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THE M-COMPETITION INDICES OF SYMMETRIC PRIMITIVE DIGRAPHS WITHOUT LOOPS*

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Abstract. For positive integers m and n with $1 \leq m \leq n$, the m-competition index (generalized competition index) of a primitive digraph D of order n is the smallest positive integer k such that for every pair of vertices x and y in D, there exist m distinct vertices $v_1, v_2, \ldots, v_m$ such that there exist walks of length k from x to v_i and from y to v_i for each $i = 1, \ldots, m$. In this paper, we study the generalized competition indices of symmetric primitive digraphs without loops. We determine the generalized competition index set and characterize the digraphs in this class with largest generalized competition index.

Key words. Competition index, m-Competition index, Scrambling index, Generalized competition index.

AMS subject classifications. 05C50, 15A48, 05C20.

1. Introduction. For terminology and notation used here, we follow those in [3, 5]. Let $D = (V, E)$ denote a digraph on n vertices with the vertex set $V = V(D)$ and the arc set $E = E(D)$. Loops are permitted but multiple arcs are not. A walk in D is a sequence $w = v_0 v_1 \cdots v_k$ such that for $1 \leq i \leq k$, there exists an arc from v_{i-1} to v_i. A digraph D is called *primitive* if there exists a positive integer k such that for any pair of vertices u and v, there is a walk of length k from vertex u to vertex v. The smallest such k is called the *exponent* of D, and it is denoted by $\exp(D)$. It is well known that D is primitive if and only if D is strongly connected and the greatest common divisor of the lengths of all the cycles in D is 1.

The length of a walk w is denoted by $l(w)$. The distance from vertex u to vertex v in D is the length of a shortest walk from u to v, and is denoted by $d_D(u, v)$ (or simply $d(u, v)$). For $X \subseteq V(D)$, set $d(u, X) = \min_{v \in X} d(u, v)$. The notation $u \xrightarrow{k} v$ indicates that there is a walk of length k from u to v. For distinct r vertices $v_1, v_2, \ldots, v_r$, the notation $C_r = v_1 v_2 \cdots v_r v_1$ means that C_r is the r-cycle consisting of the arcs (v_r, v_1) and $(v_i, v_{i+1}), 1 \leq i \leq r-1$. Let D be a primitive digraph of

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order n. For positive integers m and n with $1 \le m \le n$, we define the *m-competition index* (*generalized competition index*) of the primitive digraph D, denoted by $k_m(D)$, as the smallest positive integer k such that for every pair of vertices x and y, there exist m distinct vertices $v_1, v_2, \ldots, v_m$ such that $x \xrightarrow{k} v_i$ and $y \xrightarrow{k} v_i$ in D for each $i = 1, \ldots, m$.

Akelbek and Kirkland [1, 2] introduced the scrambling index of a primitive digraph D, denoted by $k(D)$. Kim [5] introduced the m-competition index as a generalization of the competition index. In the case of primitive digraphs, the definitions of the scrambling index and 1-competition index are identical, i.e., $k(D) = k_1(D)$.

For a positive integer k and a primitive digraph D, we define the *k-step outneighborhood* of a vertex x as

$$N^+(D^k : x) = \{v \in V(D) \mid x \xrightarrow{k} v\}.$$

The *k-step common outneighborhood* of vertices x and y is defined as

$$N^+(D^k : x, y) = N^+(D^k : x) \cap N^+(D^k : y).$$

The *local m-competition index* of vertices x and y is defined as

$$k_m(D : x, y) = \min\{k : |N^+(D^t : x, y)| \ge m, \text{ for all } t \ge k\},$$

and the *local m-competition index* of x is defined as

$$k_m(D : x) = \max_{y \in V(D)}\{k_m(D : x, y)\}.$$

Then we have

$$k_m(D) = \max_{x \in V(D)} k_m(D : x) = \max_{x,y \in V(D)} k_m(D : x, y).$$

The m-competition index is a generalization of the scrambling index and the exponent of a primitive digraph. It was known that for $1 \le m \le n$ (for example see [5]),

$$k(D) = k_1(D) \le k_2(D) \le \cdots \le k_n(D) = \exp(D).$$

A symmetric digraph is a digraph such that for any vertices u and v, (u, v) is an arc if and only if (v, u) is an arc. An undirected graph (possibly with loops) can be regarded as a symmetric digraph.

There has been interest recently in generalized competition index [5, 6, 7, 9]. Let S_n^0 denote the set of all symmetric primitive digraphs of order n without loops. In this paper, we study the m-competition indices of digraphs in S_n^0 with $n \ge 6$. We determine the m-competition index set for S_n^0, and characterize the digraphs in S_n^0 with the largest m-competition index.

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2. The generalized competition indices for special graphs. In this section, we study the generalized competition indices for some special graphs. Let $S_n(r)$ denote the set of all symmetric primitive digraphs of order n having a cycle of odd length r but no cycle of any odd length less than r.

LEMMA 2.1. *Let $G \in S_n(r)$ and C_r be an r-cycle in G. For any positive integer k and any vertex $u \in V(C_r)$,*

$$|N^+(G^k : u) \cap V(C_r)| = \min\{k+1, r\}.$$

Proof. Set $C_r = v_1 v_2 \cdots v_r v_1$. Note that for any positive integer k and any vertex $v_i \in V(C_r)$,

$$N^+(G^k : v_i) \cap V(C_r) = \begin{cases} \{v_i, v_{i+2}, \ldots, v_{i+k}, v_{i-2}, v_{i-4}, \ldots, v_{i-k}\}, & \text{if } k \text{ is even,} \\ \{v_{i+1}, v_{i+3}, \ldots, v_{i+k}, v_{i-1}, v_{i-3}, \ldots, v_{i-k}\}, & \text{if } k \text{ is odd,} \end{cases}$$

where the vertex subscripts are taken modulo r. Then

$$|N^+(G^k : v_i) \cap V(C_r)| = \begin{cases} k+1, & \text{if } 1 \le k \le r-1, \\ r, & \text{if } k \ge r, \end{cases}$$

that is, $|N^+(G^k : v_i) \cap V(C_r)| = \min\{k+1, r\}$. ☐

Now, we study the graphs in Figure 1, where r is odd with $3 \le r \le n-1$, and $1 \le l \le n-r$.

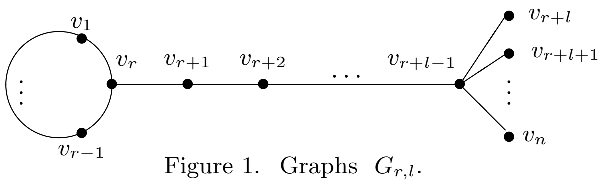

Figure 1. Graphs $G_{r,l}$.

THEOREM 2.2. *Let $G_{r,l}$ be primitive graphs as shown in Figure 1. For $2 \le m \le n-1$,*

$$k_m(G_{r,l}) = \begin{cases} l + \lfloor \frac{r+m-2}{2} \rfloor, & \textit{if } 2 \le m \le r-1, \\ l+m-1, & \textit{if } r \le m \le r+l-1, \\ 2l+r-1, & \textit{if } m \ge r+l. \end{cases}$$

Proof. We prove the statement case by case.

Case 1. $2 \le m \le r-1$.

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Let C_r be the only r-cycle in $G_{r,l}$. For any $v_i, v_j \in V(G_{r,l})$, note that

$$d(v_i, C_r) \le l, \quad d(v_j, C_r) \le l, \text{ and } \frac{r-1}{2} \le \left\lfloor \frac{r+m-2}{2} \right\rfloor \le r-2.$$

If $i \ge r+l$ and $j \ge r+l$, then by Lemma 2.1,

$$\begin{aligned}
&|N^+(G_{r,l}^{l+\lfloor \frac{r+m-2}{2} \rfloor} : v_i, v_j)| \\
&\ge |N^+(G_{r,l}^{l+\lfloor \frac{r+m-2}{2} \rfloor} : v_i, v_j) \cap V(C_r)| \\
&= |N^+(G_{r,l}^{l+\lfloor \frac{r+m-2}{2} \rfloor} : v_{r+l}) \cap V(C_r)| \\
&= |N^+(G_{r,l}^{\lfloor \frac{r+m-2}{2} \rfloor} : v_r) \cap V(C_r)| \\
&= \min\{\left\lfloor \frac{r+m-2}{2} \right\rfloor + 1, r\} \\
&= \left\lfloor \frac{r+m-2}{2} \right\rfloor + 1 \\
&\ge \left\lfloor \frac{m+1+m-2}{2} \right\rfloor + 1 = m.
\end{aligned}$$

If $i < r+l$, then $l - d(v_i, C_r) \ge 1$. By Lemma 2.1,

$$\begin{aligned}
&|N^+(G_{r,l}^{l+\lfloor \frac{r+m-2}{2} \rfloor} : v_i) \cap V(C_r)| \\
&= \begin{cases} |N^+(G_{r,l}^{l-d(v_i,C_r)+\lfloor \frac{r+m-2}{2} \rfloor} : v_r) \cap V(C_r)|, & \text{if } r \le i < r+l \\ |N^+(G_{r,l}^{l+\lfloor \frac{r+m-2}{2} \rfloor} : v_i) \cap V(C_r)|, & \text{if } i < r-1 \end{cases} \\
&= \min\{l - d(v_i, C_r) + \left\lfloor \frac{r+m-2}{2} \right\rfloor + 1, r\} \\
&\ge \min\{\left\lfloor \frac{r+m-2}{2} \right\rfloor + 2, r\} \\
&= \left\lfloor \frac{r+m-2}{2} \right\rfloor + 2.
\end{aligned}$$

So, if $i < r+l$ or $j < r+l$, then

$$\begin{aligned}
&|N^+(G_{r,l}^{l+\lfloor \frac{r+m-2}{2} \rfloor} : v_i, v_j)| \\
&\ge |N^+(G_{r,l}^{l+\lfloor \frac{r+m-2}{2} \rfloor} : v_i, v_j) \cap V(C_r)| \\
&= \left|\left(N^+(G_{r,l}^{l+\lfloor \frac{r+m-2}{2} \rfloor} : v_i) \cap V(C_r)\right) \cap \left(N^+(G_{r,l}^{l+\lfloor \frac{r+m-2}{2} \rfloor} : v_j) \cap V(C_r)\right)\right| \\
&\ge \left|N^+(G_{r,l}^{l+\lfloor \frac{r+m-2}{2} \rfloor} : v_i) \cap V(C_r)\right| + \left|N^+(G_{r,l}^{l+\lfloor \frac{r+m-2}{2} \rfloor} : v_j) \cap V(C_r)\right)\right| - r \\
&\ge 2\left\lfloor \frac{r+m-2}{2} \right\rfloor + 3 - r \ge m.
\end{aligned}$$

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Therefore, $k_m(G_{r,l}) \le l + \left\lfloor \frac{r+m-2}{2} \right\rfloor$.

We now show that $k_m(G_{r,l}) > l + \left\lfloor \frac{r+m-2}{2} \right\rfloor - 1$. If $\left\lfloor \frac{r+m-2}{2} \right\rfloor - 1$ is even, then

$$\begin{aligned}
&N^+(G_{r,l}^{\lfloor \frac{r+m-2}{2} \rfloor -1} : v_r) \cap V(C_r) \\
&= \{v_r, v_2, v_4, \ldots, v_{\lfloor \frac{r+m-2}{2} \rfloor -1}, v_{r-2}, v_{r-4}, \ldots, v_{r-\lfloor \frac{r+m-2}{2} \rfloor +1}\}, \\
&N^+(G_{r,l}^{\lfloor \frac{r+m-2}{2} \rfloor} : v_r) \cap V(C_r) \\
&= \{v_1, v_3, \ldots, v_{\lfloor \frac{r+m-2}{2} \rfloor}, v_{r-1}, v_{r-3}, \ldots, v_{r-\lfloor \frac{r+m-2}{2} \rfloor}\},
\end{aligned}$$

and

$$\begin{aligned}
&\left(N^+(G_{r,l}^{\lfloor \frac{r+m-2}{2} \rfloor -1} : v_r) \cap V(C_r)\right) \cap \left(N^+(G_{r,l}^{\lfloor \frac{r+m-2}{2} \rfloor} : v_r) \cap V(C_r)\right) \\
&= \begin{cases} \{v_{r-\lfloor \frac{r+m-2}{2} \rfloor}, \ldots, v_{\lfloor \frac{r+m-2}{2} \rfloor}\}, & \text{if } m \ge 3, \\ \phi, & \text{if } m = 2. \end{cases}
\end{aligned}$$

If $\left\lfloor \frac{r+m-2}{2} \right\rfloor - 1$ is odd, then the result follows in a similar manner. Thus,

$$\begin{aligned}
&|N^+(G_{r,l}^{l+\lfloor \frac{r+m-2}{2} \rfloor -1} : v_{r+l}, v_{r+l-1})| \\
&= |N^+(G_{r,l}^{l+\lfloor \frac{r+m-2}{2} \rfloor -1} : v_{r+l}, v_{r+l-1}) \cap V(C_r)| \\
&= \left|\left(N^+(G_{r,l}^{l+\lfloor \frac{r+m-2}{2} \rfloor -1} : v_{r+l}) \cap V(C_r)\right) \cap \left(N^+(G_{r,l}^{l+\lfloor \frac{r+m-2}{2} \rfloor -1} : v_{r+l-1}) \cap V(C_r)\right)\right| \\
&= \left|\left(N^+(G_{r,l}^{\lfloor \frac{r+m-2}{2} \rfloor -1} : v_r) \cap V(C_r)\right) \cap \left(N^+(G_{r,l}^{\lfloor \frac{r+m-2}{2} \rfloor} : v_r) \cap V(C_r)\right)\right| \\
&\le m-1.
\end{aligned}$$

So $k_m(G_{r,l}) > l + \left\lfloor \frac{r+m-2}{2} \right\rfloor - 1$. Then $k_m(G_{r,l}) = l + \left\lfloor \frac{r+m-2}{2} \right\rfloor$.

Case 2. $r \le m \le r+l-1$.

For any $v_i \in V(G_{r,l})$, we will show that $\{v_1, v_2, \ldots, v_m\} \subseteq N^+(G_{r,l}^{l+m-1} : v_i)$. First, consider a vertex $u \in \{v_{r+l}, v_{r+l+1}, \ldots, v_n\}$. Note that $w = uv_{r+l-1}\cdots v_m$ is the walk from u to v_m of length $r+l-m$. Since $r+l-m$ and $l+m-1$ have the same parity, and $r+l-m \le l+m-1$, we have $v_m \in N^+(G_{r,l}^{l+m-1} : u)$. Note that there exist two walks w_1 and w_2 from u to v_i $(1 \le i \le m-1)$ such that $l(w_1) \le l+m-1$, $l(w_2) \le l+m-1$, and $l(w_1)$ and $l(w_2)$ have different parity. So $v_i \in N^+(G_{r,l}^{l} : u)$.

Next, for any vertex v_i, $1 \le i \le r+l-1$, we can show $\{v_1, v_2, \ldots, v_m\} \subseteq N^+(G_{r,l}^{l+m-1} : v_i)$ similarly. Then $|N^+(G_{r,l}^{l+m-1} : v_i, v_j)| \ge m$ for any $v_i, v_j \in V(G_{r,l})$, and $k_m(G_{r,l}) \le l+m-1$.

We now show that $k_m(G_{r,l}) > l+m-2$. If $l+m-2$ is even, then

$$N^+(G_{r,l}^{l+m-2} : v_{r+l})$$

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$$= V(G_{r,l})\backslash\{v_k \mid m \le k \le r+l-1, \text{ and } k \equiv (l+r-1)(\bmod 2)\},$$
$$N^+(G_{r,l}^{l+m-2} : v_{r+l-1})$$
$$= \{v_1, v_2, \ldots, v_{r+l-1}\}\backslash\{v_k \mid m+1 \le k \le r+l-2, \text{ and } k \equiv (l+r-2)(\bmod 2)\}.$$

Therefore,

$$N^+(G_{r,l}^{l+m-2} : v_{r+l-1}) \cap N^+(G_{r,l}^{l+m-2} : v_{r+l}) = \{v_1, v_2, \ldots, v_{m-1}\}.$$

If $l+m-2$ is odd, then the result follows in a similar manner. So $k_m(G_{r,l}) > l+m-2$, and $k_m(G_{r,l}) = l+m-1$.

Case 3. $m \ge r+l$.

On the one hand, it is easy to see that for each vertex v_i, $N^+(G_{r,l}^{2l+r-1} : v_i) = V(G_{r,l})$, so $k_m(G_{r,l}) \le 2l+r-1$. On the other hand, since

$$|N^+(G_{r,l}^{2l+r-2} : v_{r+l})| = |\{v_1, v_2, \ldots, v_{r+l-1}\}| = r+l-1 < m,$$

we have $k_m(G_{r,l}) > 2l+r-2$, and $k_m(G_{r,l}) = 2l+r-1$. This completes the proof. $\square$

Corollary 2.3. *For* $2 \le m \le n-1$,

$$k_m(G_{r,n-r}) = \begin{cases} n-r+\left\lfloor \frac{r+m-2}{2} \right\rfloor, & \text{if } 2 \le m \le r-1, \\ n-r+m-1, & \text{if } r \le m \le n-1. \end{cases}$$

Next, we study the graphs in Figure 2 , where r is odd with $3 \le r \le n-3$, and $1 \le l \le n-r-2$.

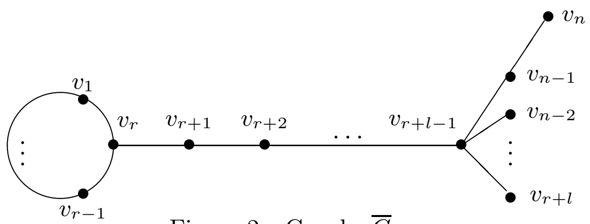

Figure 2. Graphs $\overline{G}_{r,l}$.

Theorem 2.4. *Let* $\overline{G}_{r,l}$ *be primitive graphs as shown in Figure 2. For* $2 \le m \le n-1$,

$$k_m(\overline{G}_{r,l}) = \begin{cases} l+\left\lfloor \frac{r+m-2}{2} \right\rfloor + 1, & \text{if } 2 \le m \le r-1, \\ l+m, & \text{if } r \le m \le r+l-1, \\ 2l+r, & \text{if } r+l \le m \le n-1. \end{cases}$$

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Proof. We prove the statement case by case.

Case 1. $2 \le m \le r-1$.

Let C_r be the only r-cycle in $\overline{G}_{r,l}$. Note that $d(v_n, C_r) = l+1$, $d(v_i, C_r) \le l$ for any $v_i \ne v_n$, and

$$\frac{r-1}{2} \le \left\lfloor \frac{r+m-2}{2} \right\rfloor \le r-2.$$

For any $v_i \ne v_n$, by Lemma 2.1,

$$\begin{aligned}
&|N^+(\overline{G}_{r,l}^{l+\lfloor \frac{r+m-2}{2} \rfloor+1} : v_i, v_n)| \\
&\ge |N^+(\overline{G}_{r,l}^{l+\lfloor \frac{r+m-2}{2} \rfloor+1} : v_i, v_n) \cap V(C_r)| \\
&\ge \left|N^+(\overline{G}_{r,l}^{l+\lfloor \frac{r+m-2}{2} \rfloor+1} : v_i) \cap V(C_r)\right| + \left|N^+(\overline{G}_{r,l}^{l+\lfloor \frac{r+m-2}{2} \rfloor+1} : v_n) \cap V(C_r)\right| - r \\
&= \min\{l + \left\lfloor \frac{r+m-2}{2} \right\rfloor + 1 - d(v_i, C_r) + 1, r\} + \min\{\left\lfloor \frac{r+m-2}{2} \right\rfloor + 1, r\} - r \\
&\ge \min\{\left\lfloor \frac{r+m-2}{2} \right\rfloor + 2, r\} + \left\lfloor \frac{r+m-2}{2} \right\rfloor + 1 - r \\
&= 2\left\lfloor \frac{r+m-2}{2} \right\rfloor + 3 - r \ge m.
\end{aligned}$$

For any $v_i \ne v_n$, and $v_j \ne v_n$,

$$\begin{aligned}
&|N^+(\overline{G}_{r,l}^{l+\lfloor \frac{r+m-2}{2} \rfloor+1} : v_i, v_j)| \\
&\ge |N^+(\overline{G}_{r,l}^{l+\lfloor \frac{r+m-2}{2} \rfloor+1} : v_i, v_j) \cap V(C_r)| \\
&\ge \left|N^+(\overline{G}_{r,l}^{l+\lfloor \frac{r+m-2}{2} \rfloor+1} : v_i) \cap V(C_r)\right| + \left|N^+(\overline{G}_{r,l}^{l+\lfloor \frac{r+m-2}{2} \rfloor+1} : v_j) \cap V(C_r)\Big)\right| - r \\
&\ge 2\min\{\Big(\left\lfloor \frac{r+m-2}{2} \right\rfloor + 2\Big), r\} - r \\
&= 2\Big(\left\lfloor \frac{r+m-2}{2} \right\rfloor + 2\Big) - r \ge m.
\end{aligned}$$

Therefore, $k_m(\overline{G}_{r,l}) \le l + \lfloor \frac{r+m-2}{2} \rfloor + 1$.

Now, we will show that $k_m(\overline{G}_{r,l}) > l + \lfloor \frac{r+m-2}{2} \rfloor$. If $\lfloor \frac{r+m-2}{2} \rfloor - 1$ is even, then

$$\begin{aligned}
&N^+(\overline{G}_{r,l}^{\lfloor \frac{r+m-2}{2} \rfloor-1} : v_r) \cap V(C_r) \\
&= \{v_r, v_2, v_4, \ldots, v_{\lfloor \frac{r+m-2}{2} \rfloor-1}, v_{r-2}, v_{r-4}, \ldots, v_{r-\lfloor \frac{r+m-2}{2} \rfloor+1}\}, \\
&N^+(\overline{G}_{r,l}^{\lfloor \frac{r+m-2}{2} \rfloor} : v_r) \cap V(C_r) \\
&= \{v_1, v_3, \ldots, v_{\lfloor \frac{r+m-2}{2} \rfloor}, v_{r-1}, v_{r-3}, \ldots, v_{r-\lfloor \frac{r+m-2}{2} \rfloor}\},
\end{aligned}$$

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and

$$\begin{aligned}
&\Big(N^+(\overline{G}_{r,l}^{\lfloor\frac{r+m-2}{2}\rfloor-1}:v_r)\cap V(C_r)\Big)\cap\Big(N^+(\overline{G}_{r,l}^{\lfloor\frac{r+m-2}{2}\rfloor}:v_r)\cap V(C_r)\Big)\\
&=\begin{cases}\{v_{r-\lfloor\frac{r+m-2}{2}\rfloor},\ldots,v_{\lfloor\frac{r+m-2}{2}\rfloor}\}, & \text{if } m\ge 3,\\ \phi, & \text{if } m=2.\end{cases}
\end{aligned}$$

If $\left\lfloor\frac{r+m-2}{2}\right\rfloor-1$ is odd, then the result follows in a similar manner. Thus,

$$\begin{aligned}
&|N^+(\overline{G}_{r,l}^{l+\lfloor\frac{r+m-2}{2}\rfloor}:v_n,v_{n-1})|\\
&=|N^+(\overline{G}_{r,l}^{l+\lfloor\frac{r+m-2}{2}\rfloor}:v_n,v_{n-1})\cap V(C_r)|\\
&=\Big|\Big(N^+(\overline{G}_{r,l}^{\lfloor\frac{r+m-2}{2}\rfloor-1}:v_r)\cap V(C_r)\Big)\cap\Big(N^+(\overline{G}_{r,l}^{\lfloor\frac{r+m-2}{2}\rfloor}:v_r)\cap V(C_r)\Big)\Big|\\
&\le m-1.
\end{aligned}$$

So $k_m(\overline{G}_{r,l})>l+\left\lfloor\frac{r+m-2}{2}\right\rfloor$. Then $k_m(\overline{G}_{r,l})=l+\left\lfloor\frac{r+m-2}{2}\right\rfloor+1$.

Case 2. $r\le m\le r+l-1$.

For any $v_i\in V(\overline{G}_{r,l})$, by similar arguments as in Case 2 of the proof of Theorem 2.2, we have

$$\{v_1,v_2,\ldots,v_m\}\subseteq N^+(\overline{G}_{r,l}^{l+m}:v_i).$$

So $k_m(\overline{G}_{r,l})\le l+m$.

Now, we will show that $k_m(\overline{G}_{r,l})>l+m-1$. If $l+m-1$ is even, then

$$\begin{aligned}
&N^+(\overline{G}_{r,l}^{l+m-1}:v_n)\\
&=\{v_1,v_2,\ldots,v_{r+l-1},v_n\}\backslash\{v_k\mid m\le k\le r+l-2,\ \text{and } k\equiv(l+r-2)(\mathrm{mod}\ 2)\},\\
&N^+(\overline{G}_{r,l}^{l+m-1}:v_{n-1})\\
&=\{v_1,v_2,\ldots,v_{n-1}\}\backslash\{v_k\mid m+1\le k\le r+l-1,\ \text{and } k\equiv(l+r-1)(\mathrm{mod}\ 2)\}.
\end{aligned}$$

Therefore,

$$N^+(\overline{G}_{r,l}^{l+m-1}:v_n)\cap N^+(\overline{G}_{r,l}^{l+m-1}:v_{n-1})=\{v_1,v_2,\ldots,v_{m-1}\}.$$

If $l+m-1$ is odd, then the result follows in a similar manner. So $k_m(\overline{G}_{r,l})>l+m-1$, and $k_m(\overline{G}_{r,l})=l+m$.

Case 3. $r+l\le m\le n-1$.

On the one hand, it is easy to see that for each vertex v_i, $V(\overline{G}_{r,l})\backslash\{v_n\}\subseteq N^+(\overline{G}_{r,l}^{2l+r}:v_i)$, so $k_m(\overline{G}_{r,l})\le 2l+r$. On the other hand, since

$$\begin{aligned}
&N^+(\overline{G}_{r,l}^{2l+r-1}:v_n)=\{v_1,v_2,\ldots,v_{r+l-1}\}\cup\{v_n\},\\
&N^+(\overline{G}_{r,l}^{2l+r-1}:v_{n-1})=\{v_1,v_2,\ldots,v_{n-1}\},
\end{aligned}$$

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we have

$$|N^+(\overline{G}_{r,l}^{2l+r-1} : v_n) \cap N^+(\overline{G}_{r,l}^{2l+r-1} : v_{n-1})| = |\{v_1, v_2, \ldots, v_{r+l-1}\}| = r + l - 1 < m.$$

So, $k_m(\overline{G}_{r,l}) > 2l + r - 1$, and $k_m(\overline{G}_{r,l}) = 2l + r$. This completes the proof. ◻

Now, we consider the graph G_r as shown in Figure 3, where r is odd with $3 \le r \le n$.

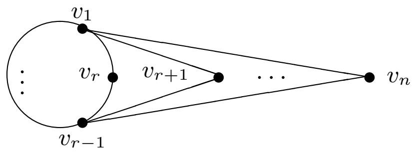

Figure 3. Graph G_r.

Theorem 2.5. *For $2 \le m \le n-1$,*

$$k_m(G_r) = \begin{cases} \left\lfloor \frac{r+m-1}{2} \right\rfloor, & \textit{if } \ 2 \le m \le r-1, \\ r-1, & \textit{if } \ r \le m \le n-1. \end{cases}$$

Proof. We prove the statement case by case.

Case 1. $2 \le m \le r-1$.

Write $C_r = v_1 v_2 \cdots v_r v_1$. Note that for any integer $1 \le l \le r-1$ and any vertex v_i, if $r \le i \le n$, by Lemma 2.1,

$$|N^+(G_r^l : v_i) \cap V(C_r)| = |N^+(G_r^l : v_r) \cap V(C_r)| = l + 1.$$

If $1 \le i \le r-1$, by Lemma 2.1,

$$|N^+(G_r^l : v_i) \cap V(C_r)| = l + 1.$$

For $2 \le m \le r-1$, and any $v_i, v_j \in V(G_r)$, noticing that $\frac{r+1}{2} \le \left\lfloor \frac{r+m-1}{2} \right\rfloor \le r-1$, we have

$$\begin{aligned} &|N^+(G_r^{\lfloor \frac{r+m-1}{2} \rfloor} : v_i, v_j)| \\ &\ge |N^+(G_r^{\lfloor \frac{r+m-1}{2} \rfloor} : v_i, v_j) \cap V(C_r)| \\ &\ge \left|N^+(G_r^{\lfloor \frac{r+m-1}{2} \rfloor} : v_i) \cap V(C_r)\right| + \left|N^+(G_r^{\lfloor \frac{r+m-1}{2} \rfloor} : v_i) \cap V(C_r)\right| - r \\ &= 2\Big(\left\lfloor \frac{r+m-1}{2} \right\rfloor + 1 \Big) - r \ge m, \end{aligned}$$

and so $k_m(G_r) \le \left\lfloor \frac{r+m-1}{2} \right\rfloor$.

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Now, we will show that $k_m(G_r) > \lfloor \frac{r+m-1}{2} \rfloor - 1$. If $\lfloor \frac{r+m-1}{2} \rfloor - 1$ is even, then

$$\begin{aligned}
&N^+(G_r^{\lfloor \frac{r+m-1}{2} \rfloor - 1} : v_r)\\
&= \{v_2, v_4, \ldots, v_{\lfloor \frac{r+m-1}{2} \rfloor - 1}, v_{r-2}, v_{r-4}, \ldots, v_{r-\lfloor \frac{r+m-1}{2} \rfloor + 1}\} \cup \{v_r, v_{r+1}, \ldots, v_n\},\\
&N^+(G_r^{\lfloor \frac{r+m-1}{2} \rfloor - 1} : v_1)\\
&= \{v_1, v_3, \ldots, v_{\lfloor \frac{r+m-1}{2} \rfloor}, v_{r-1}, v_{r-3}, \ldots, v_{r-\lfloor \frac{r+m-1}{2} \rfloor + 2}\}.
\end{aligned}$$

If $\lfloor \frac{r+m-1}{2} \rfloor - 1$ is odd, then

$$\begin{aligned}
&N^+(G_r^{\lfloor \frac{r+m-1}{2} \rfloor - 1} : v_r)\\
&= \{v_1, v_3, \ldots, v_{\lfloor \frac{r+m-1}{2} \rfloor - 1}, v_{r-1}, v_{r-3}, \ldots, v_{r-\lfloor \frac{r+m-1}{2} \rfloor + 1}\},\\
&N^+(G_r^{\lfloor \frac{r+m-1}{2} \rfloor - 1} : v_1)\\
&= \{v_2, v_4, \ldots, v_{\lfloor \frac{r+m-1}{2} \rfloor}, v_{r-2}, v_{r-4}, \ldots, v_{r-\lfloor \frac{r+m-1}{2} \rfloor + 2}\} \cup \{v_r, v_{r+1}, \ldots, v_n\}.
\end{aligned}$$

Then

$$\begin{aligned}
&N^+(G_r^{\lfloor \frac{r+m-1}{2} \rfloor - 1} : v_r, v_1)\\
&= N^+(G_r^{\lfloor \frac{r+m-1}{2} \rfloor - 1} : v_r) \cap N^+(G_r^{\lfloor \frac{r+m-1}{2} \rfloor - 1} : v_1)\\
&= \{v_{r-\lfloor \frac{r+m-1}{2} \rfloor + 1}, v_{r-\lfloor \frac{r+m-1}{2} \rfloor + 2}, \ldots, v_{\lfloor \frac{r+m-1}{2} \rfloor}\},
\end{aligned}$$

and

$$|N^+(G_r^{\lfloor \frac{r+m-1}{2} \rfloor - 1} : v_r, v_1)| = 2\left\lfloor \frac{r+m-1}{2} \right\rfloor - r \le m-1.$$

The fact that $|N^+(G_r^{\lfloor \frac{r+m-1}{2} \rfloor - 1} : v_r, v_1)| \le m-1$ implies that $k_m(G_r) > \lfloor \frac{r+m-1}{2} \rfloor - 1$. Therefore, $k_m(G_r) = \lfloor \frac{r+m-1}{2} \rfloor$, for $2 \le m \le r-1$.

Case 2. $r \le m \le n-1$.

It is easy to see that for each vertex v_i, $N^+(G_r^{r-1} : v_i) = V(G_r)$. So $r-1 = k_{r-1}(G_r) \le k_m(G_r) \le r-1$, and we have $k_m(G_r) = r-1$.

The theorem follows. ∎

Now, we consider the graph $\overline{G}_r$ as shown in Figure 4, where r is odd with $3 \le r \le n-1$.

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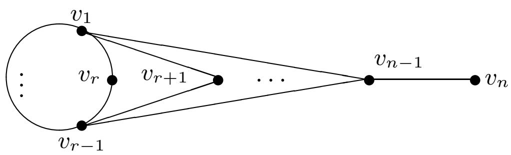

Figure 4. Graph $\overline{G}_r$.

THEOREM 2.6. *For $r \le m \le n-1$, $k_m(\overline{G}_r) = r$.*

Proof. On the one hand, it is easy to see that $N^+(\overline{G}_r^r : v_i) = V(\overline{G}_r)$ for each vertex $v_i \neq v_n$, and $N^+(\overline{G}_r^r : v_n) = V(\overline{G}_r)\backslash\{v_n\}$, so $k_m(\overline{G}_r) \le r$. On the other hand, we have

$$N^+(\overline{G}_r^{r-1} : v_n) = \{v_n, v_1, v_2, \ldots, v_{r-1}\},$$
$$N^+(\overline{G}_r^{r-1} : v_{n-1}) = \{v_1, v_2, \ldots, v_{n-1}\},$$

and

$$|N^+(\overline{G}_r^{r-1} : v_n) \cap N^+(\overline{G}_r^{r-1} : v_{n-1})| = |\{v_1, v_2, \ldots, v_{r-1}\}| = r-1 < m.$$

Therefore, $k_m(\overline{G}_r) > r-1$, and the theorem follows. ☐

3. The generalized competition index set of S_n^0. For $1 \le m \le n$, let $E_m(r) = \{k_m(G) \mid G \in S_n(r)\}$, $E_m = \{k_m(G) \mid G \in S_n^0\}$. It is known that $E_n = \{2, 3, \ldots, 2n-4\}\backslash S_1$, where S_1 is the set of all odd numbers in $\{n-2, n-1, \ldots, 2n-5\}$ ([8]), and $E_1(r) = \{\frac{r-1}{2}, \frac{r-1}{2}+1, \ldots, n-\frac{r+1}{2}\}$ (Theorem 3.3 in [4]). Note that $1 \le \frac{r-1}{2}$, and $n - \frac{r+1}{2} \le n-2$ for any odd number $r \ge 3$. We have $E_1(r) \subseteq E_1(3)$ for any odd r with $3 \le r \le n$, and so $E_1 = \{1, 2, \ldots, n-2\}$.

In this section, we show that

$$E_m = \begin{cases} \{1, 2, \ldots, n+m-4\}, & \text{if } 2 \le m \le n-2, \\ \{2, 3, \ldots, n+m-4\}, & \text{if } m = n-1. \end{cases}$$

We also characterize the graphs in S_n^0 with the largest generalized competition index $n+m-4$.

THEOREM 3.1. *For any graph $G \in S_n(r)$, $3 \le r \le n-1$, and $2 \le m \le n-1$,*

$$k_m(G) \le k_m(G_{r,n-r}) = \begin{cases} n-r+\left\lfloor \frac{r+m-2}{2} \right\rfloor, & \text{if } 2 \le m \le r-1, \\ n+m-r-1, & \text{if } r \le m \le n-1. \end{cases}$$

Proof. Let C_r be the cycle in G of length r. For any $v_i, v_j \in V(G)$, let P_i be the shortest path from v_i to C_r, P_j the shortest path from v_j to C_r, $V(P_i) \cap V(C_r) = \{u_i\}$

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and $V(P_j) \cap V(C_r) = \{u_j\}$. Then $l(P_i) \leq n-r$, and $l(P_j) \leq n-r$. Consider the following cases.

Case 1. $2 \leq m \leq r-1$.

For any $v_i, v_j \in V(G)$, it is clear that $l(P_i) \leq n-r-1$ or $l(P_j) \leq n-r-1$. Without loss of generality, we assume that $l(P_j) \leq n-r-1$ and $l(P_i) \leq n-r$. By Lemma 2.1,

$$\begin{aligned}
&|N^+(G^{n-r+\lfloor \frac{r+m-2}{2} \rfloor} : v_i) \cap V(C_r)| \\
&\geq |N^+(G^{n-r+\lfloor \frac{r+m-2}{2} \rfloor - l(P_i)} : u_i) \cap V(C_r)| \\
&= \min\{n-r+\left\lfloor \frac{r+m-2}{2} \right\rfloor - l(P_i)+1, r\} \\
&\geq \min\{n-r+\left\lfloor \frac{r+m-2}{2} \right\rfloor - (n-r)+1, r\} \\
&= \min\{\left\lfloor \frac{r+m-2}{2} \right\rfloor + 1, r\} = \left\lfloor \frac{r+m-2}{2} \right\rfloor + 1.
\end{aligned}$$

Similarly,

$$|N^+(G^{n-r+\lfloor \frac{r+m-2}{2} \rfloor} : v_j) \cap V(C_r)| \geq \left\lfloor \frac{r+m-2}{2} \right\rfloor + 2.$$

Therefore,

$$\begin{aligned}
&|N^+(G^{n-r+\lfloor \frac{r+m-2}{2} \rfloor} : v_i, v_j)| \\
&\geq |N^+(G^{n-r+\lfloor \frac{r+m-2}{2} \rfloor} : v_i, v_j) \cap V(C_r)| \\
&\geq \left|N^+(G^{n-r+\lfloor \frac{r+m-2}{2} \rfloor} : v_i) \cap V(C_r)\right| + \left|N^+(G^{n-r+\lfloor \frac{r+m-2}{2} \rfloor} : v_j) \cap V(C_r)\right| - r \\
&\geq 2\left\lfloor \frac{r+m-2}{2} \right\rfloor + 3 - r \geq m,
\end{aligned}$$

and so $k_m(G) \leq n-r+\left\lfloor \frac{r+m-2}{2} \right\rfloor$.

Case 2. $r \leq m \leq n-1$.

For a vertex $x \in V(G)$, if $d(x, C_r) \leq n-r-1$, then from x to each vertex $v \in V(C_r)$, there exist two walks of lengths l_1 and l_2, respectively, such that l_1 and l_2 have different parity and $\max\{l_1, l_2\} \leq n-1$. So $V(C_r) \subseteq N^+(G^{n-1} : x)$.

If $d(x, C_r) = n-r$, denoting by P_x the shortest path from x to C_r, $V(P_x) \cap V(C_r) = \{u_x\}$, then from x to each vertex $v \neq u_x \in V(C_r)$, there exist two walks of lengths l_1 and l_2, respectively, such that l_1 and l_2 have different parity and $\max\{l_1, l_2\} \leq n-1$. So $V(C_r) \backslash \{u_x\} \subseteq N^+(G^{n-1} : x)$. Note that $d(x, u_x) = n-r < n-1$, and $n-r$ and $n-1$ have the same parity. Hence, $\{u_x\} \subseteq N^+(G^{n-1} : x)$, and we have $V(C_r) \subseteq N^+(G^{n-1} : x)$.

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Therefore, for any $v_i, v_j \in V(G)$ and any integer k with $k \geq n-1$,

$$v_i \xrightarrow{k} u,\ v_j \xrightarrow{k} u, \text{ for each vertex } u \text{ in } C_r.$$

So, for $m = r$, $k_m(G) \leq n-1 = n+m-r-1$. For $r+1 \leq m \leq n-1$, since G is connected, there exist $m-r$ vertices $u_1, u_2, \ldots, u_{m-r} \in V(G)\backslash V(C_r)$ such that $d(u_t, C_r) \leq m-r$ for $t = 1, 2, \ldots, m-r$. Then

$$v_i \overset{n+m-r-1}{\longrightarrow} u_t,\ v_j \overset{n+m-r-1}{\longrightarrow} u_t, \text{ for } t = 1, 2, \ldots, m-r.$$

Thus, $k_m(G) \leq n+m-r-1$.

By the above discussions, we see that Theorem 3.1 holds. □

Theorem 3.2. *For any $G \in S_n^0$, and $2 \leq m \leq n-1$,*

$$k_m(G) \leq n+m-4.$$

The equality holds if and only if the graph G is isomorphic to $G_{3,n-3}$.

Proof. Let $G \in S_n^0$. Then there exists an odd number r with $3 \leq r \leq n$ such that $G \in S_n(r)$.

Case 1. $r = 3$.

By Theorem 3.1, we have $k_m(G) \leq n+m-4$ for $2 \leq m \leq n-1$.

Now, we assume that G is not isomorphic to $G_{3,n-3}$. We will show that $k_m(G) \leq n+m-5$ for $2 \leq m \leq n-1$.

Take C_3 to be a 3-cycle in G such that $\max_{u \in V(G)} d(u, C_3)$ is the smallest. Since G is not isomorphic to $G_{3,n-3}$, we have that $\max_{u \in V(G)} d(u, C_3) \leq n-4$.

For any $v_i, v_j \in V(G)$, let P_i be the shortest path from v_i to C_r, P_j the shortest path from v_j to C_r, $V(P_i) \cap V(C_r) = \{u_i\}$ and $V(P_j) \cap V(C_r) = \{u_j\}$. Then $l(P_i) \leq n-4$, and $l(P_j) \leq n-4$. Consider the following cases.

Subcase 1.1. $m = 2$.

If $l(P_i) = l(P_j) = n-4$, then G is isomorphic to $G_{3,n-4}$, and $k_m(G) = n-3$.

If $l(P_i) \leq n-5$ or $l(P_j) \leq n-5$, then, without loss of generality, we assume $l(P_j) \leq n-5$ and $l(P_i) \leq n-4$. By Lemma 2.1,

$$\begin{aligned}
&|N^+(G^{n-3} : v_i) \cap V(C_3)| \\
&\geq |N^+(G^{n-3-l(P_i)} : u_i) \cap V(C_3)| \\
&= \min\{n-3-l(P_i)+1, 3\} \\
&\geq \min\{n-2-(n-4), 3\} = 2.
\end{aligned}$$

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Similarly, $|N^+(G^{n-3}:v_j)\cap V(C_3)|=3$. Therefore,

$$\begin{aligned}&|N^+(G^{n-3}:v_i,v_j)|\\ &\geq |N^+(G^{n-3}:v_i,v_j)\cap V(C_3)|\\ &= \left|\Big(N^+(G^{n-3}:v_i)\cap V(C_3)\Big)\cap\Big(N^+(G^{n-3}:v_j)\cap V(C_3)\Big)\right|\geq 2,\end{aligned}$$

and so $k_m(G)\leq n-3$.

Subcase 1.2. $3\leq m\leq n-1$.

For a vertex $x\in V(G)$, if $d(x,C_r)\leq n-5$, then from x to each vertex $v\in V(C_r)$, there exist two walks of lengths l_1 and l_2, respectively, such that l_1 and l_2 have different parity and $\max\{l_1,l_2\}\leq n-2$. So $V(C_r)\subseteq N^+(G^{n-2}:x)$.

If $d(x,C_3)=n-4$, denoting by P_x the shortest path from x to C_3, with $V(P_x)\cap V(C_3)=\{u_x\}$, then from x to each vertex $v\neq u_x\in V(C_3)$, there exist walks of length $n-2$. So $V(C_r)\backslash\{u_x\}\subseteq N^+(G^{n-2}:x)$. Noting that $d(x,u_x)=n-4<n-2$ and $n-4$ and $n-2$ have the same parity, so $\{u_x\}\subseteq N^+(G^{n-2}:x)$, and we have $V(C_3)\subseteq N^+(G^{n-2}:x)$.

Therefore, for any $v_i,v_j\in V(G)$ and any integer k with $k\geq n-2$,

$$v_i\xrightarrow{k}u,\ v_j\xrightarrow{k}u,\ \text{for each vertex } u \text{ in } C_3.$$

So, for $m=3$, $k_m(G)\leq n-2=n+m-5$. For $r+1\leq m\leq n-1$, since G is connected, there exist $m-3$ vertices $u_1,u_2,\ldots,u_{m-3}\in V(G)\backslash V(C_3)$, such that $d(C_3,u_t)\leq m-3$ for $t=1,2,\ldots,m-3$. Then

$$v_i\overset{n+m-5}{\longrightarrow}u_t,\ v_j\overset{n+m-5}{\longrightarrow}u_t,\ \text{for } t=1,2,\ldots,m-3.$$

Thus, $k_m(G)\leq n+m-5$.

Case 2. $r\geq 5$.

If $r\leq n-1$, then by Theorem 3.1, $k_m(G)\leq n-r+\left\lfloor\frac{r+m-2}{2}\right\rfloor<n+m-4$ when $2\leq m\leq r-1$, and $k_m(G)\leq n+m-r-1<n+m-4$ when $r\leq m\leq n-1$.

If n is odd and $r=n$, by Theorem 2.5, $k_m(G)=\left\lfloor\frac{n+m-1}{2}\right\rfloor<n+m-4$.

The proof is now complete. $\square$

LEMMA 3.3. *Let r be odd with $3\leq r\leq n-1$. For $2\leq m\leq n-1$,*

$$E_m(r)\supseteq\begin{cases}\{\left\lfloor\frac{r+m-1}{2}\right\rfloor,\left\lfloor\frac{r+m-1}{2}\right\rfloor+1,\ldots,n-r+\left\lfloor\frac{r+m-2}{2}\right\rfloor\}, & \textit{if }\ 2\leq m\leq r-1,\\ \{r-1,r,\ldots,n+m-r-1\}, & \textit{if }\ r\leq m\leq n-1.\end{cases}$$

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Proof. By Theorem 2.2, taking $1 \le l \le n-r$, we have

$$E_m(r) \supseteq \begin{cases} \{1+\lfloor\frac{r+m-2}{2}\rfloor, \dots, n-r+\lfloor\frac{r+m-2}{2}\rfloor\}, & \text{if } 2 \le m \le r-1,\\ \{m, m+1, \dots, m+n-r-1\}, & \text{if } m=r,\\ \{m+1, \dots, m+n-r-1\} \cup \{r+1\}, & \text{if } m=r+1,\\ \{m+2, \dots, m+n-r-1\} \cup \{r+1, r+3\}, & \text{if } m=r+2,\\ \quad \cdots\cdots & \\ \{m+n-r-2, m+n-r-1\} & \\ \quad \cup\{r+1, r+3, \dots, 2n-r-5\}, & \text{if } m=n-2,\\ \{m+n-r-1\} \cup \{r+1, r+3, \dots, 2n-r-3\}, & \text{if } m=n-1. \end{cases}$$

By Theorem 2.4, taking $1 \le l \le n-r-2$, we have

$$E_m(r) \supseteq \begin{cases} \{1+\lfloor\frac{r+m-2}{2}\rfloor, \dots, n-r+\lfloor\frac{r+m-2}{2}\rfloor-1\}, & \text{if } 2 \le m \le r-1,\\ \{m+1, \dots, m+n-r-2\}, & \text{if } m=r,\\ \{m+2, \dots, m+n-r-2\} \cup \{r+2\}, & \text{if } m=r+1,\\ \{m+3, \dots, m+n-r-2\} \cup \{r+2, r+4\}, & \text{if } m=r+2,\\ \quad \cdots\cdots & \\ \{m+n-r-2\} \cup \{r+2, r+4, \dots, 2n-r-6\}, & \text{if } m=n-3,\\ \{r+2, r+4, \dots, 2n-r-4\}, & \text{if } m=n-2, n-1. \end{cases}$$

By Theorem 2.5,

$$E_m(r) \supseteq \begin{cases} \lfloor\frac{r+m-1}{2}\rfloor, & \text{if } \ 2 \le m \le r-1,\\ r-1, & \text{if } \ r \le m \le n-1. \end{cases}$$

By Theorem 2.6, for $r \le m \le n-1$, $E_m(r) \supseteq \{r\}$.

By the above discussions, we have

$$E_m(r) \supseteq \begin{cases} \{\lfloor\frac{r+m-1}{2}\rfloor, \lfloor\frac{r+m-1}{2}\rfloor+1, \dots, n-r+\lfloor\frac{r+m-2}{2}\rfloor\}, & \text{if } \ 2 \le m \le r-1,\\ \{r-1, r, \dots, n+m-r-1\}, & \text{if } \ r \le m \le n-1. \end{cases} \qquad \square$$

THEOREM 3.4.

$$E_m = \begin{cases} \{1, 2, \dots, n+m-4\}, & \textit{if } \ 2 \le m \le n-2,\\ \{2, 3, \dots, n+m-4\}, & \textit{if } \ m=n-1. \end{cases}$$

Proof. By Lemma 3.3 with $r=3$, we know $E_m \supseteq \{2, 3, \dots, n+m-4\}$ for $2 \le m \le n-1$. For the complete graph K_n, it is clear that

$$k_m(K_n) = \begin{cases} 1, & \text{if } \ 2 \le m \le n-2,\\ 2, & \text{if } \ m=n-1. \end{cases}$$

Note that for any $G \in S_n^0$, $k_{n-1}(G) \ge 2$. By Theorem 3.2, the theorem follows. $\square$

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