solving a general polynomial equation of degree 6.

Key words. General Polynomial, Characteristic Polynomial, Determinant, Left eigenvalue, Quaternions.

AMS subject classifications. 12E15, 16S10, 11R52.

1. Introduction.

1.1. Polynomial rings over division algebras. Let F be a field and D be a division algebra over F of degree d. We adopt the terminology in [3]. Let $D_L[z]$ denote the usual ring of polynomials over D, where the variable z commutes with every $y \in D$. When substituting a value, we consider the coefficients to be placed on the left-hand side of the variable. The substitution map $S_y : D_L[z] \to D$ is not a ring homomorphism in general. For example, if $f(z) = az$ and $ay \neq ya$, then $S_y(f^2) = S_y(a^2z^2) = a^2y^2$ while $(S_y(f))^2 = (S_y(az))^2 = (ay)^2 \neq S_y(f^2)$.

The ring $D_G[z]$ is, by definition, the (associative) ring of polynomials over D, where z is assumed to commute with every $y \in F = Z(D)$, but not with arbitrary elements of D. For example, if $y \in D$ is non-central, then yz^2, zyz, and z^2y are distinct elements of this ring. There is a ring epimorphism $D_G[z] \to D_L[z]$, defined by $z \mapsto z$ and $y \mapsto y$ for every $y \in D$, whose kernel is the ideal generated by the commutators $[y, z]$ ($y \in D$). Unlike the situation of $D_L[z]$, the substitution maps from $D_G[z]$ to D are all ring homomorphisms. Polynomials from $D_G[z]$ are called *general polynomials*; for example, $ziz + jzi + zij + 5 \in \mathbb{H}_G[z]$, where $\mathbb{H}$ is the algebra of real quaternions.

*Received by the editors on October 11, 2011. Accepted for publication on April 15, 2012. Handling Editor: Roderick Gow.

†Department of Mathematics, Bar-Ilan University, Ramat-Gan 52115, Israel (adam1chapman@yahoo.com). The author is a PhD student under the supervision of Prof. Uzi Vishne.

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A publication of the International Linear Algebra Society
Volume 23, pp. 508-513, May 2012

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Polynomials in $D_L[z]$ and polynomials in $D_G[z]$ which "look like" polynomials in $D_L[z]$, i.e., the coefficients are placed on the left-hand side of the variable, are called *left* or *standard polynomials*; for example, $z^2 + iz + j \in \mathbb{H}_G[z]$.

Let $D\langle x_1, \ldots, x_N\rangle$ be the ring of multi-variable polynomials, where for every $1 \leq i \leq N$, x_i commutes with every $y \in D$ and is not assumed to commute with x_j for $j \neq i$. This is the group ring of the free monoid $\langle x_1, \ldots, x_N\rangle$ over D. The commutative counterpart is $D_L[x_1, \ldots, x_N]$, which is the ring of multi-variable polynomials where for every $1 \leq i \leq N$, x_i commutes with every $y \in D$ and with every x_j for $j \neq i$.

For further reading on what is generally known about polynomial equations over division rings, see [4].

1.2. Left eigenvalues of matrices over division algebras. Given a matrix $A \in M_n(D)$, a *left eigenvalue* of A is an element $\lambda \in D$ for which there exists a nonzero vector $v \in D^{n\times 1}$ such that $Av = \lambda v$.

For the special case of $D = \mathbb{H}$ and $n = 2$, it was proven by Wood in [7] that the left eigenvalues of A are the roots of a standard quadratic quaternion polynomial. In [6] it is proven that for $n = 3$, the left eigenvalues of A are the roots of a general cubic quaternion polynomial.

In [5] Macías-Virgós and Pereira-Sáez gave another proof for Wood's result. Their proof makes use of the Study determinant.

Given a matrix $A \in M_n(\mathbb{H})$, there exist unique matrices $B, C \in M_n(\mathbb{C})$ such that $A = B + Cj$. The *Study determinant* of A is $\det \begin{bmatrix} B & -\overline{C} \\ C & \overline{B} \end{bmatrix}$. The *Dieudonné determinant* is (in this case) the square root of the Study determinant. (In [5] the Study determinant is defined to be what we call the Dieudonné determinant.) For further information about these determinants see [1].

2. The isomorphism between the ring of general polynomials and the group ring of the free monoid with $[D : F]$ variables. Let $N = d^2$, i.e., N is the dimension of D over its center F. In particular, there exist $a_1, \ldots, a_{N-1} \in D$ such that $D = F + a_1F + \cdots + a_{N-1}F$.

Let $h : D_G[z] \to D\langle x_1, \ldots, x_N\rangle$ be the homomorphism for which $h(y) = y$ for all $y \in D$, and $h(z) = x_1 + a_1x_2 + \cdots + a_{N-1}x_N$. $D_L[x_1, \ldots, x_N]$ is a quotient ring of $D\langle x_1, \ldots, x_N\rangle$. Let $g : D\langle x_1, \ldots, x_N\rangle \to D_L[x_1, \ldots, x_N]$ be the standard epimorphism.

In [3, Theorem 6], it says that if D is a division algebra, then the homomorphism $g \circ h : D_G[z] \to D_L[x_1, \ldots, x_N]$ is an epimorphism. The next theorem is a result of

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this fact.

THEOREM 2.1. *The homomorphism* $h : D_G[z] \to D\langle x_1, \dots, x_N\rangle$ *is an isomorphism, and therefore* $D_G[z] \cong D\langle x_1, \dots, x_N\rangle$.

Proof. The map h is well-defined because z commutes only with the center.

Both $D_G[z]$ and $D\langle x_1, \dots, x_N\rangle$ can be graded, $D_G[z] = G_0 \bigoplus G_1 \bigoplus \cdots$ and $D\langle x_1, \dots, x_N\rangle = H_0 \bigoplus H_1 \bigoplus \cdots$ such that for all n, G_n and H_n are spanned by monomials of degree n.

For every n, $h(G_n) \subseteq H_n$. Furthermore, the basis of G_n as a vector space over F is $\{b_1 z b_2 \cdots b_n z b_{n+1} : b_1, \dots, b_{n+1} \in \{1, a_1, \dots, a_{N-1}\}\}$, which means that $[G_n : F] = N^{n+1}$. Furthermore, the basis of H_n as a vector space over F is $\{b x_{k_1} x_{k_2} \cdots x_{k_n} : b \in \{1, a_1, \dots, a_{N-1}\}, \forall_j k_j \in \{1, \dots, N\}\}$, hence $[H_n : F] = N \cdot N^n = N^{n+1} = [G_n : F]$.

Consequently, it is enough to prove that $h|_{G_n} : G_n \to H_n$ is an epimorphism. For that, it is enough to prove that for each $1 \le k \le N$, x_k has a co-image in G_1. The reduced epimorphism $g|_{H_1}$ is an isomorphism (an easy exercise), and since $g \circ h|_{G_1}$ is an epimorphism, $h|_{G_1}$ is also an epimorphism. Hence, the result follows. $\square$

We propose the following algorithm for finding the co-image of x_k for any $1 \le k \le N$:

ALGORITHM 2.2. Let $p_1 = z$. Then $h(p_1) = x_1 + a_1 x_2 + \cdots + a_{N-1} x_N$. We shall define a sequence $\{p_j : j = 1, \dots, n\} \subseteq G_1$ as follows: If there exists a monomial in $h(p_j)$ whose coefficient a does not commute with the coefficient of x_k, denoted by c, then we shall define $p_{j+1} = a p_j a^{-1} - p_j$, by which we shall annihilate at least one monomial (the one whose coefficient is a), and yet the element x_k will not be annihilated, because $c x_k$ does not commute with a.

If c commutes with all the other coefficients, then we shall pick some monomial which we want to annihilate. Let b denote its coefficient. Now, we shall pick some $a \in D$ which does not commute with cb^{-1} and define $p_{j+1} = b a p_j b^{-1} a^{-1} - p_j$.

The element x_k is not annihilated in this process, because if we assume that it does at some point, let us say it is annihilated in $h(p_{j+1})$, then $bacb^{-1}a^{-1} - c = 0$. Therefore, $c^{-1}bacb^{-1}a^{-1} = 1$, and hence $cb^{-1}a^{-1} = (c^{-1}ba)^{-1} = a^{-1}b^{-1}c$. Since b commutes with c, a commutes with cb^{-1}, which is a contradiction.

In each iteration, the length of $h(p_j)$ (the number of monomials in it) decreases by at least one, and yet the element x_k always remains. Since the length of $h(p_1)$ is finite, this process will end with some p_n for which $h(p_n)$ is a monomial. In this case, $h(p_n) = c x_k$ and consequently, $x_k = h(c^{-1} p_n)$.

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2.1. Real quaternions. Let $D = \mathbb{H} = \mathbb{R} + i\mathbb{R} + j\mathbb{R} + ij\mathbb{R}$. Now
$h(z) = x_1 + x_2 i + x_3 j + x_4 ij$
$h(z - jzj^{-1}) = h(z + jzj) = 2x_2 i + 2x_4 ij$
$h((z + jzj) - ij(z + jzj)(ij)^{-1}) = 2x_2 i + 2x_4 ij - ij(2x_2 i + 2x_4 ij)(ij)^{-1} = 4x_2 i$
therefore $h^{-1}(x_2) = -\frac{1}{4} i((z + jzj) + ij(z + jzj)ij)) = -\frac{1}{4}(iz + ijzj - jzij + zi)$.

Similarly, $h^{-1}(x_1) = \frac{1}{4}(z - izi - jzj - ijzij)$, $h^{-1}(x_3) = -\frac{1}{4}(jz - ijzi + izij + zj)$ and $h^{-1}(x_4) = -\frac{1}{4}(ijz - izj + jzi + zij)$. Consequently, $\overline{z} = \overline{h^{-1}(x_1 + x_2 i + x_3 j + x_4 ij)} = h^{-1}(x_1 - x_2 i - x_3 j - x_4 ij) = -\frac{1}{2}(z + izi + jzj + ijzij)$.

3. The characteristic polynomial. Let D, F, d, N be the same as they were in the previous section.

There is an injection of D in $M_d(K)$ where K is a maximal subfield of D. (In particular, $[K : F] = d$.) More generally, there is an injection of $M_k(D)$ in $M_{kd}(K)$ for any $k \in \mathbb{N}$. Let $\widehat{A}$ denote the image of A in $M_{kd}(K)$ for any $A \in M_k(D)$.

The determinant of $\widehat{A}$ is equal to the Dieudonné determinant of A to the power of d. (The reduced norm of A is defined to be the determinant of $\widehat{A}$.)

Therefore, $\lambda \in D$ is a left eigenvalue of A if and only if $\det(\widehat{A - \lambda I}) = 0$. Considering D as an F-vector space $D = F + Fa_1 + \cdots + Fa_{N-1}$, we can write $\lambda = x_1 + x_2 a_1 + \cdots + x_N a_{N-1}$ for some $x_1, \ldots, x_N \in F$. Then $\det(\widehat{A - \lambda I}) \in F[x_1, \ldots, x_N]$. It can also be considered as a polynomial in $D\langle x_1, \ldots, x_N\rangle$. Now, there is an isomorphism $h : D_G[z] \to D\langle x_1, \ldots, x_N\rangle$, and so $h^{-1}(\det(\widehat{A - \lambda I})) \in D_G[z]$.

Defining $p_A(z) = h^{-1}(\det(\widehat{A - \lambda I}))$ to be the *characteristic polynomial* of A, the left eigenvalues of A are precisely the roots of $p_A(z)$. The degree of the characteristic polynomial of A is therefore kd.

REMARK 3.1. If one proves that the Dieudonné determinant of $A - \lambda I$ is the absolute value of some polynomial $q(x_1, \ldots, x_N) \in D_L[x_1, \ldots, x_N]$, then we will be able to define the characteristic polynomial to be $h^{-1}(q(x_1, \ldots, x_N))$ and obtain a characteristic polynomial of degree k.

4. The left eigenvalues of a 4×4 quaternion matrix. Let Q be a quaternion division F-algebra. Calculating the roots of the characteristic polynomial as defined in Section 3 is not always the best way to obtain the left eigenvalues of a given matrix.

The reductions, which Wood did in [7] and So did in [6], suggest that in order to obtain the left eigenvalues of a 2×2 or 3×3 matrix, one can calculate the roots of a polynomial of degree 2 or 3, respectively, instead of calculating the roots of the characteristic polynomial whose degree is d times greater.

In the next proposition, we show how (under a certain condition) the eigenval-

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ues of a 4×4 quaternion matrix can be obtained by calculating the roots of three polynomials of degree 2 and one of degree 6.

PROPOSITION 4.1. *Let $A, B, C, D \in M_2(Q)$, and suppose that C is invertible. If $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, then λ is a left eigenvalue of M if and only if either $e(\lambda) = f(\lambda)g(\lambda) = 0$ or $e(\lambda) \neq 0$ and $e(\lambda)\overline{e(\lambda)}h(\lambda) - g(\lambda)\overline{e(\lambda)}f(\lambda) = 0$, where $C(A - \lambda I)C^{-1}(D - \lambda I) - CB = \begin{bmatrix} e(\lambda) & f(\lambda) \\ g(\lambda) & h(\lambda) \end{bmatrix}$.*

Proof. Let M be a 4×4 quaternion matrix. Therefore, $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where A, B, C, D are 2×2 quaternion matrices.

An element λ is a left eigenvalue if $\det(M - \lambda I) = 0$. Assuming that $\det(C) \neq 0$, we have $\det(M - \lambda I) = \det(C(A - \lambda I)C^{-1}(D - \lambda I) - CB)$. (This is an easy result of the Schur complements identity for complex matrices extended to quaternion matrices in [2].)

The matrix $C(A - \lambda I)C^{-1}(D - \lambda I) - CB$ is equal to $\begin{bmatrix} e(\lambda) & f(\lambda) \\ g(\lambda) & h(\lambda) \end{bmatrix}$ for some quadratic polynomials e, f, g, h.

Now, if $e(\lambda) \neq 0$, then $\det \begin{bmatrix} e(\lambda) & f(\lambda) \\ g(\lambda) & h(\lambda) \end{bmatrix} = 0$ if and only if
$h(\lambda) - g(\lambda)e(\lambda)^{-1}f(\lambda) = 0$.

This happens if and only if $e(\lambda)\overline{e(\lambda)}h(\lambda) - g(\lambda)\overline{e(\lambda)}f(\lambda) = 0$. $\square$

As we saw in Subsection 2.1, $\overline{e(\lambda)}$ is also a quadratic polynomial, which means that $e(\lambda)\overline{e(\lambda)}h(\lambda) - g(\lambda)\overline{e(\lambda)}f(\lambda)$ is a polynomial of degree 6, while the characteristic polynomial of M as defined in Section 3 is of degree 8.

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Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 508-513, May 2012

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