Representations for the Drazin Inverse of Block Cyclic Matrices

M. Catral† and P. Van Den Driessche‡

Abstract. A formula for the Drazin inverse of a block $k$-cyclic ($k \geq 2$) matrix $A$ with nonzeros only in blocks $A_{i,i+1}$, for $i = 1, \ldots, k$ (mod $k$) is presented in terms of the Drazin inverse of a smaller order product of the nonzero blocks of $A$, namely $B_i = A_{i,i+1} \cdots A_{i-1,i}$, for some $i$. Bounds on the index of $A$ in terms of the minimum and maximum indices of these $B_i$ are derived. Illustrative examples and special cases are given.

Key words. Drazin inverse, Block cyclic matrix, Index.

AMS subject classifications. 15A09.

1. Introduction. We consider $k$-cyclic ($k \geq 2$) block real or complex matrices of the form

$$A = \begin{bmatrix}
0 & A_{12} & 0 & \cdots & 0 \\
0 & 0 & A_{23} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & A_{k-1,k} \\
A_{k1} & 0 & 0 & \cdots & 0
\end{bmatrix},$$

where $A_{12}, \ldots, A_{k1}$ are block submatrices and the diagonal zero blocks are square. It is easily verified that for any matrix $A$ of the form (1.1), the Moore-Penrose inverse $A^\dagger$ of $A$ is given by

$$A^\dagger = \begin{bmatrix}
0 & 0 & \cdots & 0 & A_{k1}^\dagger \\
A_{12}^\dagger & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & A_{k-1,k}^\dagger & 0
\end{bmatrix},$$

Received by the editors on February 6, 2012. Accepted for publication on April 15, 2012. Handling Editor: Leslie Hogben.

†Department of Mathematics and Computer Science, Xavier University, Cincinnati, OH 45207, USA (catralm@xavier.edu).

‡Department of Mathematics and Statistics, University of Victoria, Victoria BC V8W 3R4, Canada (pvd@math.uvic.ca).
where $A_{ij}^\dagger$ denotes the Moore-Penrose inverse of the block submatrix $A_{ij}$. Note that if each of the blocks $A_{ij}$ is square and invertible, then (1.2) gives the formula for the usual inverse $A^{-1}$ of $A$. We present a block formula for another type of generalized inverse, the Drazin inverse, of matrices of the form (1.1). Unlike the Moore-Penrose inverse, the Drazin inverse is defined only for square matrices.

Let $A$ be a real or complex square matrix. The Drazin inverse of $A$ is the unique matrix $A^D$ satisfying

\begin{align}
AA^D &= A^DA \\
A^DAA^D &= A^D \\
A^{q+1}A^D &= A^q,
\end{align}

where $q = \text{index } A$, the smallest nonnegative integer $q$ such that $\text{rank } A^{q+1} = \text{rank } A^q$. If index $A = 0$, then $A$ is nonsingular and $A^D = A^{-1}$. If index $A = 1$, then $A^D = A^\#$, the group inverse of $A$. See [1], [2], [6] and references therein for applications of the Drazin inverse.

**Theorem 1.1.** [2, Theorem 7.2.3] Let $A$ be a square matrix with index $A = q$. If $p$ is a nonnegative integer and $X$ is a matrix satisfying $XAX = X$, $AX =XA$, and $A^{q+1}X = A^p$, then $p \geq q$ and $X = A^D$.

The problem of finding explicit representations for the Drazin inverse of a general $2 \times 2$ block matrix of the form

\begin{equation}
A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}
\end{equation}

in terms of its blocks was posed by Campbell and Meyer in [2] and special cases of this problem were the focus of several recent papers, including [3]–[10], [13], [14] and [15]. In [4] and [13], representations for $2 \times 2$ block matrices matrices of the form (1.6) with $A_{11}$ and $A_{22}$ being square zero diagonal blocks were presented. Such block matrices were called bipartite (or 2-cyclic), and in this article, we extend the results given in [4] to general block $k$-cyclic matrices as defined in (1.1).

2. **Drazin inverse formula for block cyclic matrices.** Let $A$ be a block $k$-cyclic matrix of the form given in (1.1). For our Drazin inverse formula we introduce some notation that is also used in writing powers of $A$. For $i = 2, \ldots, k-1$, let $B_i$ be the square matrix defined by

\begin{equation}
B_i = A_{i,i+1} \cdots A_{k-1,k} A_{k1} A_{12} \cdots A_{i-1,i},
\end{equation}

with $B_1 = A_{12} A_{23} \cdots A_{k-1,k} A_{k1}$ and $B_k = A_{k1} A_{12} \cdots A_{k-1,k}$, i.e., subscripts are taken mod $k$. For ease of notation, we define the matrix product

\begin{equation}
A_{i\to j} := A_{i,i+1} A_{i+1,i+2} \cdots A_{j-1,j},
\end{equation}

where $A_{ij}^\dagger$ denotes the Moore-Penrose inverse of the block submatrix $A_{ij}$. Note that if each of the blocks $A_{ij}$ is square and invertible, then (1.2) gives the formula for the usual inverse $A^{-1}$ of $A$. We present a block formula for another type of generalized inverse, the Drazin inverse, of matrices of the form (1.1). Unlike the Moore-Penrose inverse, the Drazin inverse is defined only for square matrices.

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B_i = A_{i,i+1} \cdots A_{k-1,k} A_{k1} A_{12} \cdots A_{i-1,i},
\end{equation}

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\begin{equation}
A_{i\to j} := A_{i,i+1} A_{i+1,i+2} \cdots A_{j-1,j},
\end{equation}
for \( j \neq i \). Whenever it arises, we use the convention \( A_{i \rightarrow i} = I \), an identity matrix. For example, if \( k = 4 \) then \( A_{2 \rightarrow 3} = A_{23}, A_{3 \rightarrow 2} = A_{34}A_{41}A_{12}, A_{2 \rightarrow 1} = A_{23}A_{34}A_{41}, \) and by Lemma 2.2, \( B_3 = A_{34}A_{41}A_{12}A_{23} \). Observe that \( B_i = A_{i \rightarrow j}A_{j \rightarrow i} \), for any \( j \in \{1, \ldots, k\} \setminus \{i\} \).

**Lemma 2.1.** For \( A \) given in (I,1) and with the notation above, for \( p \geq 0 \),

\[
A^{kp} = \begin{bmatrix}
B_1^p & 0 & \cdots & 0 \\
0 & B_2^p & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & B_k^p
\end{bmatrix},
\]

(2.3) \[ A^{kp+1} = \begin{bmatrix}
0 & B_1^pA_{1 \rightarrow 2} & 0 & \cdots & 0 \\
0 & 0 & B_2^pA_{2 \rightarrow 3} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & B_{k-1}^pA_{k-1 \rightarrow k} \\
B_k^pA_{k \rightarrow 1} & 0 & 0 & \cdots & 0
\end{bmatrix}, \]

(2.4) \[ A^{kp+2} = \begin{bmatrix}
0 & 0 & B_1^pA_{1 \rightarrow 3} & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & B_{k-2}^pA_{k-2 \rightarrow k} \\
B_k^pA_{k-1 \rightarrow 1} & 0 & 0 & \cdots & 0 \\
0 & B_k^pA_{k \rightarrow 2} & 0 & \cdots & 0
\end{bmatrix} \]

and so on, until

\[
A^{kp+k-1} = \begin{bmatrix}
0 & 0 & \cdots & 0 & B_1^pA_{1 \rightarrow k} \\
B_2^pA_{2 \rightarrow 1} & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & B_{k-1}^pA_{k-1 \rightarrow k-1} & 0 \\
0 & 0 & \cdots & B_k^pA_{k \rightarrow k-1} & 0
\end{bmatrix},
\]

(2.5)

**Lemma 2.2.** For all \( i \neq j \), \( B_i^kA_{i \rightarrow j} = A_{i \rightarrow j}B_j^k \).

*Proof.* \( B_i^kA_{i \rightarrow j} = (A_{i \rightarrow j}A_{j \rightarrow i})^kA_{i \rightarrow j} = A_{i \rightarrow j}A_{j \rightarrow i}(A_{i \rightarrow j}A_{j \rightarrow i})^{k-1}A_{i \rightarrow j} = A_{i \rightarrow j}(A_{j \rightarrow i}A_{i \rightarrow j})^k = A_{i \rightarrow j}B_i^k. \)

**Lemma 2.3.** For all \( i \neq j \), \( B_i^D A_{i \rightarrow j} = A_{i \rightarrow j}B_j^D \). Hence, if \( \ell \neq i,j \) satisfies \( A_{i \rightarrow j} = A_{i \rightarrow \ell}A_{\ell \rightarrow j} \), then \( B_i^D A_{i \rightarrow j} = A_{i \rightarrow j}B_j^D = A_{i \rightarrow \ell}B_{\ell}^D A_{\ell \rightarrow j}. \)
Proof.  $B_i^D A_{i \rightarrow j} = (A_{i \rightarrow j} A_{j \rightarrow i})^D A_{i \rightarrow j} = A_{i \rightarrow j} (A_{j \rightarrow i} A_{i \rightarrow j})^D = A_{i \rightarrow j} B_j^D$, where the second equality is due to \cite[Lemma 2.4]{I}.

With the above notation, we now give a formula for the Drazin inverse of a $k$-cyclic matrix $A$ given by (1.1).

**Theorem 2.4.** Let $A$ be as in (1.1) with associated matrices $B_i$ defined as in (2.1) and $A_{i \rightarrow j}$ defined in (2.2). Then, for all $i = 1, \ldots, k$,

$$
D = A_i \rightarrow j B_j D A_i \rightarrow k
$$

Moreover, if $\text{index } B_i = s_i$, then $\text{index } A \leq ks_i + k - 1$.

**Proof.** Denote the matrix on the right hand side of (2.6) by $X$. Performing block multiplication gives

$$
AX = \begin{bmatrix}
A_{12} A_{2 \rightarrow 1} A_{1 \rightarrow 1} & 0 & \cdots & 0 \\
0 & A_{23} A_{3 \rightarrow 1} A_{1 \rightarrow 2} & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_{k1} A_{1 \rightarrow i} A_{i \rightarrow k}
\end{bmatrix}
$$

(by Lemma 2.3)

$$
= \begin{bmatrix}
B_1 B_1^D & 0 & \cdots & 0 \\
0 & B_2 B_2^D & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & B_k B_k^D
\end{bmatrix},
$$
and by using Lemma 2.3 again

\[
XA = \begin{bmatrix}
A_{1\to i}, B_i^D A_{i\to k} A_{k1} & 0 & \cdots & 0 \\
0 & A_{2\to i}, B_i^D A_{i\to 12} & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & A_{k\to i}, B_i^D A_{i\to k-1} A_{k-1,k}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
B_i^D A_{1\to i}, A_{i\to k} A_{k1} & 0 & \cdots & 0 \\
0 & B_2^D A_{2\to i}, A_{i\to 12} & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & B_k^D A_{k\to i}, A_{i\to k-1} A_{k-1,k}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
B_i^D B_1 & 0 & \cdots & 0 \\
0 & B_2^D B_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & B_k^D B_k
\end{bmatrix}
\begin{bmatrix}
B_i B_1^D & 0 & \cdots & 0 \\
0 & B_2 B_2^D & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & B_k B_k^D
\end{bmatrix}
\]

\[
= AX,
\]

since \( B_i^D B_i = B_i B_i^D \) by (1.3). Also, block-multiplying \( X \) with \( AX \) gives

\[
XAX = X(AX)
\]

\[
= \begin{bmatrix}
0 & 0 & \cdots & 0 & A_1, B_1^D B_k B_1 \\
0 & A_2, B_2^D B_1 B_2 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & A_{k-1}, B_{k-1}^D B_{k-1} B_{k-1} & 0
\end{bmatrix}
\]

\[
= X, \text{ by Lemma 2.3 and since } B_i^D B_i B_i^D = B_i^D \text{ by (1.4).}
\]

Let \( i \) be any integer in \( \{1, \ldots, k\} \) and suppose that index \( B_i = s_i = s \). Then using (2.3) and Lemma 2.2

\[
A^{k+k} X = A^{k(s+1)} X
\]

\[
= \begin{bmatrix}
0 & 0 & \cdots & 0 & A_{1\to s} B_{s+1}^D A_{s\to k} \\
0 & A_{2\to s} B_{s+1}^D A_{s\to 1} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & A_{k\to s} B_{s+1}^D A_{s\to k-1} & 0
\end{bmatrix}
\]

\[\cdots \]
Since index $B_i = s$, it follows by (1.5) that $B_i^{s+1} = B_i^s$. Thus, using Lemma 2.2 and $A_{\ell \to i, 1 \to j} = A_{\ell \to j}$ for $\ell \neq j$, 

$$
A^{ks+k} X = 
\begin{bmatrix}
0 & 0 & \cdots & 0 & A_{1 \to k} B_k^s \\
A_{2 \to 1} B_1^s & 0 & \cdots & 0 & 0 \\
0 & A_{3 \to 2} B_2^s & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & A_{k \to k-1} B_{k-1}^s & 0 \\
\end{bmatrix}
$$

= $A^{ks+k-1}$, 

from (2.5) by using Lemma 2.2. By Theorem 1.1, index $A \leq ks + k - 1$ and $X = A^D$. 

Thus, the Drazin inverse of a $k$-cyclic matrix is reduced to calculating the Drazin inverse of the smallest order Drazin inverse of any of the matrix products $B_i$.

**Corollary 2.5.** If $A$ of the form in (1.1) is nonnegative and has at least one $B_i D_i \geq 0$, then $A^D$ is nonnegative.

The following example illustrates Theorem 2.4 and Corollary 2.5.

**Example 2.6.** Let 

$$
A = 
\begin{bmatrix}
0 & \frac{1}{2} & \frac{1}{2} & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
\end{bmatrix}
= 
\begin{bmatrix}
0 & A_{12} & 0 \\
0 & 0 & A_{23} \\
A_{31} & 0 & 0 \\
\end{bmatrix}
.$$ 

Then $B_1 = A_{12} A_{23} A_{31} = 1, B_2 = A_{23} A_{31} A_{12} = \frac{1}{2} J_2$ (where $J_2$ is a $2 \times 2$ all ones matrix) and $B_3 = A_{31} A_{12} A_{23} = 1$. Note that index $B_1 = 0$ and $B_1^D = B_1^{-1} = 1$.

Using Theorem 2.4, 

$$
A^D = 
\begin{bmatrix}
0 & 0 & B_1^D A_{12} A_{23} \\
A_{24} A_{31} B_1^D & 0 & 0 \\
0 & A_{31} B_1^D A_{12} & 0 \\
\end{bmatrix}
= 
\begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
0 & \frac{1}{2} & \frac{1}{2} \\
\end{bmatrix}
= A^2.
$$

In fact, rank $A = rank A^2$, hence $A^D = A^\# = A^2$ agreeing with Theorem 2.2 in [11].

3. Index of $A$ in relation to the indices of the block products. With $A$ as in (1.1), for $j \geq 0$, by (2.9) and (2.4),

(3.1) $\text{rank } A^{kj} = \text{rank } B_i^j + \text{rank } B_2^j + \cdots + \text{rank } B_k^j$

(3.2) $\text{rank } A^{kj+1} = \text{rank } B_i^j A_{12} + \text{rank } B_2^j A_{23} + \cdots + \text{rank } B_k^j A_{k1}$.
The following rank inequality is used throughout the proof of Lemma 3.2 and can be found in standard linear algebra texts (see, e.g., [12, page 13]).

**Lemma 3.1. (Frobenius Inequality)** If $U$ is $m \times n$, $V$ is $n \times p$ and $W$ is $p \times q$, then

$$\text{rank} UV + \text{rank} VW \leq \text{rank} V + \text{rank} UVW.$$

**Lemma 3.2.** Let $A$ be as in (2.1) with associated matrices $B_i$ defined in (2.1), and let $s = \text{index } B_i \geq 1$ for some $i \in \{1, \ldots, k\}$. Then $\text{rank } A^{ks-k+1} < \text{rank } A^{ks-k}$.

**Proof.** Let $s = \text{index } B_i$ for some $i \in \{1, \ldots, k\}$. From (3.2),

$$\text{rank } A^{ks-k+1} = \text{rank } A^{k(s-1)+1} = \text{rank } B_i^{s-1}A_{12} + \text{rank } B_2^{s-1}A_{23} + \text{rank } B_3^{s-1}A_{34} + \cdots + \text{rank } B_k^{s-1}A_{ki},$$

where the terms can be reordered as

$$\text{rank } B_i^{s-1}A_{i,i+1} + \text{rank } B_{i+1}^{s-1}A_{i+1,i+2} + \cdots + \text{rank } B_k^{s-1}A_{k1} + \text{rank } B_1^{s-1}A_{12} + \cdots + \text{rank } B_{i-1}^{s-1}A_{i-1,j}.$$  

(3.3)

Using Lemma 2.2 the first two terms in the expression in (3.3) can be written as

$$\text{rank } A_{i,i+1}B_{i+1}^{s-1} + \text{rank } B_{i+1}^{s-1}A_{i+1,i+2},$$

and using the Frobenius inequality (Lemma 3.1),

$$\text{rank } B_i^{s-1}A_{i,i+1} + \text{rank } B_{i+1}^{s-1}A_{i+1,i+2} \leq \text{rank } B_{i+1}^{s-1} + \text{rank } A_{i,i+1}B_{i+1}^{s-1}A_{i+1,i+2} = \text{rank } B_{i+1}^{s-1} + \text{rank } B_i^{s-1}A_{i+1,i+2},$$

where the equality is again due to Lemma 2.2. Thus,

$$\text{rank } A^{ks-k+1} \leq \text{rank } B_{i+1}^{s-1} + \text{rank } B_i^{s-1}A_{i,i+1} + \text{rank } B_{i+2}^{s-1}A_{i+2,i+3} + \cdots + \text{rank } B_{i+1}^{s-1}A_{i,1,j}.$$  

(3.4)

Applying Lemma 2.2 and the Frobenius inequality again to the second and third terms on the right-hand side of the inequality in (3.1) gives

$$\text{rank } B_i^{s-1}A_{i,i+2} + \text{rank } B_i^{s-1}A_{i+2,i+3} \leq \text{rank } B_i^{s-1} + \text{rank } A_{i,i+3}B_i^{s-1}.$$  

Continuing in this manner gives

$$\text{rank } A^{ks-k+1} \leq \text{rank } B_{i+1}^{s-1} + \text{rank } B_{i+2}^{s-1} + \cdots + \text{rank } B_i^{s-1} + \text{rank } A_{i,i+1}B_i^{s-1}A_{i-1,i}.\quad (3.5)$$
Using Lemma 2.2, the last term on the righthand side of the inequality in (3.5) becomes
\[ \text{rank } B_i^{-1}A_{i-1}A_{i-1} = \text{rank } B_i^{-1} = \text{rank } B_i^{s-1}, \]
since index \( B_i = s \). Thus,
\[ \text{rank } A^{k_{s_{i-k+1}}} < \text{rank } A^{k_{s_{i-k+1}}} + \text{rank } B_i^{s-1} + \cdots \text{rank } B_i^{s-1} + \cdots \]
\[ = \text{rank } A^{k_{s_{i-k+1}}} = \text{rank } A^{k_{s-k}}, \]
where the equality follows from (3.1).

**THEOREM 3.3.** Let \( A \) be as in (1.1) with associated matrices \( B_i \) defined in (2.1).

Then, the following statements hold.

(i) If index \( B_i = 0 \) for all \( i = 1, \ldots, k \), then \( A \) is nonsingular and index \( A = 0 \).

(ii) If index \( B_i = s_i \geq 1 \) for some \( i \in \{1, \ldots, k\} \), then \( \text{index } A \geq k s_i - k + 1 \).

**Proof.** The first statement follows immediately from (2.3) and (3.1). For the second statement, let index \( B_i = s_i \geq 1 \) for some \( i \in \{1, \ldots, k\} \). Then \( \text{rank } A^{k_{s_i-k+1}} < \text{rank } A^{k_{s-k}} \), by Lemma 2.2. From the strict inequality, \( \text{index } A \geq k s_i - k + 1 \).

The next result follows immediately from Theorem 3.3(ii).

**COROLLARY 3.4.** Let \( A \) be as in (1.1) with associated matrices \( B_i \) defined in (2.1). If index \( A \leq 1 \), then index \( B_i \leq 1 \) for all \( i = 1, \ldots, k \). That is, if the group inverse \( A^\# \) exists, then the group inverses \( B_i^\# \) exist for all \( i = 1, \ldots, k \).

Note however that the converse to Corollary 3.4 is false (see, e.g., [4, Example 4.3]).

**REMARK 3.5.** If \( A \) of the form (1.1) is nonnegative and all matrices with the same +,0 sign pattern as \( A \) that have index 1 have at least one \( B_i^\# \) nonnegative, then these group inverses are nonnegative (Corollary 2.5) and \( A \) is conditionally \( S^2GI \) in the notation of Zhou et al. [15].

**COROLLARY 3.6.** Let \( A \) be as in (1.1) with associated matrices \( B_i \) defined in (2.1), and let \( s = \min\text{index } B_i \) and \( s' = \max\text{index } B_i > 0 \). Then \( ks' - k + 1 \leq \text{index } A \leq ks + k - 1 \). If \( s' = 0 \), then \( \text{index } A = 0 \).

Corollary 3.6 leads to a result about the indices of \( B_i \) that is of independent interest.

**THEOREM 3.7.** Let \( A \) be as in (1.1) with associated matrices \( B_i \) defined in (2.1), and let \( s_\ell = \text{index } B_\ell \) for \( \ell \in \{1, \ldots, k\} \). Then \( |s_i - s_j| \leq 1 \) for all \( i, j \in \{1, \ldots, k\} \).
Proof. Let \( s = \min_{1 \leq i \leq k} \text{index } B_i \) and \( s' = \max_{1 \leq i \leq k} \text{index } B_i \), and suppose that \( s' = s + t \) where \( t \geq 0 \). By Corollary 3.6
\[
k(s + t) - k + 1 \leq \text{index } A \leq ks + k - 1.
\]
It follows that
\[
k(s + t) - k + 1 \leq ks + k - 1,
\]
or equivalently,
\[
k(t - 2) + 2 \leq 0.
\]
As \( k \geq 2 \), the inequality above is possible only if \( t \leq 1 \). Thus, \( s' - s = t \leq 1 \) and \( |\text{index } B_i - \text{index } B_j| \leq 1 \) for all \( i, j \).

The next result gives tight bounds on index \( A \) in terms of the minimum index of the block products \( B_i \). The proof is immediate from Corollary 3.6 and Theorem 3.7.

**Theorem 3.8.** Let \( A \) be as in (1.1) with associated matrices \( B_i \) defined in (2.1), and let \( s = \min_{1 \leq i \leq k} \text{index } B_i \). Then, exactly one of the following holds:
(i) \( \text{index } B_i = s \) for all \( i = 1, \ldots, k \), or
(ii) \( \text{index } B_i = s + 1 \) for some \( i = 1, \ldots, k \).
If (i) holds, then \( ks - k + 1 \leq \text{index } A \leq ks + k - 1 \). If (ii) holds, then \( ks + 1 \leq \text{index } A \leq ks + k - 1 \).

The above result generalizes bounds found in [4, Section 3] and shows that if \( k = 2 \) and (ii) holds, then index \( A = 2s + 1 \).

We now give examples that illustrate Theorem 3.8.

**Example 3.9.** Let \( A \) be the matrix in Example 2.6. Using the notation in Theorem 3.8, \( s = 0 = \text{index } B_1 = \text{index } B_3 \) and \( \text{index } B_2 = 1 = s + 1 \). Applying the result with \( k = 3 \) gives the bounds \( 1 \leq \text{index } A \leq 2 \). Since rank \( A = \text{rank } A^2 \), index \( A = 1 = ks + 1 \), which is the lower bound of Theorem 3.8 case (ii).

**Example 3.10.** Let
\[
A = \begin{bmatrix}
0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 1 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
= \begin{bmatrix}
0 & A_{12} & 0 \\
0 & 0 & A_{23} \\
0 & A_{31} & 0 & 0
\end{bmatrix}.
\]
Then $B_1 = 3, B_2 = \begin{bmatrix} 3 & -3 \\ 0 & 0 \end{bmatrix}$ and $B_3 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$. Note that $B_1 = 0$ and $B_1^{-1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Using Theorem 2.4,

$$A^D = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 \end{bmatrix}.$$ 

Using the notation in Theorem 3.8, $s = 0 = \text{index } B_1$ and $\text{index } B_2 = \text{index } B_3 = 1 = s + 1$. Applying the theorem with $k = 3$ gives the bounds $1 \leq \text{index } A \leq 2$. It can be computed that $\text{index } A = 2 = ks + k - 1$, which is the upper bound of Theorem 3.8 case (ii).

**Example 3.11.** Let

$$A = \begin{bmatrix} 0 & B & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ I & 0 & 0 & \cdots & 0 \end{bmatrix},$$

where $B$ is a square matrix and $I$ is an identity matrix of the same order as $B$. Note that $B_i = B$ for all $i$. Suppose that $\text{index } B = s$. Then $\text{index } A = ks$, the midpoint of the interval $[ks - k + 1, ks + k - 1]$ in Theorem 3.8 case (i), and from Theorem 2.4

$$A^D = \begin{bmatrix} 0 & 0 & \cdots & 0 & B^D B \\ 0 & 0 & \cdots & 0 & 0 \\ B^D & 0 & \cdots & 0 & 0 \\ 0 & B^D B & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B^D B & 0 \end{bmatrix}.$$
Example 3.12. Let

\[ A = \begin{bmatrix} 0 & F & 0 & \cdots & 0 \\ 0 & 0 & F & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & F \\ F & 0 & 0 & \cdots & 0 \end{bmatrix}, \]

where $F$ is a square matrix. Then index $A = \text{index } F$ and $B_i = F^k$ for $i = 1, \ldots, k$. Setting index $A = \ell$ and index $B_i = s$ gives $s = \lceil \frac{\ell}{k} \rceil$. Thus, index $A$ can take any value in the interval $[ks - k + 1, ks]$, which is half the range given in Theorem 3.8 case (i).

Examples 3.11 and 3.12 have $B_i$, and thus index $B_i$, the same for all $i$. The following result determines index $A$ in this case, and the necessary and sufficient conditions reduce to the result of [3], Theorem 3.5] for $k = 2$.

Theorem 3.13. Let $A$ be a block $k$-cyclic matrix of the form in (1.1) with associated matrices $B_i$ defined in (2.1), and suppose that $s = \min_{1 \leq i \leq k} \text{index } B_i \geq 1$. Then index $A = ks$ if and only if

(i) index $B_i = s$ for all $i = 1, \ldots, k$, and
(ii) rank $B_j^s < \text{rank } B_j^{s+1}$ for some $j \in \{1, \ldots, k\}$.

If (i) holds, then rank $B_i^s = \text{rank } B_j^s$ for all $i, j = 1, \ldots, k$. If (i) holds but (ii) does not hold, then index $A < ks$.

Proof. Suppose that index $A = ks$. Then rank $A^{ks} < \text{rank } A^{ks-1}$. It follows, using (2.3), (2.5) and (5.1), that $\sum_{i=1}^{k} \text{rank } B_i^s < \sum_{i=1}^{k} \text{rank } B_i^{s-1} A_{i-1}$. Thus, rank $B_j^s < \text{rank } B_j^{s-1} A_{j-1}$ for some $j \in \{1, \ldots, k\}$, hence (ii) holds. Suppose on the contrary that (i) does not hold. Then, for some $j \in \{1, \ldots, k\}$, index $B_j = s + 1$ (by Theorem 3.8). Thus, rank $B_j^s > \text{rank } B_j^{s+1}$. Hence by (2.5) rank $A^{ks} = \sum_{i=1}^{k} \text{rank } B_i^s > \sum_{i=1}^{k} \text{rank } B_i^{s+1} = \text{rank } A^{k(s+1)}$. This implies that rank $A^{ks} > \text{rank } A^{ks+k}$, so index $A > ks$, a contradiction. Hence, (i) and (ii) must hold.

For the reverse implication, suppose that (i) and (ii) hold. Then rank $A^{ks} = \sum_{i=1}^{k} \text{rank } B_i^s < \sum_{i=1}^{k} \text{rank } B_i^{s-1} A_{i-1} = \text{rank } A^{k(s-1)+(k-1)} = \text{rank } A^{ks-1}$, where the strict inequality is due to (ii). Thus, index $A \geq ks$. Note that since rank $B_i^s \geq \text{rank } B_i^s A_{i-1} \geq \text{rank } B_i^{s+1}$ and rank $B_i^s A_{i-1} = \text{rank } A_{i-1} B_i^s$ (by Lemma 2.2), it follows using (i) that rank $B_i^{s+1} = \text{rank } B_i^s = \text{rank } B_i^s A_{i-1} = \text{rank } A_{i-1} B_i^s = \text{rank } B_i^s$ for all $i, j$. Thus, rank $A^{ks} = \sum_{i=1}^{k} \text{rank } B_i^s = \sum_{i=1}^{k} \text{rank } B_i^s A_{i-1} = \text{rank } A^{ks+1}$, using (3.1) and (3.2). Hence, rank $A^{ks} = \text{rank } A^{ks+1}$, and so index $A \leq ks$. This
proves that index $A = ks$. The last two statements of the theorem follow from the proof above.

The result of Theorem 3.13 is illustrated by Example 3.11 since rank $B_2^s < rank B_2^{s-1} = rank B_2^{s-2}$, it follows that rank $A = ks$. Example 3.12 also illustrates Theorem 3.13 since rank $A^{ks} = rank A^{k+1}$ and rank $F^{ks} < rank F^{(ks-1)}F^{(k-1)}$ = rank $F^{ks-1}$ if and only if index $F = index A = ks$; otherwise index $A < ks$.

Acknowledgement. The research of PvdD was supported in part by an NSERC Discovery grant. The authors thank D.D. Olesky for helpful discussions.

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