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THE SMALLEST SIGNLESS LAPLACIAN EIGENVALUE OF GRAPHS UNDER PERTURBATION*

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Abstract. In this paper, we investigate how the smallest signless Laplacian eigenvalue of a graph behaves when the graph is perturbed by deleting a vertex, subdividing edges or moving edges.

Key words. Graph spectra, Signless Laplacian, Smallest eigenvalue.

AMS subject classifications. 05C50.

1. Introduction. Let $G = (V, E)$ be a simple graph of order n, where $V = \{v_1, v_2, \ldots, v_n\}$ is the vertex set of G and E is the edge set of G. Let $D_G = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ be the degree matrix of G, i.e., d_i is the degree of the vertex v_i. The *Laplacian* matrix L_G and the *signless Laplacian* matrix Q_G of a graph G are defined by

$$L_G = D_G - A_G \qquad \text{and} \qquad Q_G = D_G + A_G,$$

where A_G is the adjacency matrix of G. It is well known that L_G and Q_G are positive semi-definite and hence, all eigenvalues of L_G and Q_G are non-negative.

Assume that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ are all eigenvalues of L_G and $q_1 \geq q_2 \geq \cdots \geq q_n \geq 0$ are all eigenvalues of Q_G. It is not difficult to check that L_G has an eigenvector $(1, 1, \ldots, 1)^T$ with the eigenvalue $\lambda_n = 0$. Fiedler [1] showed that $\lambda_{n-1} = 0$ if and only if G is disconnected. Thus, the second smallest eigenvalue of L_G, λ_{n-1}, is popularly known as the *algebraic connectivity* of G and is usually denoted by $\alpha(G)$. Similarly, we may consider the smallest eigenvalue of Q_G. Define $q(G) = q_n$. It is well known that $q(G) \geq 0$, and the equality occurs if and only if some connected component is bipartite. The multiplicity of $q(G) = 0$ is equal to the number of bipartite connected components of G.

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For an edge e of G, let $G-e$ denote the graph obtained from G by deleting e. And for a vertex v of G, let $G-v$ denote the graph arising from G by deleting the vertex v and all its incident edges. In [2], Guo investigated how $\alpha(G)$ changes when G is perturbed by separating an edge. In [3], S. Kirkland considered the functions $\Phi(v)=\alpha(G)-\alpha(G-v)$ and $k(v)=\frac{\alpha(G-v)}{\alpha(G)}$, found upper and lower bounds on both functions, and characterized the equality cases in those bounds. For recent developments on $q(G)$, see, e.g., [4]–[8].

Motivated by the works on $\alpha(G)$, in this paper, we shall investigate how $q(G)$ changes when G is perturbed by deleting a vertex, subdividing edges or moving edges. For a vertex v of G, let $N(v)$ denote the set of all neighbors of v in G. For a vector $\mathbf{w}$, the *support* of $\mathbf{w}$ is the set of indices corresponding to the nonzero entries of $\mathbf{w}$.

THEOREM 1.1. *Let G be a graph, and let v be a vertex of G. Then*

$$q(G)\leq q(G-v)+1. \tag{1.1}$$

Furthermore, the equality holds if and only if there is unit eigenvector for $q(G)$ whose support is a subset of $N(v)$.

Theorem 1.1 gives an upper bound of $q(G)-q(G-v)$. The following theorem gives a lower bound of $q(G)-q(G-v)$.

THEOREM 1.2. *Let G be a connected graph on $n\geq 3$ vertices. Then, for each vertex v of G,*

$$q(G)-q(G-v)\geq \frac{3-\sqrt{4n^2-20n+33}}{2}. \tag{1.2}$$

Furthermore, the equality holds if and only if the degree of v is one and $G-v$ is a complete subgraph of G.

The proofs of Theorems 1.1 and 1.2 are given in the next two sections. In addition, we will discuss the behaviors of $q(G)$ after some edge operations in Section 4.

2. Proof of Theorem 1.1. We begin with two well-known lemmas.

LEMMA 2.1 (Interlacing Theorem, [10]). *Let e be an edge of the graph G. Let $q_1\geq q_2\geq\cdots\geq q_n$ and $s_1\geq s_2\geq\cdots\geq s_n$ be the eigenvalues of Q_G and Q_{G-e}, respectively. Then*

$$q_1\geq s_1\geq q_2\geq s_2\geq\cdots\geq q_n\geq s_n\geq 0.$$

From the above lemma, we know that if two graphs G_1 and G_2 have the same vertex set and $E(G_1)\subseteq E(G_2)$, then $q(G_1)\leq q(G_2)$.

LEMMA 2.2 ([9]). *If A is a symmetric n-by-n matrix with eigenvalues $\lambda_1\geq\lambda_2\geq$*

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$\cdots \geq \lambda_n$, then, for any non-zero vector $\mathbf{x} \in R^n$,

$$\mathbf{x}^T A\mathbf{x} \geq \lambda_n \mathbf{x}^T \mathbf{x}.$$

Furthermore, the equality holds if and only if $\mathbf{x}$ is an eigenvector of A corresponding to the smallest eigenvalue λ_n.

Let G_1 and G_2 be two graphs with disjoint vertex sets. The join of G_1 and G_2, which we denote by $G_1 \vee G_2$, is the graph formed from the disjoint union of G_1 and G_2 by adding all possible edges between the vertices of G_1 and the vertices of G_2. For convenience, we denote the $n \times 1$ vectors $(1, 1, \ldots, 1)^T$ and $(0, 0, \ldots, 0)^T$ as $\mathbf{1}_n$ and $\mathbf{0}_n$, respectively.

LEMMA 2.3. *For a graph G, $q(G \vee K_1) \leq q(G) + 1$.*

Proof. Suppose that G has n vertices. Let $\mathbf{y}$ be a unit eigenvector of Q_G corresponding to eigenvalue $q(G)$. Clearly,

$$Q_{G\vee K_1} = \begin{pmatrix} Q_G + I_n & \mathbf{1}_n \\ \mathbf{1}_n^T & n \end{pmatrix}.$$

Let $\tilde{\mathbf{y}} = \begin{pmatrix} \mathbf{y} \\ 0 \end{pmatrix}$. Then, by Lemma 2.2,

$$\begin{aligned} q(G \vee K_1) &\leq \tilde{\mathbf{y}}^T Q_{G\vee K_1} \tilde{\mathbf{y}} \\ &= \left(\mathbf{y}^T, 0\right) \begin{pmatrix} Q_G + I_n & \mathbf{1}_n \\ \mathbf{1}_n^T & n \end{pmatrix} \begin{pmatrix} \mathbf{y} \\ 0 \end{pmatrix} \\ &= \mathbf{y}^T Q_G \mathbf{y} + \mathbf{y}^T \mathbf{y} \\ &= q(G) + 1. \quad \square \end{aligned}$$

Using Lemma 2.3, we can derive the inequality (1.1). Since G is a spanning subgraph of $(G - v) \vee K_1$, we have

$$q(G) \leq q((G - v) \vee K_1) \leq q(G - v) + 1.$$

Now, suppose that $q(G) = q(G-v)+1$. Recall that the vertex set $V = \{v_1, \ldots, v_n\}$ and $N(v)$ is the set of all neighbors of v. Suppose that the degree of v is m. Without loss of generality, assume that $N(v) = \{v_1, \ldots, v_m\}$ and $v = v_n$. Write

$$Q_{G-v} = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}$$

and

$$Q_G = \begin{pmatrix} Q_{11} + I_m & Q_{12} & \mathbf{1}_m \\ Q_{21} & Q_{22} & \mathbf{0}_{n-m-1} \\ \mathbf{1}_m^T & \mathbf{0}_{n-m-1}^T & m \end{pmatrix},$$

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where Q_{11} is an $m \times m$ matrix and Q_{22} is an $(n-m-1) \times (n-m-1)$ matrix.

Suppose that $\mathbf{w} = \begin{pmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{pmatrix}$ be a unit eigenvector corresponding to $q(G-v)$, where $\mathbf{w}_1$ is an $m \times 1$ vector. Let $\mathbf{z}$ be the vector $\begin{pmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{w} \\ 0 \end{pmatrix}$. Clearly, by Lemma 2.2,

$$q(G-v)\mathbf{z}^T\mathbf{z} + \mathbf{w}_1^T\mathbf{w}_1 + \mathbf{w}_2^T\mathbf{w}_2 = (q(G-v)+1)\mathbf{z}^T\mathbf{z} = q(G)\mathbf{z}^T\mathbf{z} \le \mathbf{z}^T Q_G \mathbf{z},$$

and the equality holds in the last inequality if and only if $\mathbf{z}$ is an eigenvector corresponding to $q(G)$. Furthermore, we have

$$\mathbf{z}^T Q_G \mathbf{z} = \mathbf{w}^T Q_{G-v}\mathbf{w} + \mathbf{w}_1^T\mathbf{w}_1 = q(G-v)\mathbf{w}^T\mathbf{w} + \mathbf{w}_1^T\mathbf{w}_1 = q(G-v)\mathbf{z}^T\mathbf{z} + \mathbf{w}_1^T\mathbf{w}_1.$$

Hence, we must have that $\mathbf{w}_2 = 0$ and $\mathbf{z}$ is an eigenvector for $q(G)$.

Conversely, assume that Q_G has a unit eigenvector $\mathbf{y}$ for $q(G)$, and the support of y is a subset of $N(v)$. Write

$$Q_G = \begin{pmatrix} Q_{G-v} + D & \mathbf{x} \\ \mathbf{x}^T & m \end{pmatrix},$$

where D is a diagonal matrix with ones in the diagonal positions corresponding to vertices in $N(v)$, and zero elsewhere. Let $\mathbf{y} = \begin{pmatrix} \tilde{\mathbf{y}} \\ 0 \end{pmatrix}$. Since $\mathbf{y}^T Q_G \mathbf{y} = q(G)\mathbf{y}^T\mathbf{y}$, we have

$$\tilde{\mathbf{y}}^T Q_{G-v}\tilde{\mathbf{y}} + \tilde{\mathbf{y}}^T D\tilde{\mathbf{y}} = \mathbf{y}^T Q_G \mathbf{y} = q(G)\mathbf{y}^T\mathbf{y} = q(G)\tilde{\mathbf{y}}^T\tilde{\mathbf{y}}.$$

Hence,

$$\tilde{\mathbf{y}}^T Q_{G-v}\tilde{\mathbf{y}} = q(G)\tilde{\mathbf{y}}^T\tilde{\mathbf{y}} - \tilde{\mathbf{y}}^T D\tilde{\mathbf{y}} = (q(G)-1)\tilde{\mathbf{y}}^T\tilde{\mathbf{y}},$$

where the last equality follows from the fact that $\mathbf{y}$ (and hence, $\tilde{\mathbf{y}}$) has support in $N(v)$. By Lemma 2.2,

$$q(G-v) = q(G-v)\tilde{\mathbf{y}}^T\tilde{\mathbf{y}} \le \tilde{\mathbf{y}}^T Q_{G-v}\tilde{\mathbf{y}} = (q(G)-1)\tilde{\mathbf{y}}^T\tilde{\mathbf{y}} = q(G) - 1.$$

Since $q(G-v) \ge q(G) - 1$ by (1.1), we have $q(G-v) = q(G) - 1$.

Remark 2.4. The inequality of Theorem 1.1, as well as its proof, is carried out in the same way as in the case of the algebraic connectivity in [1]. And, the discussion of the equality case in Theorem 1.1, as well as its proof, is very similar to that of Theorem 2.1 in [3].

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3. Proof of Theorem 1.2. We begin with lemmas that are needed for the proof of Theorem 1.2.

LEMMA 3.1. *Let G be the graph arising from K_{n-1} by adding a vertex v and m edges between v and vertices of K_{n-1}. Then*

$$q(G) = \frac{2n+m-4-\sqrt{m^2+4m(4-n)+4(n-2)^2}}{2}.$$

Proof. We use J to denote the all-ones matrix. Clearly,

$$\det(qI_n - Q_G)$$
$$= \begin{vmatrix} q-m & -\mathbf{1}_m^T & \mathbf{0}_{n-m-1}^T \\ -\mathbf{1}_m & -J_m+(q-n+2)I_m & -J_{n-m-1} \\ \mathbf{0}_{n-m-1} & -J_{n-m-1} & -J_{n-m-1}+(q-n+3)I_m \end{vmatrix}.$$

Subtract the $(i-1)$-th row from the i-th row, where i runs from 3 to $m+1$, and subtract the $(j-1)$-th row from the j-th row, where j runs from $m+3$ to n. Thus, we get that $\underbrace{n-2,\dots,n-2}_{m-1}$, $\underbrace{n-3,\dots,n-3}_{n-m-2}$ are eigenvalues of Q_G. Let α, β, γ be the other three eigenvalues of Q_G. Then

$$\alpha+\beta+\gamma = (n-1)(n-2)+2m-(m-1)(n-2)-(n-m-2)(n-3) = 3n+m-6.$$

If $\mathbf{x} = (x_1,\dots,x_n)^T$ is an eigenvector of Q_G corresponding to the eigenvalue q, then we have

$$Q_G\mathbf{x} = q\mathbf{x},$$

or equivalently

$$\begin{aligned} &mx_1 + x_2 + \cdots + x_{m+1} = qx_1, \\ &(n-1)x_i + x_1 + x_2 + \cdots + x_n - x_i = qx_i, \qquad \text{for all } i = 2,\dots,m+1, \\ &(n-2)x_j + x_2 + \cdots + x_n - x_j = qx_j, \qquad \text{for all } j = m+2,\dots,n. \end{aligned}$$

Thus, we can conclude that

$$q^2 - 2nq - mq + 4q + 2mn - 6m = 0. \tag{3.1}$$

Without loss of generality, let

$$\alpha,\beta = \frac{2n+m-4 \pm \sqrt{m^2+4m(4-n)+4(n-2)^2}}{2}$$

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be the two roots of (3.1). Then

$$\gamma = 3n + m - 6 - (\alpha + \beta) = n - 2.$$

Therefore,

$$\frac{2n + m - 4 - \sqrt{m^2 + 4m(4-n) + 4(n-2)^2}}{2}$$

is the smallest signless Laplacian eigenvalue of G. □

With a similar discussion, we can get the following result.

Lemma 3.2.

$$q(K_{n-m} \vee (mK_1)) = \frac{3n - 2m - 2 - \sqrt{n^2 + 4(m-1)(n-m-1)}}{2}.$$

Corollary 3.3. *Let G be a graph with n vertices, containing an independent set of m vertices. Then*

$$q(G) \le \frac{3n - 2m - 2 - \sqrt{n^2 + 4(m-1)(n-m-1))}}{2}.$$

Proof. If G contains an independent set of m vertices, then G is a spanning subgraph of $K_{n-m} \vee (mK_1)$. Thus, by Lemma 3.2,

$$q(G) \le q(K_{n-m} \vee (mK_1)) = \frac{3n - 2m - 2 - \sqrt{n^2 + 4(m-1)(n-m-1)}}{2}. \quad \square$$

Corollary 3.4. *If G is not a complete graph with n vertices. Then*

$$q(G) \le \frac{3n - 6 - \sqrt{n^2 + 4n - 12}}{2}.$$

Proof. Since G is not complete, G contains an independent set of at least two vertices. The result follows from Corollary 3.3. □

Proof of Theorem 1.2. If $G - v \ne K_{n-1}$, then, by Corollary 3.4,

$$q(G - v) \le \frac{3n - 9 - \sqrt{n^2 + 2n - 15}}{2}.$$

Thus,

$$q(G) - q(G - v) \ge 0 - \frac{3n - 9 - \sqrt{n^2 + 2n - 15}}{2} > \frac{3 - \sqrt{4n^2 - 20n + 33}}{2}.$$

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Next, suppose that $G - v = K_{n-1}$, and that $d_G(v) = m$. From Lemma 3.1, it follows that

$$q(G) = \frac{2n + m - 4 - \sqrt{m^2 + 4m(4-n) + 4(n-2)^2}}{2}.$$

Hence,

$$q(G) - q(G - v) = \frac{m + 2 - \sqrt{m^2 + 4m(4-n) + 4(n-2)^2}}{2} \geq \frac{3 - \sqrt{4n^2 - 20n + 33}}{2},$$

by noting that $m - \sqrt{m^2 + 4m(4-n) + 4(n-2)^2}$ is a strictly increasing function of m when $n \geq 3$. Thus,

$$q(G) - q(G - v) \geq \frac{3 - \sqrt{4n^2 - 20n + 33}}{2},$$

and equality holds if and only if $m = 1$. ∎

4. The smallest signless Laplacian eigenvalue of a graph under edge operations. Suppose that G is a graph with at least one edge uv. For $k \geq 1$, let G' be the graph obtained from G by deleting the edge uv, inserting k new vertices $v_1, v_2, \ldots, v_k$ and adding edges $uv_1, v_1v_2, \ldots, v_{k-1}v_k, v_kv$. Then we call G' a *k-subdivision graph* of G, and say that G' is derived from G by k-subdividing the edge uv.

LEMMA 4.1 ([11]). *Let A, B, and C be n-by-n Hermitian matrices satisfying $A = B + C$. Denote the eigenvalues of A and B by $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ and $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$, respectively. If C has exactly t positive eigenvalues, then $\beta_k \geq \alpha_{k+t}$ for all $1 \leq k \leq n - t$.*

LEMMA 4.2. *Let G be a graph with order n and G' be a 1-subdivision graph of G. Then $q(G') \leq q_{n-1}(G)$, where $q_{n-1}(G)$ is the second smallest eigenvalue of Q_G.*

Proof. Let $Q_0 = (0) \oplus Q_G$, where $\oplus$ denotes the direct sum of matrices. Let

$$P = \begin{pmatrix} P_{11} & \mathbf{0}_{3\times(n-2)} \\ \mathbf{0}_{(n-2)\times 3} & \mathbf{0}_{(n-2)\times(n-2)} \end{pmatrix},$$

where

$$P_{11} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

and $\mathbf{0}_{s\times t}$ denotes the $s \times t$ zero matrix. Then we have that $Q_{G'} = Q_0 + P$. By a simple calculation, we know that the non-zero eigenvalues of P are 1 and $\frac{1}{2}(1 \pm \sqrt{17})$,

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i.e., P has exactly two positive eigenvalues. Substituting $A = Q_{G'}$, $B = Q_0$, and $C = P$ in Lemma 4.1, we get

$$q(G') = q_{n+1}(G') \leq q_{n-1}(G). \quad \square$$

REMARK 4.3. However, we cannot directly compare $q(G)$ and $q(G')$ in the following sense: If G is a bipartite graph and uv is a cut edge, then $q(G') = q(G)$. If G is a bipartite graph and uv is not a cut edge, then G' is a non-bipartite graph and $q(G') > q(G)$. If G is a non-bipartite graph which contains a unique cycle, and uv is an edge of this cycle, then G' is bipartite and $q(G') < q(G)$.

LEMMA 4.4. *Let G be a graph and G' be a 2-subdivision graph of G. Then $q(G') \leq q(G)$.*

Proof. Let $Q_0 = \mathbf{0}_{2\times 2} \oplus Q_G$. Then $Q_{G'} = Q_0 + P$, where

$$P = \begin{pmatrix} \begin{matrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{matrix} & \mathbf{0}_{4\times(n-2)} \\ \mathbf{0}_{(n-2)\times 4} & \mathbf{0}_{(n-2)\times(n-2)} \end{pmatrix}.$$

It is easy to see that the non-zero eigenvalues of P are 2, $1 \pm \sqrt{5}$, whose algebraic multiplicity are all one. Then, by Lemma 4.1, we have $q(G') \leq q(G)$. $\square$

Thus, combining Lemmas 4.2 and 4.4, we have the following theorem.

THEOREM 4.5. *Let G be a graph with n vertices and G' be a k-subdivision graph of G. If k is odd, then*

$$q(G') \leq q_{n-1}(G).$$

If k is even, then

$$q(G') \leq q(G).$$

Proof. Using Lemmas 4.2 and 4.4, the results easily follow from induction on k. $\square$

Next, we shall investigate how the smallest signless Laplacian eigenvalue of a graph behaves when the graph is perturbed by moving edges.

LEMMA 4.6 ([11]). *Let A, B and C be n-by-n Hermitian matrices satisfying $A = B + C$. Denote the eigenvalues of A and B by $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ and $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$, respectively. If C has exactly one positive eigenvalue and one negative eigenvalue, then $\alpha_k \geq \beta_{k+1}$ and $\beta_k \geq \alpha_{k+1}$ for all $1 \leq k < n$.*

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THEOREM 4.7. *Let G be a graph with n vertices. Suppose that u and v are two vertices of G, and $\{u_1, \ldots, u_k\} \subseteq N(u)\backslash N(v)$. Let*

$$G_v(u) = G - uu_1 - \cdots - uu_k + vu_1 + \cdots + vu_k.$$

Then, for all $1 \leq k < n$, we have

$$q_k(G) \geq q_{k+1}(G_v(u))$$

and

$$q_k(G_v(u)) \geq q_{k+1}(G).$$

Proof. Let

$$P = \begin{pmatrix} -k & 0 & -\mathbf{1}_k^T & \\ 0 & k & \mathbf{1}_k^T & \mathbf{0}_{2\times(n-k-2)} \\ -\mathbf{1}_k & \mathbf{1}_k & \mathbf{0}_{k\times k} & \\ & \mathbf{0}_{(n-k-2)\times 2} & & \mathbf{0}_{(n-k-2)\times(n-k-2)} \end{pmatrix}.$$

Then $Q_{G'} = Q_G + P$ and the non-zero eigenvalues of P are $\pm\sqrt{k(k+2)}$ (with the algebraic multiplicity one). Then, by Lemma 4.6, we have $q_k(G) \geq q_{k+1}(G_v(u))$ and $q_k(G_v(u)) \geq q_{k+1}(G)$ for all $1 \leq k < n$. □

COROLLARY 4.8. *Let G be a graph on n vertices. Suppose that u, v are two vertices of G and $\{u_1, \ldots, u_k\} \subseteq N(u)\backslash N(v)$. Let*

$$G_v(u) = G - uu_1 - \cdots - uu_k + vu_1 + \cdots + vu_k.$$

Then $q(G_v(u)) \leq q_{n-1}(G)$.

Let $e = uv$ be a cut edge of a graph G. Let G' be the graph arising from G by contracting the edge e into a new vertex u_e which becomes adjacent to all the former neighbors of u and of v, and adding a new pendant edge $u_e v_e$, where v_e is a new pendant vertex. Then we say that G' is constructed from G by separating a cut edge uv (see Fig. 1).

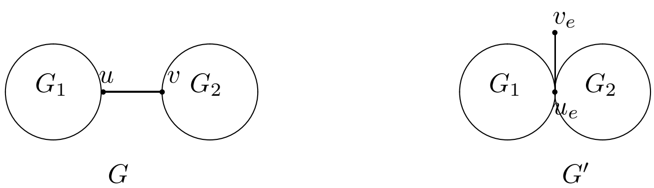

Fig. 1. Separating a cut edge uv.

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By Theorem 4.7, we have that

COROLLARY 4.9. *Let G be a connected graph of order n, and uv be a cut edge of G. Let G' be obtained from G by separating edge uv. Then $q_k(G) \geq q_{k+1}(G')$ and $q_k(G') \geq q_{k+1}(G)$, for all $1 \leq k < n$.*

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