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REPRESENTATIONS FOR THE DRAZIN INVERSE OF BLOCK CYCLIC MATRICES*

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Abstract. A formula for the Drazin inverse of a block k-cyclic ($k \geq 2$) matrix A with nonzeros only in blocks $A_{i,i+1}$, for $i = 1, \ldots, k$ (mod k) is presented in terms of the Drazin inverse of a smaller order product of the nonzero blocks of A, namely $B_i = A_{i,i+1} \cdots A_{i-1,i}$ for some i. Bounds on the index of A in terms of the minimum and maximum indices of these B_i are derived. Illustrative examples and special cases are given.

Key words. Drazin inverse, Block cyclic matrix, Index.

AMS subject classifications. 15A09.

1. Introduction. We consider k-*cyclic* ($k \geq 2$) block real or complex matrices of the form

$$A = \begin{bmatrix} 0 & A_{12} & 0 & \cdots & 0 \\ 0 & 0 & A_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{k-1,k} \\ A_{k1} & 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{1.1}$$

where $A_{12}, \ldots, A_{k1}$ are block submatrices and the diagonal zero blocks are square. It is easily verified that for any matrix A of the form (1.1), the Moore-Penrose inverse $A^\dagger$ of A is given by

$$A^\dagger = \begin{bmatrix} 0 & 0 & \cdots & 0 & A_{k1}^\dagger \\ A_{12}^\dagger & 0 & \cdots & 0 & 0 \\ 0 & A_{23}^\dagger & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_{k-1,k}^\dagger & 0 \end{bmatrix}, \tag{1.2}$$

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where $A_{ij}^{\dagger}$ denotes the Moore-Penrose inverse of the block submatrix A_{ij}. Note that if each of the blocks A_{ij} is square and invertible, then (1.2) gives the formula for the usual inverse A^{-1} of A. We present a block formula for another type of generalized inverse, the Drazin inverse, of matrices of the form (1.1). Unlike the Moore-Penrose inverse, the Drazin inverse is defined only for square matrices.

Let A be a real or complex square matrix. The *Drazin inverse* of A is the unique matrix A^D satisfying

$$AA^D = A^D A \tag{1.3}$$

$$A^D AA^D = A^D \tag{1.4}$$

$$A^{q+1}A^D = A^q, \tag{1.5}$$

where $q = \operatorname{index} A$, the smallest nonnegative integer q such that $\operatorname{rank} A^{q+1} = \operatorname{rank} A^q$. If $\operatorname{index} A = 0$, then A is nonsingular and $A^D = A^{-1}$. If $\operatorname{index} A = 1$, then $A^D = A^{\#}$, the *group inverse* of A. See [1], [2], [6] and references therein for applications of the Drazin inverse.

Theorem 1.1. [2, Theorem 7.2.3] *Let A be a square matrix with* $\operatorname{index} A = q$. *If p is a nonnegative integer and X is a matrix satisfying $XAX = X$, $AX = XA$, and $A^{p+1}X = A^p$, then $p \geq q$ and $X = A^D$.*

The problem of finding explicit representations for the Drazin inverse of a general 2×2 block matrix of the form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{1.6}$$

in terms of its blocks was posed by Campbell and Meyer in [2] and special cases of this problem were the focus of several recent papers, including [3]–[10], [13], [14] and [15]. In [4] and [14], representations for 2×2 block matrices matrices of the form (1.6) with A_{11} and A_{22} being square zero diagonal blocks were presented. Such block matrices were called bipartite (or 2-cyclic), and in this article, we extend the results given in [4] to general block k-cyclic matrices as defined in (1.1).

2. Drazin inverse formula for block cyclic matrices.

Let A be a block k-cyclic matrix of the form given in (1.1). For our Drazin inverse formula we introduce some notation that is also used in writing powers of A. For $i = 2, \ldots, k-1$, let B_i be the square matrix defined by

$$B_i = A_{i,i+1} \cdots A_{k-1,k} A_{k1} A_{12} \cdots A_{i-1,i}, \tag{2.1}$$

with $B_1 = A_{12}A_{23} \cdots A_{k-1,k}A_{k1}$ and $B_k = A_{k1}A_{12} \cdots A_{k-1,k}$, i.e, subscripts are taken mod k. For ease of notation, we define the matrix product

$$A_{i\to j} \ := \ A_{i,i+1}A_{i+1,i+2} \cdots A_{j-1,j}, \tag{2.2}$$

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for $j \neq i$. Whenever it arises, we use the convention $A_{i\to i} = I$, an identity matrix. For example, if $k = 4$ then $A_{2\to 3} = A_{23}$, $A_{3\to 2} = A_{34}A_{41}A_{12}$, $A_{2\to 1} = A_{23}A_{34}A_{41}$, and by (2.1) $B_3 = A_{34}A_{41}A_{12}A_{23}$. Observe that $B_i = A_{i\to j}A_{j\to i}$, for any $j \in \{1,\ldots,k\}\backslash\{i\}$.

LEMMA 2.1. *For A given in (1.1) and with the notation above, for $p \geq 0$,*

$$A^{kp} = \begin{bmatrix} B_1^p & 0 & \cdots & 0 \\ 0 & B_2^p & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_k^p \end{bmatrix}, \tag{2.3}$$

$$A^{kp+1} = \begin{bmatrix} 0 & B_1^pA_{1\to 2} & 0 & \cdots & 0 \\ 0 & 0 & B_2^pA_{2\to 3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & B_{k-1}^pA_{k-1\to k} \\ B_k^pA_{k\to 1} & 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{2.4}$$

$$A^{kp+2} = \begin{bmatrix} 0 & 0 & B_1^pA_{1\to 3} & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & B_{k-2}^pA_{k-2\to k} \\ B_{k-1}^pA_{k-1\to 1} & 0 & 0 & \cdots & 0 \\ 0 & B_k^pA_{k\to 2} & 0 & \cdots & 0 \end{bmatrix},$$

and so on, until

$$A^{kp+k-1} = \begin{bmatrix} 0 & 0 & \cdots & 0 & B_1^pA_{1\to k} \\ B_2^pA_{2\to 1} & 0 & \cdots & 0 & 0 \\ 0 & B_3^pA_{3\to 2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B_k^pA_{k\to k-1} & 0 \end{bmatrix}. \tag{2.5}$$

LEMMA 2.2. *For all $i \neq j$, $B_i^kA_{i\to j} = A_{i\to j}B_j^k$.*

Proof. $B_i^kA_{i\to j} = (A_{i\to j}A_{j\to i})^kA_{i\to j} = A_{i\to j}A_{j\to i}(A_{i\to j}A_{j\to i})^{k-1}A_{i\to j} = A_{i\to j}(A_{j\to i}A_{i\to j})^k = A_{i\to j}B_j^k$. ∎

LEMMA 2.3. *For all $i \neq j$, $B_i^DA_{i\to j} = A_{i\to j}B_j^D$. Hence, if $\ell \neq i,j$ satisfies $A_{i\to j} = A_{i\to \ell}A_{\ell\to j}$, then $B_i^DA_{i\to j} = A_{i\to j}B_j^D = A_{i\to \ell}B_\ell^DA_{\ell\to j}$.*

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Proof. $B_i^D A_{i\to j} = (A_{i\to j}A_{j\to i})^D A_{i\to j} = A_{i\to j}(A_{j\to i}A_{i\to j})^D = A_{i\to j}B_j^D$, where the second equality is due to [4, Lemma 2.4]. ▯

With the above notation, we now give a formula for the Drazin inverse of a k-cyclic matrix A given by (1.1).

THEOREM 2.4. *Let A be as in (1.1) with associated matrices B_i defined as in (2.1) and $A_{i\to j}$ defined in (2.2). Then, for all $i = 1, \ldots, k$,*

$$A^D = \begin{bmatrix} 0 & 0 & \cdots & 0 & A_{1\to i}B_i^D A_{i\to k} \\ A_{2\to i}B_i^D A_{i\to 1} & 0 & \cdots & 0 & 0 \\ 0 & A_{3\to i}B_i^D A_{i\to 2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_{k\to i}B_i^D A_{i\to k-1} & 0 \end{bmatrix}. \tag{2.6}$$

Moreover, if $\operatorname{index} B_i = s_i$, *then* $\operatorname{index} A \le ks_i + k - 1$.

Proof. Denote the matrix on the right hand side of (2.6) by X. Performing block multiplication gives

$$\begin{aligned} AX &= \begin{bmatrix} A_{12}A_{2\to i}B_i^D A_{i\to 1} & 0 & \cdots & 0 \\ 0 & A_{23}A_{3\to i}B_i^D A_{i\to 2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_{k1}A_{1\to i}B_i^D A_{i\to k} \end{bmatrix} \\ &= \begin{bmatrix} A_{12}A_{2\to i}A_{i\to 1}B_1^D & 0 & \cdots & 0 \\ 0 & A_{23}A_{3\to i}A_{i\to 2}B_2^D & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_{k1}A_{1\to i}A_{i\to k}B_k^D \end{bmatrix} \\ &\quad \text{(by Lemma 2.3)} \\ &= \begin{bmatrix} B_1B_1^D & 0 & \cdots & 0 \\ 0 & B_2B_2^D & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_kB_k^D \end{bmatrix}, \end{aligned}$$

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and by using Lemma 2.3 again

$$\begin{aligned}
XA &= \begin{bmatrix} A_{1\to i}B_i^D A_{i\to k}A_{k1} & 0 & \cdots & 0 \\ 0 & A_{2\to i}B_i^D A_{i\to 1}A_{12} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_{k\to i}B_i^D A_{i\to k-1}A_{k-1,k} \end{bmatrix} \\
&= \begin{bmatrix} B_1^D A_{1\to i}A_{i\to k}A_{k1} & 0 & \cdots & 0 \\ 0 & B_2^D A_{2\to i}A_{i\to 1}A_{12} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & B_k^D A_{k\to i}A_{i\to k-1}A_{k-1,k} \end{bmatrix} \\
&= \begin{bmatrix} B_1^D B_1 & 0 & \cdots & 0 \\ 0 & B_2^D B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_k^D B_k \end{bmatrix} = \begin{bmatrix} B_1 B_1^D & 0 & \cdots & 0 \\ 0 & B_2 B_2^D & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_k B_k^D \end{bmatrix} \\
&= AX,
\end{aligned}$$

since $B_i^D B_i = B_i B_i^D$ by (1.3). Also, block-multiplying X with AX gives

$$\begin{aligned}
XAX &= X(AX) \\
&= \begin{bmatrix} 0 & 0 & \cdots & 0 & A_{1\to k}B_k^D B_k B_k^D \\ A_{2\to 1}B_1^D B_1 B_1^D & 0 & \cdots & 0 & 0 \\ 0 & A_{3\to 2}B_2^D B_2 B_2^D & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_{k\to k-1}B_{k-1}^D B_{k-1}B_{k-1}^D & 0 \end{bmatrix} \\
&= X, \text{ by Lemma 2.3 and since } B_i^D B_i B_i^D = B_i^D \text{ by (1.4).}
\end{aligned}$$

Let i be any integer in $\{1, \dots, k\}$ and suppose that $\operatorname{index} B_i = s_i = s$. Then using (2.3) and Lemma 2.2,

$$\begin{aligned}
A^{ks+k}X &= A^{k(s+1)}X \\
&= \begin{bmatrix} 0 & 0 & \cdots & 0 & A_{1\to i}B_i^{s+1}B_i^D A_{i\to k} \\ A_{2\to i}B_i^{s+1}B_i^D A_{i\to 1} & 0 & \cdots & 0 & 0 \\ 0 & A_{3\to i}B_i^{s+1}B_i^D A_{i\to 2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_{k\to i}B_i^{s+1}B_i^D A_{i\to k-1} & 0 \end{bmatrix}.
\end{aligned}$$

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Since index $B_i = s$, it follows by (1.5) that $B_i^{s+1} B_i^D = B_i^s$. Thus, using Lemma 2.2 and $A_{\ell\to i} A_{i\to j} = A_{\ell\to j}$ for $\ell \neq j$,

$$\begin{aligned} A^{ks+k}X &= \begin{bmatrix} 0 & 0 & \cdots & 0 & A_{1\to k}B_k^s \\ A_{2\to 1}B_1^s & 0 & \cdots & 0 & 0 \\ 0 & A_{3\to 2}B_2^s & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_{k\to k-1}B_{k-1}^s & 0 \end{bmatrix} \\ &= A^{ks+k-1}, \end{aligned}$$

from (2.5) by using Lemma 2.2. By Theorem 1.1, index $A \leq ks+k-1$ and $X = A^D$. □

Thus, the Drazin inverse of a k-cyclic matrix is reduced to calculating the Drazin inverse of the smallest order Drazin inverse of any of the matrix products B_i.

Corollary 2.5. *If A of the form in (1.1) is nonnegative and has at least one $B_i^D \geq 0$, then A^D is nonnegative.*

The following example illustrates Theorem 2.4 and Corollary 2.5.

Example 2.6. Let

$$A = \left[\begin{array}{c|cc|c} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{c|c|c} 0 & A_{12} & 0 \\ \hline 0 & 0 & A_{23} \\ \hline A_{31} & 0 & 0 \end{array}\right].$$

Then $B_1 = A_{12}A_{23}A_{31} = 1, B_2 = A_{23}A_{31}A_{12} = \frac{1}{2}J_2$ (where J_2 is 2×2 all ones matrix) and $B_3 = A_{31}A_{12}A_{23} = 1$. Note that index $B_1 = 0$ and $B_1^D = B_1^{-1} = 1$. Using Theorem 2.4,

$$A^D = \left[\begin{array}{c|c|c} 0 & 0 & B_1^D A_{12}A_{23} \\ \hline A_{23}A_{31}B_1^D & 0 & 0 \\ \hline 0 & A_{31}B_1^D A_{12} & 0 \end{array}\right] = \left[\begin{array}{c|cc|c} 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{array}\right] = A^2.$$

In fact, rank A = rank A^2, hence $A^D = A^{\#} = A^2$ agreeing with Theorem 2.2 in [11].

3. Index of A in relation to the indices of the block products. With A as in (1.1), for $j \geq 0$, by (2.3) and (2.4),

$$\text{rank}\, A^{kj} = \text{rank}\, B_1^j + \text{rank}\, B_2^j + \cdots + \text{rank}\, B_k^j \tag{3.1}$$

$$\text{rank}\, A^{kj+1} = \text{rank}\, B_1^j A_{12} + \text{rank}\, B_2^j A_{23} + \cdots + \text{rank}\, B_k^j A_{k1}. \tag{3.2}$$

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The following rank inequality is used throughout the proof of Lemma 3.2 and can be found in standard linear algebra texts (see, e.g., [12, page 13]).

LEMMA 3.1. (Frobenius Inequality) *If U is $m \times n$, V is $n \times p$ and W is $p \times q$, then*

$$\operatorname{rank} UV + \operatorname{rank} VW \leq \operatorname{rank} V + \operatorname{rank} UVW.$$

LEMMA 3.2. *Let A be as in (1.1) with associated matrices B_i defined in (2.1), and let $s = \operatorname{index} B_i \geq 1$ for some $i \in \{1, \ldots, k\}$. Then $\operatorname{rank} A^{ks-k+1} < \operatorname{rank} A^{ks-k}$.*

Proof. Let $s = \operatorname{index} B_i$ for some $i \in \{1, \ldots, k\}$. From (3.2),

$$\begin{aligned} \operatorname{rank} A^{ks-k+1} &= \operatorname{rank} A^{k(s-1)+1} \\ &= \operatorname{rank} B_1^{s-1} A_{12} + \operatorname{rank} B_2^{s-1} A_{23} + \operatorname{rank} B_3^{s-1} A_{34} + \cdots \\ &\quad + \operatorname{rank} B_k^{s-1} A_{k1}, \end{aligned}$$

where the terms can be reordered as

$$\begin{aligned} &\operatorname{rank} B_i^{s-1} A_{i,i+1} + \operatorname{rank} B_{i+1}^{s-1} A_{i+1,i+2} + \cdots + \operatorname{rank} B_k^{s-1} A_{k1} + \operatorname{rank} B_1^{s-1} A_{12} + \cdots \\ &\qquad + \operatorname{rank} B_{i-1}^{s-1} A_{i-1,i}. \end{aligned} \tag{3.3}$$

Using Lemma 2.2, the first two terms in the expression in (3.3) can be written as

$$\operatorname{rank} A_{i,i+1} B_{i+1}^{s-1} + \operatorname{rank} B_{i+1}^{s-1} A_{i+1,i+2},$$

and using the Frobenius inequality (Lemma 3.1),

$$\begin{aligned} \operatorname{rank} B_i^{s-1} A_{i,i+1} + \operatorname{rank} B_{i+1}^{s-1} A_{i+1,i+2} &\leq \operatorname{rank} B_{i+1}^{s-1} + \operatorname{rank} A_{i,i+1} B_{i+1}^{s-1} A_{i+1,i+2} \\ &= \operatorname{rank} B_{i+1}^{s-1} + \operatorname{rank} B_i^{s-1} A_{i \to i+2}, \end{aligned}$$

where the equality is again due to Lemma 2.2. Thus,

$$\begin{aligned} \operatorname{rank} A^{ks-k+1} &\leq \operatorname{rank} B_{i+1}^{s-1} + \operatorname{rank} B_i^{s-1} A_{i \to i+2} + \operatorname{rank} B_{i+2}^{s-1} A_{i+2,i+3} + \cdots \\ &\quad + \operatorname{rank} B_{i-1}^{s-1} A_{i-1,i}. \end{aligned} \tag{3.4}$$

Applying Lemma 2.2 and the Frobenius inequality again to the second and third terms on the righthand side of the inequality in (3.4) gives

$$\operatorname{rank} B_i^{s-1} A_{i \to i+2} + \operatorname{rank} B_{i+2}^{s-1} A_{i+2,i+3} \leq \operatorname{rank} B_{i+2}^{s-1} + \operatorname{rank} A_{i \to i+3} B_{i+3}^{s-1}.$$

Continuing in this manner gives

$$\begin{aligned} \operatorname{rank} A^{ks-k+1} &\leq \operatorname{rank} B_{i+1}^{s-1} + \operatorname{rank} B_{i+2}^{s-1} + \cdots + \operatorname{rank} B_{i-1}^{s-1} + \\ &\quad \operatorname{rank} A_{i \to i-1} B_{i-1}^{s-1} A_{i-1,i}. \end{aligned} \tag{3.5}$$

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Using Lemma 2.2, the last term on the righthand side of the inequality in (3.5) becomes

$$\operatorname{rank} B_i^{s-1} A_{i\to i-1} A_{i-1,i} = \operatorname{rank} B_i^{s-1} B_i = \operatorname{rank} B_i^s < \operatorname{rank} B_i^{s-1},$$

since $\operatorname{index} B_i = s$. Thus,

$$\begin{aligned}\operatorname{rank} A^{ks-k+1} &< \operatorname{rank} B_{i+1}^{s-1} + \operatorname{rank} B_{i+2}^{s-1} + \cdots \operatorname{rank} B_k^{s-1} + \operatorname{rank} B_1^{s-1} + \cdots \\ &\quad \operatorname{rank} B_{i-1}^{s-1} + \operatorname{rank} B_i^{s-1} \\ &= \operatorname{rank} A^{k(s-1)} = \operatorname{rank} A^{ks-k},\end{aligned}$$

where the equality follows from (3.1). ∎

Theorem 3.3. *Let A be as in (1.1) with associated matrices B_i defined in (2.1). Then, the following statements hold.*

(i) If $\operatorname{index} B_i = 0$ *for all* $i = 1, \ldots, k$*, then* A *is nonsingular and* $\operatorname{index} A = 0$.
(ii) If $\operatorname{index} B_i = s_i \geq 1$ *for some* $i \in \{1, \ldots, k\}$*, then* $\operatorname{index} A \geq ks_i - k + 1$.

Proof. The first statement follows immediately from (2.3) and (3.1). For the second statement, let $\operatorname{index} B_i = s_i \geq 1$ for some $i \in \{1, \ldots, k\}$. Then $\operatorname{rank} A^{ks_i-k+1} < \operatorname{rank} A^{ks_i-k}$, by Lemma 3.2. From the strict inequality, $\operatorname{index} A \geq ks_i - k + 1$. ∎

The next result follows immediately from Theorem 3.3(ii).

Corollary 3.4. *Let A be as in (1.1) with associated matrices B_i defined in (2.1). If* $\operatorname{index} A \leq 1$*, then* $\operatorname{index} B_i \leq 1$ *for all $i = 1, \ldots, k$. That is, if the group inverse $A^\#$ exists, then the group inverses $B_i^\#$ exist for all $i = 1, \ldots, k$.*

Note however that the converse to Corollary 3.4 is false (see, e.g., [4, Example 4.3]).

Remark 3.5. If A of the form (1.1) is nonnegative and all matrices with the same $+, 0$ sign pattern as A that have index 1 have at least one $B_i^\#$ nonnegative, then these group inverses are nonnegative (Corollary 2.5) and A is *conditionally* S^2GI in the notation of Zhou et al. [15].

Corollary 3.6. *Let A be as in (1.1) with associated matrices B_i defined in (2.1), and let* $s = \min\limits_{1\leq i\leq k} \operatorname{index} B_i$ *and* $s' = \max\limits_{1\leq i\leq k} \operatorname{index} B_i > 0$*. Then* $ks' - k + 1 \leq \operatorname{index} A \leq ks + k - 1$*. If $s' = 0$, then* $\operatorname{index} A = 0$.

Corollary 3.6 leads to a result about the indices of B_i that is of independent interest.

Theorem 3.7. *Let A be as in (1.1) with associated matrices B_i defined in (2.1), and let* $s_\ell = \operatorname{index} B_\ell$ *for $\ell \in \{1, \ldots, k\}$. Then $|s_i - s_j| \leq 1$ for all $i, j \in \{1, \ldots, k\}$.*

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Proof. Let $s = \min_{1\le i\le k} \operatorname{index} B_i$ and $s' = \max_{1\le i\le k} \operatorname{index} B_i$, and suppose that $s' = s+t$ where $t \geq 0$. By Corollary 3.6,

$$k(s+t) - k + 1 \leq \operatorname{index} A \leq ks + k - 1.$$

It follows that

$$k(s+t) - k + 1 \leq ks + k - 1,$$

or equivalently,

$$k(t-2) + 2 \leq 0.$$

As $k \geq 2$, the inequality above is possible only if $t \leq 1$. Thus, $s' - s = t \leq 1$ and $|\operatorname{index} B_i - \operatorname{index} B_j| \leq 1$ for all i, j. □

The next result gives tight bounds on $\operatorname{index} A$ in terms of the minimum index of the block products B_i. The proof is immediate from Corollary 3.6 and Theorem 3.7.

THEOREM 3.8. *Let A be as in (1.1) with associated matrices B_i defined in (2.1), and let $s = \min_{1\le i\le k} \operatorname{index} B_i$. Then, exactly one of the following holds:*
(i) $\operatorname{index} B_i = s$ *for all* $i = 1, \ldots, k$, *or*
(ii) $\operatorname{index} B_i = s + 1$ *for some* $i = 1, \ldots, k$.
If (i) holds, then $ks - k + 1 \leq \operatorname{index} A \leq ks + k - 1$. *If (ii) holds, then* $ks + 1 \leq \operatorname{index} A \leq ks + k - 1$.

The above result generalizes bounds found in [4, Section 3] and shows that if $k = 2$ and (ii) holds, then $\operatorname{index} A = 2s + 1$.

We now give examples that illustrate Theorem 3.8.

EXAMPLE 3.9. Let A be the matrix in Example 2.6. Using the notation in Theorem 3.8, $s = 0 = \operatorname{index} B_1 = \operatorname{index} B_3$ and $\operatorname{index} B_2 = 1 = s + 1$. Applying the result with $k = 3$ gives the bounds $1 \leq \operatorname{index} A \leq 2$. Since $\operatorname{rank} A = \operatorname{rank} A^2$, $\operatorname{index} A = 1 = ks + 1$, which is the lower bound of Theorem 3.8, case (ii).

EXAMPLE 3.10. Let

$$A = \left[\begin{array}{c|cc|ccc} 0 & 1 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{c|c|c} 0 & A_{12} & 0 \\ \hline 0 & 0 & A_{23} \\ \hline A_{31} & 0 & 0 \end{array}\right].$$

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Then $B_1 = 3, B_2 = \begin{bmatrix} 3 & -3 \\ 0 & 0 \end{bmatrix}$ and $B_3 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$. Note that index $B_1 = 0$ and $B_1^D = B_1^{-1} = \frac{1}{3}$. Using Theorem 2.4,

$$A^D = \left[\begin{array}{c|cc|ccc} 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & \frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 \end{array}\right].$$

Using the notation in Theorem 3.8, $s = 0 = \operatorname{index} B_1$ and $\operatorname{index} B_2 = \operatorname{index} B_3 = 1 = s + 1$. Applying the theorem with $k = 3$ gives the bounds $1 \le \operatorname{index} A \le 2$. It can be computed that $\operatorname{index} A = 2 = ks + k - 1$, which is the upper bound of Theorem 3.8, case (ii).

Example 3.11. Let

$$A = \begin{bmatrix} 0 & B & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ I & 0 & 0 & \cdots & 0 \end{bmatrix},$$

where B is a square matrix and I is an identity matrix of the same order as B. Note that $B_i = B$ for all i. Suppose that $\operatorname{index} B = s$. Then $\operatorname{index} A = ks$, the midpoint of the interval $[ks - k + 1,\ ks + k - 1]$ in Theorem 3.8 case (i), and from Theorem 2.4

$$A^D = \begin{bmatrix} 0 & 0 & \cdots & 0 & B^DB \\ B^D & 0 & \cdots & 0 & 0 \\ 0 & B^DB & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B^DB & 0 \end{bmatrix}.$$

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EXAMPLE 3.12. Let

$$A = \begin{bmatrix} 0 & F & 0 & \cdots & 0 \\ 0 & 0 & F & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & F \\ F & 0 & 0 & \cdots & 0 \end{bmatrix},$$

where F is a square matrix. Then $\operatorname{index} A = \operatorname{index} F$ and $B_i = F^k$ for $i = 1, \ldots, k$. Setting $\operatorname{index} A = \ell$ and $\operatorname{index} B_i = s$ gives $s = \left\lceil \frac{\ell}{k} \right\rceil$. Thus, $\operatorname{index} A$ can take any value in the interval $[ks - k + 1, ks]$, which is half the range given in Theorem 3.8 case (i).

Examples 3.11 and 3.12 have B_i, and thus $\operatorname{index} B_i$, the same for all i. The following result determines $\operatorname{index} A$ in this case, and the necessary and sufficient conditions reduce to the result of [4, Theorem 3.5] for $k = 2$.

THEOREM 3.13. *Let A be a block k-cyclic matrix of the form in (1.1) with associated matrices B_i defined in (2.1), and suppose that $s = \min\limits_{1 \le i \le k} \operatorname{index} B_i \ge 1$. Then $\operatorname{index} A = ks$ if and only if*
(i) $\operatorname{index} B_i = s$ for all $i = 1, \ldots, k$, and
(ii) $\operatorname{rank} B_j^s < \operatorname{rank} B_j^{s-1} A_{j \to j-1}$ for some $j \in \{1, \ldots, k\}$.
If (i) holds, then $\operatorname{rank} B_i^s = \operatorname{rank} B_j^s$ for all $i, j = 1, \ldots, k$. If (i) holds but (ii) does not hold, then $\operatorname{index} A < ks$.

Proof. Suppose that $\operatorname{index} A = ks$. Then $\operatorname{rank} A^{ks} < \operatorname{rank} A^{ks-1}$. It follows, using (2.3), (2.5) and (3.1), that $\sum_{i=1}^{k} \operatorname{rank} B_i^s < \sum_{i=1}^{k} \operatorname{rank} B_i^{s-1} A_{i \to i-1}$. Thus, $\operatorname{rank} B_j^s < \operatorname{rank} B_j^{s-1} A_{j \to j-1}$ for some $j \in \{1, \ldots, k\}$, hence (ii) holds. Suppose on the contrary that (i) does not hold. Then, for some $j \in \{1, \ldots, k\}$, $\operatorname{index} B_j = s + 1$ (by Theorem 3.8). Thus, $\operatorname{rank} B_j^s > \operatorname{rank} B_j^{s+1}$, hence by (2.3) $\operatorname{rank} A^{ks} = \sum_{i=1}^{k} \operatorname{rank} B_i^s > \sum_{i=1}^{k} \operatorname{rank} B_i^{s+1} = \operatorname{rank} A^{k(s+1)}$. This implies that $\operatorname{rank} A^{ks} > \operatorname{rank} A^{ks+k}$, so $\operatorname{index} A > ks$, a contradiction. Hence, (i) and (ii) must hold.

For the reverse implication, suppose that (i) and (ii) hold. Then $\operatorname{rank} A^{ks} = \sum_{i=1}^{k} \operatorname{rank} B_i^s < \sum_{i=1}^{k} \operatorname{rank} B_i^{s-1} A_{i \to i-1} = \operatorname{rank} A^{k(s-1)+(k-1)} = \operatorname{rank} A^{ks-1}$, where the strict inequality is due to (ii). Thus, $\operatorname{index} A \ge ks$. Note that since $\operatorname{rank} B_i^s \ge \operatorname{rank} B_i^s A_{i \to j} \ge \operatorname{rank} B_i^{s+1}$ and $\operatorname{rank} B_i^s A_{i \to j} = \operatorname{rank} A_{i \to j} B_j^s$ (by Lemma 2.2), it follows using (i) that $\operatorname{rank} B_i^{s+1} = \operatorname{rank} B_i^s = \operatorname{rank} B_i^s A_{i \to j} = \operatorname{rank} A_{i \to j} B_j^s = \operatorname{rank} B_j^{s+1}$ for all i, j. Thus, $\operatorname{rank} A^{ks} = \sum_{i=1}^{k} \operatorname{rank} B_i^s = \sum_{i=1}^{k} \operatorname{rank} B_i^s A_{i \to i+1} = \operatorname{rank} A^{ks+1}$, using (3.1) and (3.2) . Hence, $\operatorname{rank} A^{ks} = \operatorname{rank} A^{ks+1}$, and so $\operatorname{index} A \le ks$. This

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proves that $\operatorname{index} A = ks$. The last two statements of the theorem follow from the proof above. ∎

The result of Theorem 3.13 is illustrated by Example 3.11; since $\operatorname{rank} B_2^s < \operatorname{rank} B_2^{s-1} A_{2\to 1} = \operatorname{rank} B_2^{s-1}$, it follows that $\operatorname{rank} A = ks$. Example 3.12 also illustrates Theorem 3.13, since $\operatorname{rank} A^{ks} = \operatorname{rank} A^{ks+1}$ and $\operatorname{rank} F^{ks} < \operatorname{rank} F^{k(s-1)} F^{k-1} = \operatorname{rank} F^{ks-1}$ if and only if $\operatorname{index} F = \operatorname{index} A = ks$; otherwise $\operatorname{index} A < ks$.

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