

## AN EXTENSION OF THE CLASS OF MATRICES ARISING IN THE NUMERICAL SOLUTION OF PDES\*

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**Abstract.** This paper studies block matrices  $A = [A_{ij}] \in C^{km \times km}$ , where every block  $A_{ij} \in C^{k \times k}$  for  $i, j \in \langle m \rangle = \{1, 2, \dots, m\}$  and  $A_{ii}$  is non-Hermitian positive definite for all  $i \in \langle m \rangle$ . Such a matrix is called an extended  $H$ -matrix if its block comparison matrix is a generalized  $M$ -matrix. Matrices of this type are an extension of generalized  $M$ -matrices proposed by Elsner and Mehrmann [L. Elsner and V. Mehrmann. Convergence of block iterative methods for linear systems arising in the numerical solution of Euler equations. *Numer. Math.*, 59:541–559, 1991.] and generalized  $H$ -matrices by Nabben [R. Nabben. On a class of matrices which arise in the numerical solution of Euler equations. *Numer. Math.*, 63:411–431, 1992.]. This paper also discusses some properties including positive definiteness and invariance under block Gaussian elimination of a subclass of extended  $H$ -matrices, especially, convergence of some block iterative methods for linear systems with such a subclass of extended  $H$ -matrices. Furthermore, the incomplete  $LDU$ -factorization of these matrices is investigated and applied to establish some convergent results on some iterative methods. Finally, this paper generalizes theory on generalized  $H$ -matrices and answers the open problem proposed by R. Nabben.

**Key words.** Extended  $H$ -matrices, Generalized  $M$ -matrices, Generalized  $H$ -matrices.

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**1. Introduction.** Elsner and Mehrmann in [5, 6] proposed a generalization of  $Z$ -matrices. They call a block matrix  $A = [A_{ij}] \in C^{km \times km}$  a generalized  $Z$ -matrix if the blocks  $A_{ij} \in C^{k \times k}$  are Hermitian and the off-diagonal block matrices  $A_{ij}$ ,  $i \neq j$  are negative semidefinite. This class of matrices is denoted by  $Z_m^k$ . They also propose a generalization of  $M$ -matrices, i.e., a block matrix  $A = [A_{ij}] \in Z_m^k$  is called a generalized  $M$ -matrix if there exists a positive vector  $u = (u_1, u_2, \dots, u_m)^T$  such

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that the matrix  $\sum_{j=1}^m u_j A_{ij}$  is positive definite for all  $i \in \langle m \rangle = \{1, 2, \dots, m\}$ . The class of generalized  $M$ -matrices is denoted by  $M_m^k$ . In [5], the properties of this class of matrices were discussed, especially, the convergence of some block iterative methods for generalized  $M$ -matrices was proved. These classes of matrices arise not only in the numerical solution of 2D and 3D Euler equations in fluid dynamics [2, 9] and in the study of invariant of dynamical systems [14] but also in the discretizations of a class of PDEs associated to invariant tori [3, 4].

An extension for generalized  $M$ -matrices was presented by Nabben in [14, 15, 16] and Huang et al. in [13]. Let  $D_m^k := \{A = [A_{ij}] \in C^{km \times km} \mid A_{ij} \in C^{k \times k} \text{ is Hermitian for } i, j \in \langle m \rangle \text{ and } A_{ii} \text{ is positive definite for all } i \in \langle m \rangle\}$ . A block matrix  $A = [A_{ij}] \in D_m^k$  is called a generalized  $H$ -matrix if there exists a positive vector  $u = (u_1, u_2, \dots, u_m)^T$  such that the matrix  $u_i |A_{ii}| - \sum_{j=1, j \neq i}^m u_j |A_{ij}|$  is positive definite for all  $i \in \langle m \rangle$ , where  $|A_{ij}| := (A_{ij}^H A_{ij})^{\frac{1}{2}}$ . Furthermore, Nabben [14] gave some significant results for this class of matrices, such as the convergence of the associated block Gauss-Seidel method, the incomplete block  $LDU$ -factorization, the invariance under Gaussian elimination and an equivalence theorem for a subclass of generalized  $H$ -matrices. Recently, Huang et al. [13] presented some new and interesting equivalent conditions for generalized  $H$ -matrices and gave an improvement on Definition 5.1 in [14]. Zhang et al. in [19] study the convergence of some block iterative methods including block Jacobi method, block Gauss-Seidel methods, block JOR-method, the block SOR-method and the block AOR-method for the solution of linear systems when the coefficient matrices are generalized  $H$ -matrices.

However, these classes of matrices, such as generalized  $Z$ -matrices, generalized  $M$ -matrices and generalized  $H$ -matrices, are some very special classes of matrices with very strict conditions. For example, the off-diagonal block entries of this class of matrices need to be Hermitian and the diagonal blocks need to be (Hermitian) positive definite. But, for a general matrix, the results about these classes of matrices can not hold (see [2, 8, 10]). As was proposed by R. Nabben [14], *it is an open problem if this construction can be generalized to the class of matrices with non-Hermitian off-diagonal blocks, and if similar results can be proved for such matrices.*

The purpose of this paper is to give a further extension of generalized  $H$ -matrices and to propose a class of extended  $H$ -matrices ( $EH$ -matrices) with non-Hermitian off-diagonal blocks and non-Hermitian positive definite diagonal blocks, and furthermore, to discuss some properties of a subclass of  $H$ -matrices including positive definiteness, invariance under block Gaussian elimination, especially, convergence of some block iterative methods for linear systems with such a class of matrices. Lastly, this paper also investigates the incomplete  $LDU$ -factorization of these matrices. Hence, this paper answers the open problem of R. Nabben.

The paper is organized as follows. After introducing some notations and preliminary results about generalized  $H$ -matrices and extended  $H$ -matrices in Section 2, we discuss in Section 3 some general properties of the subclass of  $EH_m^k$ . We establish that the class of block matrices  $A \in \Omega_m^k$  satisfying  $\eta(A) + \eta(A^H) \in M_m^k$  is non-Hermitian positive definite. In Section 4, we show that the subclass of  $EH_m^k$  is invariant under block Gaussian elimination. And in Section 5, we study some iterative methods, particularly, the block Jacbi method, block Gauss-Seidel method, block JOR-method, block SOR-method and block AOR-method as well. In the rest of Section 5, we investigate the incomplete  $LDU$ -factorization for a subclass of  $EH$ -matrices, which is applied to establish some results on convergent iterative methods. Conclusions are given in Section 6.

**2. Preliminaries.** Let  $C^{n \times n}$  ( $R^{n \times n}$ ) be the set of all  $n \times n$  complex (real) matrices. We denote by  $C^n$  the set of all  $n$ -dimensional complex vectors;  $R_+^n$  the set of positive vectors in  $R^n$ ;  $A^T$  the transpose of  $A$ ;  $A^H$  the conjugate transpose of  $A$ ;  $\rho(A)$  the spectral radius of  $A$ ;  $Re(z)$  the real part of the complex number  $z$ .

DEFINITION 2.1.

1. A matrix  $A \in C^{n \times n}$  is called Hermitian if  $A^H = A$ , skew-Hermitian if  $A^H = -A$ .
2. A Hermitian matrix  $A \in C^{n \times n}$  is called Hermitian positive (negative) definite if  $x^H A x > 0$  ( $x^H A x < 0$ ) for all nonzero  $x \in C^n$  and Hermitian positive (negative) semidefinite if  $x^H A x \geq 0$  ( $x^H A x \leq 0$ ) for all  $x \in C^n$ .
3. A matrix  $A \in C^{n \times n}$  is called positive (negative) definite if  $Re(x^H A x) > 0$  ( $Re(x^H A x) < 0$ ) for all nonzero  $x \in C^n$  and positive (negative) semidefinite if  $Re(x^H A x) \geq 0$  ( $Re(x^H A x) \leq 0$ ) for all  $x \in C^n$ .

Let  $A > 0$  and  $A \geq 0$  denote  $A$  being (Hermitian) positive definite and (Hermitian) positive semidefinite. Analogously we write  $A < 0$  if  $-A > 0$  and  $A \leq 0$  if  $-A \geq 0$ . Furthermore, for  $A, B \in C^{n \times n}$ , we write  $A > B$ ,  $A \geq B$ ,  $A < B$  and  $A \leq B$  if  $A - B > 0$ ,  $A - B \geq 0$ ,  $A - B < 0$  and  $A - B \leq 0$ .

DEFINITION 2.2. (See [12, 13, 14])

1. Let  $A = (a_{ij}) \in C^{n \times n}$  be given. Then there exist two unitary matrices  $P \in C^{n \times n}$  and  $Q \in C^{n \times n}$  such that  $A = P \Sigma Q^H$ , where  $\Sigma = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_n) \in R^{n \times n}$  with  $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n \geq 0$ .
2. Let  $A = (a_{ij}) \in C^{n \times n}$  be Hermitian positive semidefinite. Then there exists a unitary matrix  $U$  such that  $A = U \Lambda U^H$ , where  $\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) \in R^{n \times n}$ . We define  $\sqrt{A} := U \sqrt{\Lambda} U^H$ , where  $\sqrt{\Lambda} = \text{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \dots, \sqrt{\lambda_n})$ .

3. Let  $A = (a_{ij}) \in C^{n \times n}$  be given. Then we define

$$(2.1) \quad \begin{aligned} |A| &:= \sqrt{A^H A} = Q \Sigma Q^H \in C^{n \times n} \text{ and} \\ |A^H| &:= \sqrt{A A^H} = P \Sigma P^H \in C^{n \times n}, \end{aligned}$$

where  $P$ ,  $Q$  and  $\Sigma$  are defined as 1. It follows from (2.1) that  $|A^H| = |A| = \sqrt{A A^H}$  if  $A$  is normal. In particular, we have  $|A| = A$  if  $A$  is Hermitian positive semidefinite.

LEMMA 2.3. (See [11]) *Let  $A \in C^{n \times n}$ . Then  $A$  is positive definite if and only  $A^H + A$  is Hermitian positive definite.*

DEFINITION 2.4. (See [1, 18]) Let  $A = (a_{ij}) \in R^{n \times n}$ . Then

1.  $A$  is called a  $Z$ -matrix if  $a_{ij} \leq 0$  for  $i \neq j$ ;  $i, j = 1, 2, \dots, n$ ;
2.  $A$  is called an  $M$ -matrix if  $A$  is a  $Z$ -matrix,  $A^{-1} = (\hat{a}_{ij})$  exists and  $\hat{a}_{ij} \geq 0$  for all  $i, j = 1, 2, \dots, n$ ;
3.  $A \in C^{n \times n}$  is called an  $H$ -matrix if  $\mu(A) = (\mu_{ij})$ , where

$$\mu_{ij} = \begin{cases} |a_{ii}|, & \text{if } i = j, \\ -|a_{ij}|, & \text{if } i \neq j, \end{cases}$$

is an  $M$ -matrix.

We denote the class of  $n \times n$   $M$ -matrices and the class of  $n \times n$   $H$ -matrices by  $M_n$  and  $H_n$ , respectively.

DEFINITION 2.5. We set  $P(m) = \{(i, j) \mid i, j \in \langle m \rangle, i \neq j\}$ . For a subset  $E \subseteq P(m)$  and a matrix  $A \in C^{km \times km}$ , we define a block decomposition  $A = M_E - N_E$  of  $A$  by the following block matrices  $M_E = [M_{ij}]$  with

$$(2.2) \quad M_{ij} = \begin{cases} A_{ij}, & \text{if } (i, j) \in E \text{ or } i = j, \\ 0, & \text{otherwise,} \end{cases}$$

and  $N_E = [N_{ij}]$  with

$$(2.3) \quad N_{ij} = \begin{cases} 0, & \text{if } (i, j) \in E, \\ -A_{ij}, & \text{otherwise.} \end{cases}$$

For  $A \in C^{km \times km}$ , we use the standard block decomposition  $A = D - L - U$  with

$$(2.4) \quad \begin{aligned} D &= \begin{bmatrix} A_{11} & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & A_{s-1,s-1} & 0 \\ 0 & \cdots & 0 & A_{ss} \end{bmatrix}, \\ -L &= \begin{bmatrix} 0 & \cdots & 0 & 0 \\ A_{21} & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ A_{s1} & \cdots & A_{s-1,2} & 0 \end{bmatrix}, \quad -U = \begin{bmatrix} 0 & A_{12} & \cdots & A_{1s} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & A_{s-1,s} \\ 0 & 0 & \cdots & 0 \end{bmatrix}. \end{aligned}$$

The following notation and definitions for block matrices was introduced by Elsner and Mehrmann [5, 6] and Nabben [14].

DEFINITION 2.6. Let  $A = [A_{ij}] \in C^{km \times km}$  with  $A_{ij} \in C^{k \times k}$ . Then, we define the block graph  $G_A$  of  $A$  as the nondirected graph of vertices  $1, 2, \dots, m$  and edges  $\{i, j\}$ ,  $i \neq j$ , where  $\{i, j\}$  is an edge of  $G_A$  if  $A_{ij} \neq 0$  or  $A_{ji} \neq 0$ . By  $E(G_A)$  we denote the edge set of  $G_A$ .  $A$  is called block acyclic if  $G_A$  is a forest, i.e.,  $G_A$  is either a tree or a collection of trees. A vertex of  $G_A$  that has less than two neighbors is called a leaf.

DEFINITION 2.7.

1.  $Z_m^k = \{A = [A_{ij}] \in C^{km \times km} \mid A_{ij} \in C^{k \times k} \text{ is Hermitian for all } i, j \in \langle m \rangle \text{ and } A_{ii} \leq 0 \text{ for all } i \in \langle m \rangle\}$ .
2.  $\hat{Z}_m^k = \{A = [A_{ij}] \in Z_m^k \mid A_{ii} > 0, i \in \langle m \rangle\}$ .
3.  $M_m^k = \{A \in \hat{Z}_m^k \mid \text{there exists } u \in R_+^m \text{ such that } \sum_{j=1}^m u_j A_{ij} > 0 \text{ for all } i \in \langle m \rangle\}$ , where  $R_+^m$  denotes all positive vectors in  $R$ .
4.  $D_m^k = \{A = [A_{ij}] \in C^{km \times km} \mid A_{ij} \in C^{k \times k} \text{ is Hermitian for all } i, j \in \langle m \rangle \text{ and } A_{ii} > 0 \text{ for all } i \in \langle m \rangle\}$ .
5.  $H_m^k = \{A \in D_m^k \mid \mu(A) \in M_m^k\}$ , where  $\mu(A) = [M_{ij}] \in C^{mk \times mk}$  is defined as

$$M_{ij} := \begin{cases} |A_{ii}|, & \text{if } i = j, \\ -|A_{ij}|, & \text{if } i \neq j. \end{cases}$$

We now present a further extension of definitions such as generalized  $H$ -matrices.

DEFINITION 2.8.

1.  $\Omega_m^k = \{A = [A_{ij}] \in C^{km \times km} \mid A_{ij} \in C^{k \times k} \text{ for all } i, j \in \langle m \rangle \text{ and } A_{ii} \text{ is non-Hermitian positive definite for all } i \in \langle m \rangle\}$ .

2. For  $A \in \Omega_m^k$ , we denote by  $\eta(A) = [\eta_{ij}] \in C^{mk \times mk}$  the block comparison matrix of  $A$ , which we define as

$$\eta_{ij} := \begin{cases} H(A_{ii}), & \text{if } i = j, \\ -|A_{ij}|, & \text{if } i \neq j, \end{cases}$$

where

$$H(A_{ii}) := \frac{1}{2}(A_{ii} + A_{ii}^H),$$

the Hermitian part of the matrix  $A_{ii}$  for all  $i \in \langle m \rangle$ .

3.  $EH_m^k = \{A \in \Omega_m^k \mid \eta(A) \in M_m^k\}$ . Moreover, a block matrix  $A$  is called an  $EH$ -matrix if  $A \in EH_m^k$ .

According to Definition 2.7 and Definition 2.8,  $D_m^k \subset \Omega_m^k$ , and consequently,  $H_m^k \subset EH_m^k$ .

**3. Positive definiteness.** In this section, some results on positive definiteness for the matrices in  $\Omega_m^k$  are presented to generalize the results of [14]. The following lemma will be used in this section.

LEMMA 3.1. Let  $A \in C^{n \times n}$  and  $0 \leq \alpha \leq 1$ . Then, the matrix  $\tilde{A}_\alpha(t)$  defined by

$$\tilde{A}_\alpha(t) = \begin{bmatrix} |A| & \alpha e^{-it} A^H \\ \alpha e^{it} A & |A^H| \end{bmatrix}$$

is positive semidefinite for all  $t \in R$ .

*Proof.* It follows from Definition 2.2 that there exist two unitary matrices  $P \in C^{n \times n}$  and  $Q \in C^{n \times n}$  such that  $A = P\Sigma Q^H$ , where  $\Sigma = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_n) \in R^{n \times n}$  with  $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n \geq 0$ , and (2.1) holds. Then it is easy to see that

$$\begin{aligned} \tilde{A}_\alpha(t) &= \begin{bmatrix} |A| & \alpha e^{-it} A^H \\ \alpha e^{it} A & |A^H| \end{bmatrix} = \begin{bmatrix} Q & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} \Sigma & \alpha e^{-it} \Sigma \\ \alpha e^{it} \Sigma & \Sigma \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & P \end{bmatrix}^H \\ &= \mathcal{C} \begin{bmatrix} \Sigma & \alpha e^{-it} \Sigma \\ \alpha e^{it} \Sigma & \Sigma \end{bmatrix} \mathcal{C}^H, \end{aligned} \quad (3.1)$$

where  $\mathcal{C} = \begin{bmatrix} Q & 0 \\ 0 & P \end{bmatrix}$  is nonsingular since  $Q$  and  $P$  are both unitary. Then we have with (3.1) that

$$\mathcal{C}^{-1} \tilde{A}_\alpha(t) (\mathcal{C}^{-1})^H = \begin{bmatrix} \Sigma & \alpha e^{-it} \Sigma \\ \alpha e^{it} \Sigma & \Sigma \end{bmatrix}.$$

Let  $X = \begin{bmatrix} I & 0 \\ -\alpha e^{it}\Sigma\Sigma^+ & I \end{bmatrix}$ , where  $I$  is the  $n \times n$  identity matrix and  $\Sigma^+$  is the Moore-Penrose generalized inverse of the matrix  $\Sigma$ . Since

$$\begin{aligned} -\alpha e^{it}\Sigma\Sigma^+\Sigma + \alpha e^{it}\Sigma &= -\alpha e^{it}P\Sigma + \alpha e^{it}\Sigma = 0, \\ -\alpha e^{-it}\Sigma\Sigma^+\Sigma + \alpha e^{-it}\Sigma &= (-\alpha e^{it}\Sigma\Sigma^+\Sigma + \alpha e^{it}\Sigma)^H = 0 \quad \text{and} \\ \Sigma - \alpha^2\Sigma\Sigma^+\Sigma &= \Sigma - \alpha^2\Sigma \geq \Sigma - \Sigma = 0, \end{aligned}$$

one has

$$\begin{aligned} (3.2) \quad X\mathcal{C}^{-1}\tilde{A}_\alpha(t)(\mathcal{C}^{-1})^HX^H &= \begin{bmatrix} \Sigma & -\Sigma\Sigma^+\Sigma + \Sigma \\ -\Sigma\Sigma^+\Sigma + \Sigma & \Sigma - \Sigma\Sigma^+\Sigma \end{bmatrix} \\ &\geq \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Let  $U = X\mathcal{C}^{-1}$ . Since  $\Sigma \geq 0$ , it follows from (3.2) that

$$U\tilde{A}_\alpha(t)U^H = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \geq 0,$$

and consequently,  $\tilde{A}_\alpha(t) \geq 0$ .  $\square$

**THEOREM 3.2.** Let a block matrix  $A = [A_{ij}] \in \Omega_m^k$ .

1. If there exists a vector  $v = (v_1, v_2, \dots, v_m)^T \in R_+^m$  such that

$$(3.3) \quad v_i(A_{ii} + A_{ii}^H) - \sum_{j=1, j \neq i}^m v_j(|A_{ij}| + |A_{ji}^H|) \geq 0$$

for all  $i \in \langle m \rangle$ , then  $A \geq 0$ .

2. If  $\eta(A) + \eta(A^H) \in M_m^k$ , then  $A > 0$ .

*Proof.* We define  $V = \text{diag}(v_1 I_k, v_2 I_k, \dots, v_m I_k)$ , where  $I_k$  is  $m \times m$  identity matrix. Multiplying the inequality (3.3) by  $v_i$ , we have

$$v_i^2(A_{ii} + A_{ii}^H) - \sum_{j=1, j \neq i}^m v_i(|A_{ij}| + |A_{ji}^H|)v_j \geq 0, \quad i = 1, 2, \dots, m,$$

which shows that  $VAV$  satisfies

$$(3.4) \quad v_i^2(A_{ii} + A_{ii}^H) - \sum_{j=1, j \neq i}^m (|v_i A_{ij} v_j| + |(v_j A_{ji} v_i)^H|) \geq 0$$

for all  $i \in \langle m \rangle$ . Let  $B = VAV = [B_{ij}]$ , where  $B_{ij} = v_i A_{ij} v_j$  for all  $i, j \in \langle m \rangle$ . Since  $v_i$  is a positive real number for each  $i \in \langle m \rangle$ ,  $B \in \Omega_m^k$ . It follows from (3.4) that the matrix  $B$  satisfies

$$(3.5) \quad R_i(B) = (B_{ii} + B_{ii}^H) - \sum_{j=1, j \neq i}^m (|B_{ij}| + |B_{ji}|) \geq 0, \quad i = 1, \dots, m.$$

As a result,

$$B + B^H = \Delta + \sum_{i>j} R_{ij} + \sum_{i<j} S_{ij},$$

where

$$(3.6) \quad R_{ij} = \begin{bmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & |B_{ij}| & 0 & \cdots & B_{ij}^H & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & B_{ij} & 0 & \cdots & |B_{ij}^H| & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}$$

and

$$(3.7) \quad S_{ij} = \begin{bmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & |B_{ij}^H| & 0 & \cdots & B_{ij} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & B_{ij}^H & 0 & \cdots & |B_{ij}| & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Since (3.5) implies  $\Delta \geq 0$  and Lemma 3.1 gives  $R_{ij} \geq 0$  and  $S_{ij} \geq 0$ ,  $B + B^H \geq 0$  and consequently  $A = V^{-1}BV^{-1} \geq 0$ . This completes the proof of 1. Using the same method as one of the proof of 1, we can prove the conclusion of 2.  $\square$

**THEOREM 3.3.** Let  $A = [A_{ij}] \in \Omega_m^k$ . For  $E \subseteq P(m)$  and  $t \in R$ , define

$$(3.8) \quad A_t = M_E + M_E^H - (e^{it}N_E + e^{-it}N_E^H),$$

where  $A = M_E - N_E$  and  $M_E, N_E$  as in (2.2) and (2.3). If  $\eta(A) + \eta(A^H) \in M_m^k$ , then  $A_t > 0$  for all  $t \in R$ .



*Proof.* Since  $\eta(A) + \eta(A^H) \in M_m^k$ , there exists a vector  $v = (v_1, v_2, \dots, v_m)^T \in R_+^m$  such that

$$(3.9) \quad v_i(A_{ii} + A_{ii}^H) - \sum_{j=1, j \neq i}^m v_j(|A_{ij}| + |A_{ji}^H|) > 0$$

for all  $i \in \langle m \rangle$ . Similar to the proof of Theorem 3.2, multiply the inequality (3.9) by  $v_i$  and define  $V = \text{diag}(v_1 I_k, v_2 I_k, \dots, v_m I_k)$ , where  $I_k$  is  $m \times m$  identity matrix, such that  $B = VAV$  satisfies

$$(3.10) \quad v_i^2(A_{ii} + A_{ii}^H) - \sum_{j=1, j \neq i}^m (|v_i A_{ij} v_j| + |(v_j A_{ji} v_i)^H|) > 0$$

for all  $i \in \langle m \rangle$ . Let  $B = VAV = [B_{ij}]$  with  $B_{ij} = v_i A_{ij} v_j$  for all  $i, j \in \langle m \rangle$ . Then following (3.10), we have

$$(3.11) \quad R_i(B) = (B_{ii} + B_{ii}^H) - \sum_{j=1, j \neq i}^m (|B_{ij}| + |B_{ji}|) > 0, \quad i = 1, \dots, m.$$

Furthermore, according to (3.8), we have

$$B_t = V A_t V = V [M_E + M_E^H - (e^{it} N_E + e^{-it} N_E^H)] V$$

and

$$B_t = \Delta + \sum_{\substack{(i,j) \in E \\ i > j}} R_{ij} + \sum_{\substack{(i,j) \in E \\ i < j}} S_{ij} + \sum_{\substack{(i,j) \notin E \\ i > j}} \tilde{R}_{ij} + \sum_{\substack{(i,j) \notin E \\ i < j}} \tilde{S}_{ij},$$

where  $\Delta$ ,  $R_{ij}$  and  $S_{ij}$  are defined in (3.6) and (3.7), respectively, and

$$\tilde{R}_{ij} = \begin{bmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & |B_{ij}| & 0 & \cdots & e^{-i\phi} B_{ij}^H & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & e^{i\phi} B_{ij} & 0 & \cdots & |B_{ij}^H| & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$\tilde{S}_{ij} = \begin{bmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & |B_{ij}^H| & 0 & \cdots & e^{i\varphi} B_{ij} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & e^{-i\varphi} B_{ij}^H & 0 & \cdots & |B_{ij}| & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Since (3.11) yields  $\Delta > 0$  and Lemma 3.1 indicates  $R_{ij} \geq 0$ ,  $S_{ij} \geq 0$ ,  $\tilde{R}_{ij} \geq 0$  and  $\tilde{S}_{ij} \geq 0$ ,  $B_t > 0$  and hence  $A_t > 0$ . This completes the proof.  $\square$

COROLLARY 3.4. For every  $A = [A_{ij}] \in \Omega_m^k$  with  $\eta(A) + \eta(A^H) \in M_m^k$  and every  $E \subseteq P(m)$ , we have

1.  $M_E + N_E > 0$ . In particular,  $A > 0$ ,  $2D - A > 0$ ,  $D - L + U > 0$ ,  $D - U + L > 0$ , where  $D$ ,  $L$  and  $U$  as in (2.4).
2.  $M_E > 0$ .

*Proof.* Using Theorem 3.3, we can obtain the proof of this corollary.  $\square$

THEOREM 3.5. Let  $A = [A_{ij}] \in \Omega_m^k$  with  $\eta(A) + \eta(A^H) \in M_m^k$ . Define

$$\hat{A}(\phi, \varphi) = D + D^H - \alpha(e^{i\phi} L + e^{-i\phi} L^H) - \beta(e^{i\varphi} U + e^{-i\varphi} U^H),$$

where  $A = D - L - U$ ,  $D$ ,  $L$  and  $U$  as in (2.4). If  $0 < \alpha \leq 1$  and  $0 < \beta \leq 1$ , then  $\hat{A}(\phi, \varphi) > 0$ .

*Proof.* The proof of this theorem follows the proof of Theorem 3.3.  $\eta(A) + \eta(A^H) \in M_m^k$  implies that there exists a vector  $v = (v_1, v_2, \dots, v_m)^T \in R_+^m$  such that (3.9) holds for all  $i \in \langle m \rangle$ . Multiply the inequality (3.9) by  $v_i$  and define  $V = \text{diag}(v_1 I_k, \dots, v_m I_k)$ , where  $I_k$  is  $m \times m$  identity matrix, such that  $B = VAV$  satisfies (3.10). Let  $B = VAV = [B_{ij}]$  with  $B_{ij} = v_i A_{ij} v_j$  for all  $i, j \in \langle m \rangle$ . Then, (3.10) yields (3.11). Since  $A = D - L - U$ , we have  $B = D_B - L_B - U_B = VDV - VLE - VUV$ . As a result,

$$\begin{aligned} \hat{B}(\phi, \varphi) &= D_B + D_B^H - \alpha(e^{i\phi} L_B + e^{-i\phi} L_B^H) - \beta(e^{i\varphi} U_B + e^{-i\varphi} U_B^H) \\ &= V\hat{A}(\phi, \varphi)V \end{aligned}$$

and we have

$$(3.12) \quad \hat{B}(\phi, \varphi) = \Delta + \sum_{i>j} \hat{R}_{ij} + \sum_{i<j} \hat{S}_{ij},$$

where  $\Delta$  is defined in (3.6) and

$$\hat{R}_{ij} = \begin{bmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & |B_{ij}| & 0 & \cdots & \alpha e^{-i\phi} B_{ij}^H & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \alpha e^{i\phi} B_{ij} & 0 & \cdots & |B_{ij}^H| & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$\hat{S}_{ij} = \begin{bmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & |B_{ij}^H| & 0 & \cdots & \beta e^{i\phi} B_{ij} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \beta e^{-i\phi} B_{ij}^H & 0 & \cdots & |B_{ij}| & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

(3.11) yields  $\Delta > 0$  and Lemma 3.1 indicates  $\hat{R}_{ij} \geq 0$  and  $\hat{S}_{ij} \geq 0$ . As a result,  $\hat{B}(\phi, \varphi) > 0$  follows directly from (3.12) and consequently  $\hat{A}(\phi, \varphi) > 0$ .  $\square$

**4. Invariance under block Gaussian elimination.** Similarly, the results proposed by Elsner and Mehrmann [5] and Nabben [14] can be generalized to show that a class of acyclic matrices in  $EH_m^k$  are invariant under block Gaussian elimination.

LEMMA 4.1. (See [14]) An  $(m+n) \times (m+n)$  matrix  $P$  is partitioned as

$$P = \begin{bmatrix} A & D^H \\ B & C \end{bmatrix},$$

where  $A \in C^{m \times m}$ ,  $C \in C^{n \times n}$  and  $B, D \in C^{m \times n}$ . If  $P$  is positive definite, then the Schur complement with respect to  $A$ ,  $P/A = C - BA^{-1}D^H$  is positive definite.

THEOREM 4.2. Let  $A = [A_{ij}] \in \Omega_m^k$  be block acyclic with  $\eta(A) + \eta(A^H) \in M_m^k$ . Assume that  $\{s, l\}$  is an edge of  $G_A$ , where  $s$  is a vertex that has only one neighbor,

and let  $L = [L_{ij}]$ ,  $U = [U_{ij}] \in C^{km \times km}$  be defined by

$$L_{ij} := \begin{cases} I, & \text{if } i = j, \\ -A_{ls}A_{ss}^{-1}, & \text{if } i = l, j = s, \\ 0, & \text{otherwise,} \end{cases}$$

$$U_{ij} := \begin{cases} I, & \text{if } i = j, \\ -A_{ss}^{-1}A_{sl}, & \text{if } i = l, j = s, \\ 0, & \text{otherwise.} \end{cases}$$

Then,

$$\tilde{A} = [\tilde{A}_{ij}] = LAU \in \Omega_m^k, \quad \eta(\tilde{A}) + \eta(\tilde{A}^H) \in M_m^k$$

and  $\tilde{A}$  is block acyclic.

*Proof.* The proof of this theorem follows the proofs of Theorem 3.24 in [5] and Theorem 3.6 in [14]. Multiplication with  $L$  from the left changes only elements in row  $l$  and multiplication with  $U$  from the right changes only elements in column  $l$ . Thus,

$$\tilde{A}_{ij} := \begin{cases} A_{ij}, & \text{if } i, j \neq l, \\ A_{ij} - A_{is}A_{ss}^{-1}A_{sl}, & \text{if } i = l \text{ or } j = l. \end{cases}$$

Now suppose that  $\{l, j\}$  is an edge of  $G_A$  for  $j \neq l, s$ . Then  $\{j, s\}$  is not an edge of  $G_A$ , since otherwise  $\{l, j\}, \{s, j\}, \{l, s\}$  would be a cycle of  $G_A$ . Thus, the only blocks in  $\tilde{A}$  which are different from the corresponding blocks in  $A$  are

$$(4.1) \quad \tilde{A}_{ls} = 0, \quad \tilde{A}_{sl} = 0, \quad \tilde{A}_{ll} = A_{ll} - A_{ls}A_{ss}^{-1}A_{sl}.$$

Again,  $\tilde{A}_{ll} = A_{ll} - A_{ls}A_{ss}^{-1}A_{sl} = [A(s, l)]/A_{ss}$  is the Schur complement of the matrix  $A(s, l) = \begin{bmatrix} A_{ss} & A_{sl} \\ A_{ls} & A_{ll} \end{bmatrix} \in \Omega_2^k$  with respect to the non-Hermitian positive definite matrix  $A_{ss}$ . Since  $\eta(A) + \eta(A^H) \in M_m^k$ , Theorem 3.2 shows  $A > 0$  and hence  $A(s, l) > 0$  for  $A(s, l)$  is a block principal submatrix of  $A$ . Therefore, Lemma 4.1 gives that  $\tilde{A}_{ll} = [A(s, l)]/A_{ss} > 0$ . Obviously,  $\tilde{A}$  is block acyclic and  $\tilde{A} = [\tilde{A}_{ij}] = LAU \in \Omega_m^k$ . It remains to show that  $\eta(\tilde{A}) + \eta(\tilde{A}^H) \in M_m^k$ . So we have to show that there is a positive vector  $v = (v_1, \dots, v_m)^T \in R^m$  such that

$$v_i(\tilde{A}_{ii} + \tilde{A}_{ii}^H) - \sum_{j=1, j \neq i}^m v_j(|\tilde{A}_{ij}| + |\tilde{A}_{ji}^H|) > 0$$

for all  $i \in \langle m \rangle$ . Since  $\eta(A) + \eta(A^H) \in M_m^k$ , there exists a vector  $u = (u_1, \dots, u_m)^T \in R^m$  such that

$$(4.2) \quad u_i(A_{ii} + A_{ii}^H) - \sum_{j=1, j \neq i}^m u_j(|A_{ij}| + |A_{ji}^H|) > 0$$

for all  $i \in \langle m \rangle$ . Setting  $v = u$  we have for  $i \neq l, s$ ,

$$v_i(\tilde{A}_{ii} + \tilde{A}_{ii}^H) - \sum_{j=1, j \neq i}^m v_j(|\tilde{A}_{ij}| + |\tilde{A}_{ji}^H|) > 0,$$

and for  $i = s$ , it follows from (4.1) that

$$v_s(\tilde{A}_{ii} + \tilde{A}_{ii}^H) - \sum_{j=1, j \neq s}^m v_j(|\tilde{A}_{sj}| + |\tilde{A}_{js}^H|) = v_s(\tilde{A}_{ii} + \tilde{A}_{ii}^H) > 0.$$

For  $i = l$ , from (4.1) we have

$$\begin{aligned} & v_l(\tilde{A}_{ii} + \tilde{A}_{ii}^H) - \sum_{j=1, j \neq l}^m v_j(|\tilde{A}_{lj}| + |\tilde{A}_{jl}^H|) \\ &= v_l(\tilde{A}_{ii} + \tilde{A}_{ii}^H) - \sum_{j=1, j \neq l, s}^m v_j(|\tilde{A}_{lj}| + |\tilde{A}_{jl}^H|) \\ (4.3) \quad &= v_l(\tilde{A}_{ii} + \tilde{A}_{ii}^H) + (|A_{ls}| + |A_{sl}^H|) - \sum_{j=1, j \neq l}^m v_j(|\tilde{A}_{lj}| + |\tilde{A}_{jl}^H|) \\ &= v_l(A_{ii} + A_{ii}^H) - \sum_{j=1, j \neq l}^m v_j(|A_{lj}| + |A_{jl}^H|) \\ &\quad + v_s(|A_{ls}| + |A_{sl}^H|) - v_l[A_{ls}A_{ss}^{-1}A_{sl} + (A_{ls}A_{ss}^{-1}A_{sl})^H]. \end{aligned}$$

Since  $\eta(A) + \eta(A^H) \in M_m^k$ , from (4.2), setting  $v_i = u_i$  for all  $i \in \langle m \rangle$ , we have

$$v_l(A_{ii} + A_{ii}^H) - \sum_{j=1, j \neq l}^m v_j(|A_{lj}| + |A_{jl}^H|) > 0.$$

Then, the sum of the last two lines of (4.3) is Hermitian positive definite if

$$(4.4) \quad v_s(|A_{ls}| + |A_{sl}^H|) - v_l[A_{ls}A_{ss}^{-1}A_{sl} + (A_{ls}A_{ss}^{-1}A_{sl})^H]$$

is Hermitian positive semidefinite. Let  $B = \begin{bmatrix} v_s A_{ss} & v_l A_{sl} \\ v_l A_{ls} & \frac{1}{2} v_l (|A_{ls}| + |A_{sl}^H|) \end{bmatrix} \in \Omega_2^k$ .

Then, with  $\eta(A) + \eta(A)^H \in M_m^k$ , the matrix

$$\eta(B) + \eta(B)^H = \begin{bmatrix} v_s(A_{ss} + A_{ss}^H) & v_l(|A_{ls}| + |A_{sl}^H|) \\ v_l(|A_{ls}| + |A_{sl}^H|) & v_l(|A_{ls}| + |A_{sl}^H|) \end{bmatrix}$$

satisfies the condition (3.3) of Theorem 3.2. It then follows from Theorem 3.2 that the matrix  $B \geq 0$ . Hence, the Schur complement of  $B$  with respect to the matrix  $A_{ss}$

$$B/[v_s(A_{ss})] = \frac{1}{2} v_l(|A_{ls}| + |A_{sl}^H|) - \frac{v_l^2}{v_s} A_{ls} A_{ss}^{-1} A_{sl} \geq 0$$

comes from Lemma 4.1. Thus,

$$\frac{1}{2} v_s(|A_{ls}| + |A_{sl}^H|) - A_{ls} A_{ss}^{-1} A_{sl} \geq 0$$

and then

$$v_s(|A_{ls}| + |A_{sl}^H|) - v_l[A_{ls}A_{ss}^{-1}A_{sl} + (A_{ls}A_{ss}^{-1}A_{sl})^H] \geq 0$$

which indicates that (4.4) holds.  $\square$

This theorem shows that the class of block acyclic matrices satisfying  $\eta(A) + \eta(A^H) \in M_m^k$  is invariant under block Gaussian elimination.

**5. Convergence on iterative methods and the incomplete block  $LDU$ -factorization.** Consider the solution methods for the system of  $km$  linear equations

$$(5.1) \quad Ax = b,$$

where  $A = [A_{ij}] \in C^{km \times km}$  is an  $m \times m$  block matrix with all the blocks  $A_{ij} \in C^{k \times k}$ ,  $b, x \in C^{km \times 1}$ . The class of systems arises not only in the numerical solution of 2D and 3D Euler equations in fluid dynamics [2, 9, 14], but also in the discretizations of PDEs associated to invariant tori [3, 4].

In order to solve system (5.1) using block iterative methods, the coefficient matrix  $A = [A_{ij}] \in C^{km \times km}$  is split into

$$(5.2) \quad A = M - N,$$

where  $M \in C^{km \times km}$  is nonsingular and  $N \in C^{km \times km}$ . Then, the general form of block iterative methods for (5.1) can be described as follows:

$$(5.3) \quad x^{(i+1)} = M^{-1}Nx^{(i)} + M^{-1}b, \quad i = 0, 1, 2, \dots$$

The matrix  $H = M^{-1}N$  is called the iterative matrix of the iteration (5.3). It is well-known that (5.3) converges for any given  $x^{(0)}$  if and only if  $\rho(H) < 1$  (see [17]), where  $\rho(H)$  denotes the spectral radius of the matrix  $H$ . Thus, to establish the convergence results of block iterative methods, we study the spectral radius of the iteration matrix in iteration (5.3).

In the following, the splitting and the iteration matrices for some special block iterative methods of (5.1) are listed, respectively.

Let  $\langle m \rangle = \{1, 2, \dots, m\}$  and  $E \subset P(m) = \{(i, j) \mid i, j \in \langle m \rangle, i \neq j\}$ . Consider the  $E$ -block iterative method that is defined by the splitting

$$A = M_E - N_E \quad \text{and} \quad H_E = M_E^{-1}N_E$$

is the iteration matrix, where  $M_E$  and  $N_E$  are shown in (2.2) and (2.3).

In the case of standard block decomposition  $A = D - L - U$ , the block Jacobi method is defined by the splitting (5.2), where

$$M = D \quad N = L + U, \quad \text{and} \quad H_J = D^{-1}(L + U),$$

is the iteration matrix, and the forward and backward block Gauss-Seidel methods are defined by the splitting (5.2), where

$$M = D - L \quad N = U \quad \text{and} \quad M = D - U \quad N = L,$$

respectively, where  $D$ ,  $L$  and  $U$  as in (2.4), and

$$H_{BGS} = (D - U)^{-1}L$$

are the iteration matrices of the forward and backward block Gauss-Seidel methods, respectively. The Jacobi overrelaxation method (JOR-method) is defined by the splitting (5.2), where

$$M = \frac{1}{\omega}D, \quad N = \left[\left(\frac{1}{\omega} - 1\right)D + L + U\right],$$

where  $\omega \in R$  and  $D$ ,  $L$ ,  $U$  as in (2.4), and

$$(5.4) \quad H_{JOR(\omega)} = M^{-1}N = (1 - \omega)I + \omega D^{-1}(L + U)$$

is the iteration matrix. The SOR-method (see [16]) is defined by the splitting (5.2), where

$$M = \frac{1}{\omega}D - L, \quad N = \left[\left(\frac{1}{\omega} - 1\right)D + U\right],$$

and

$$(5.5) \quad H_{SOR(\omega)} = M^{-1}N = (D - \omega L)^{-1}[(1 - \omega)D + \omega U]$$

is the iteration matrix. The AOR-method (see [7]) is defined by the splitting (5.2), where

$$M = \frac{1}{\omega}(D - rL), \quad N = \frac{1}{\omega}[(1 - \omega)D + (\omega - r)L + \omega U],$$

and

$$(5.6) \quad H_{AOR(r, \omega)} = M^{-1}N = (D - rL)^{-1}[(1 - \omega)D + (\omega - r)L + \omega U],$$

is the iteration matrix, where  $r$  and  $\omega$  are the acceleration parameter and the overrelaxation parameter, respectively.

In [5, 14, 19], the convergence results on some block iterative methods including the  $E$ -block iterative method, the block Jacobi method, the forward and backward block Gauss-Seidel methods, the block JOR-method and the block SOR-method are established for the class of generalized  $M$ -matrices and generalized  $H$ -matrices. In this section, we not only establish convergence of these iterative methods for matrices

in  $EH_m^k$ , but also discuss the convergent behavior of the block AOR-method. Furthermore, we will show that the incomplete  $LDU$ -factorization leads to convergent iterative methods.

The following lemma will be used in this section.

LEMMA 5.1. (See [14]) Let  $A, M, N \in C^{n \times n}$  with  $A = M - N$ . If for all  $t \in R$

$$(5.7) \quad A_t := M + M^H - (e^{it}N + e^{-it}N^H) > 0,$$

then  $\rho(M^{-1}N) < 1$ . If  $A_t \geq 0$  for all  $t \in R$ , then  $\rho(M^{-1}N) \leq 1$ .

THEOREM 5.2. Let  $A = [A_{ij}] \in \Omega_m^k$ . For  $E \subseteq P(m)$ , let  $M_E, N_E$  as in (2.2) and (2.3). If  $\eta(A) + \eta(A^H) \in M_m^k$ , then  $\rho(M_E^{-1}N_E) < 1$ . In particular we have:

$$\rho(H_J) < 1, \quad \rho(H_{FGS}) < 1, \quad \rho(H_{BGS}) < 1.$$

*Proof.* Using Theorem 3.3 and Lemma 5.1, the proof of this theorem can be obtained immediately.  $\square$

The following theorems will present the convergence results on the JOR-method and SOR-method for the matrices in  $\Omega_m^k$ .

THEOREM 5.3. Let  $A = [A_{ij}] \in \Omega_m^k$  with  $\eta(A) + \eta(A^H) \in M_m^k$ . Let  $H_{JOR(\omega)}$  and  $H_{SOR(\omega)}$  be as in (5.4) and (5.5). If  $0 < \omega \leq 1$ , then

$$\rho(H_{JOR(\omega)}) < 1, \quad \rho(H_{SOR(\omega)}) < 1.$$

*Proof.* Assume that  $\lambda$  and  $\mu$  are any eigenvalues of  $H_{JOR(\omega)}$  and  $H_{SOR(\omega)}$ , respectively, and  $|\lambda| \geq 1, |\mu| \geq 1$ . For these eigenvalues the relationships below holds

$$(5.8) \quad \det(H_{JOR(\omega)} - \lambda I) = 0 \quad \text{and} \quad \det(H_{SOR(\omega)} - \mu I) = 0$$

or after performing a simple series of transformations

$$(5.9) \quad \det(P) = 0 \quad \text{and} \quad \det(Q) = 0,$$

where

$$(5.10) \quad P = D - \frac{\omega}{\lambda - 1 + \omega}(L + U)$$

and

$$(5.11) \quad Q = D - \frac{\omega\mu}{\mu - 1 + \omega}L - \frac{\omega}{\mu - 1 + \omega}U.$$

Suppose

$$(5.12) \quad \alpha e^{i\phi} = \frac{\omega}{\lambda - 1 + \omega}$$



and

$$(5.13) \quad \beta e^{i\varphi} = \frac{\mu\omega}{\mu - 1 + \omega}, \quad \gamma e^{i\psi} = \frac{\omega}{\mu - 1 + \omega}$$

We will prove

$$\alpha = |\alpha e^{i\phi}| = \frac{|\omega|}{|\lambda - 1 + \omega|} \leq 1$$

and

$$\beta = |\beta e^{i\varphi}| = \frac{|\omega|}{|\lambda - 1 + \omega|} \leq 1, \quad \gamma = |\gamma e^{i\psi}| = \frac{|\omega|}{|\mu - 1 + \omega|} \leq 1.$$

To prove these inequalities it is sufficient and necessary to prove that

$$(5.14) \quad |\lambda - 1 + \omega| \geq |\omega|$$

and

$$(5.15) \quad |\mu - 1 + \omega| \geq |\mu\omega|, \quad |\mu - 1 + \omega| \geq |\omega|.$$

We prove (5.14). If  $\lambda^{-1} = qe^{i\theta}$ , where  $q$  and  $\theta$  are real with  $0 < q \leq 1$ , then the inequality in (5.14) is equivalent to

$$1 + q^2 - 2q(1 - \omega) \cos \theta - 2q^2\omega \geq 0.$$

Since the expression in the brackets above is nonnegative, (5.14) holds for all real  $\theta$  if and only if it holds for  $\cos \theta = 1$ . Thus, we have

$$(1 - q)[(1 - q) + 2q\omega] \geq 0$$

which is true.

Next, let us prove (5.15). Set  $\mu^{-1} = pe^{i\vartheta}$ , where  $p$  and  $\vartheta$  are real with  $0 < p \leq 1$ . Then, the first inequality in (5.15) is equivalent to

$$(5.16) \quad (1 + \omega) + (1 + \omega^2)p^2 - 2p \cos \vartheta - 2q^2\omega \geq 0.$$

Since the expression in the brackets above is nonnegative, (5.16) holds for all real  $\theta$  if and only if it holds for  $\cos \theta = 1$ . Thus, (5.16) is equivalent to

$$(1 - p)[(1 + \omega)(1 - p) + 2p\omega] \geq 0,$$

which is true.

The proof of the second inequality in (5.15) is similar to the proof of the inequality (5.14). Thus,  $0 < \alpha \leq 1$ ,  $0 < \beta \leq 1$  and  $0 < \gamma \leq 1$ . It follows from (5.10) and (5.12) that

$$P = D - \alpha e^{i\phi}(L + U)$$

and it follows from (5.11) and (5.13) that

$$Q = D - \beta e^{i\varphi} L - \gamma e^{i\psi} U.$$

Hence, it follows from Theorem 3.5 that both  $P^H + P > 0$  and  $Q^H + Q > 0$ . So  $P > 0$  and  $Q > 0$  come from lemma 2.3 and consequently, both  $P$  and  $Q$  are nonsingular which contradicts (5.9) and consequently (5.8). Thus,  $\rho(H_{JOR(\omega)}) < 1$ ,  $\rho(H_{SOR(\omega)}) < 1$ .  $\square$

Now, we establish the convergence result of the block AOR-method (See [7]).

**THEOREM 5.4.** *Let  $A = [A_{ij}] \in \Omega_m^k$  with  $\eta(A) + \eta(A^H) \in M_m^k$ . Let  $H_{AOR(r,\omega)}$  be as in (5.6). If  $0 \leq r \leq 1$  and  $0 < \omega \leq 1$ , then  $\rho(H_{AOR(r,\omega)}) < 1$ .*

*Proof.* The conclusion can be proved by contradiction. We assume that there exists an eigenvalue  $\lambda$  of  $H_{AOR(r,\omega)}$  such that  $|\lambda| \geq 1$ . Since

$$H_{AOR(r,\omega)} = (D - rL)^{-1}[(1 - \omega)D + (\omega - r)L + \omega U],$$

we have

$$\begin{aligned} & \det(H_{AOR(r,\omega)} - \lambda I) \\ &= \det \left\{ (D - rL)^{-1}[(1 - \omega)D + (\omega - r)L + \omega U - \lambda(D - rL)] \right\} \\ &= \frac{\det[(\lambda - 1 + \omega)D - (r(\lambda - 1) + \omega)L - \omega U]}{\det(D - rL)} \\ (5.17) \quad &= \frac{\det \left[ D - \frac{r(\lambda - 1) + \omega}{\lambda - 1 + \omega} L - \frac{\omega}{\lambda - 1 + \omega} U \right]}{(\lambda - 1 + \omega) \det(D - rL)} \\ &= 0. \end{aligned}$$

Since  $\det(D - rL) \neq 0$  and  $\lambda - 1 + \omega \neq 0$  for  $|\lambda| \geq 1$  and  $0 < \omega \leq 1$ , we have that

$$(5.18) \quad \det(Q) = 0,$$

where

$$(5.19) \quad Q = D - \frac{r(\lambda - 1) + \omega}{\lambda - 1 + \omega} L - \frac{\omega}{\lambda - 1 + \omega} U.$$

In fact, the coefficients of  $L$  and  $U$  in (5.19) are not more than 1 in modulus. To prove this it is sufficient and necessary to prove that

$$(5.20) \quad |\lambda - 1 + \omega| \geq |r(\lambda - 1) + \omega| \quad \text{and} \quad |\lambda - 1 + \omega| \geq |\omega|.$$

Let  $\lambda^{-1} = qe^{i\theta}$ , where  $q$  and  $\theta$  are real with  $0 < q \leq 1$  since  $|\lambda| \geq 1$ . Then the first inequality in (5.20) is equivalent to

$$\begin{aligned} & |\lambda - 1 + \omega|^2 - |r(\lambda - 1) + \omega|^2 \\ (5.21) \quad &= |\lambda|^2 [1 - (1 - \omega)\lambda^{-1}]^2 - |r - (r - \omega)\lambda^{-1}|^2 \\ &= q^{-2} [1 - (1 - \omega)qe^{i\theta}]^2 - |r - (r - \omega)qe^{i\theta}|^2 \\ &\geq 0. \end{aligned}$$

Since

$$\begin{aligned} |1 - (1 - \omega)qe^{i\theta}|^2 &= [1 - (1 - \omega)q \cos \theta]^2 + [(1 - \omega)q \sin \theta]^2 \\ &= 1 - 2(1 - \omega)q \cos \theta + (1 - \omega)^2 q^2 \end{aligned}$$

and

$$\begin{aligned} |r - (r - \omega)qe^{i\theta}|^2 &= [r - (r - \omega)q \cos \theta]^2 + [(r - \omega)q \sin \theta]^2 \\ &= r^2 - 2r(r - \omega)q \cos \theta + (r - \omega)^2 q^2, \end{aligned}$$

(5.21) is equivalent to

$$\begin{aligned} (5.22) \quad & q^2 \left[ |\lambda - 1 + \omega|^2 - |r(\lambda - 1) + \omega|^2 \right] \\ &= \left[ 1 - 2(1 - \omega)q \cos \theta + (1 - \omega)^2 q^2 \right] - \left[ r^2 - 2r(r - \omega)q \cos \theta + (r - \omega)^2 q^2 \right] \\ &= (1 - r^2) + (1 - r^2)q^2 - (1 - r^2)2q \cos \theta + (1 - r)2q\omega \cos \theta - (1 - r)2q^2\omega \\ &\geq 0. \end{aligned}$$

(5.22) clearly holds for  $r = 1$ . For  $0 \leq r < 1$ , we have  $1 - r > 0$ . Therefore,

$$\begin{aligned} (5.23) \quad & q^2 \left[ |\lambda - 1 + \omega|^2 - |r(\lambda - 1) + \omega|^2 \right] \\ &= (1 - r^2) + (1 - r^2)q^2 - (1 - r^2)2q \cos \theta + (1 - r)2q\omega \cos \theta - (1 - r)2q^2\omega \\ &= (1 - r) \{ (1 + r) + (1 + r)q^2 - [(1 + r) - \omega]2q \cos \theta - 2q^2\omega \} \\ &\geq 0. \end{aligned}$$

(5.23) is equivalent to

$$(5.24) \quad (1 + r) + (1 + r)q^2 - [(1 + r) - \omega]2q \cos \theta - 2q^2\omega \geq 0.$$

Since  $0 \leq r \leq 1$ ,  $0 < \omega \leq 1$  and  $0 < q \leq 1$ ,  $(1 + r) - \omega > 0$  and consequently

$$[(1 + r) - \omega]2q \cos \theta \leq [(1 + r) - \omega]2q.$$

As a result,

$$\begin{aligned} & (1 + r) + (1 + r)q^2 - [(1 + r) - \omega]2q \cos \theta - 2q^2\omega \\ & \geq (1 + r) + (1 + r)q^2 - [(1 + r) - \omega]2q - 2q^2\omega \\ & = (1 + r) + (1 + r)q^2 - (1 + r)2q \\ & = (1 + r)(1 - q)^2 \geq 0, \end{aligned}$$

which shows that (5.24) holds for all real  $\theta$ .

The second inequality in (5.20) is equivalent to

$$(5.25) \quad 1 + q^2 - 2q(1 - \omega) \cos \theta - 2q^2\omega \geq 0.$$

Since  $0 < \omega \leq 1$  and  $0 < q \leq 1$ ,  $2q(1 - \omega) > 0$  and consequently

$$2q(1 - \omega) \cos \theta \leq (1 - \omega)2q.$$

Therefore,

$$\begin{aligned} & 1 + q^2 - 2q(1 - \omega) \cos \theta - 2q^2\omega \\ & \geq 1 + q^2 - 2q(1 - \omega) - 2q^2\omega \\ & = (1 - q)^2 + 2q\omega(1 - q) \geq 0, \end{aligned}$$

which shows that (5.25) holds for all real  $\theta$ . This proves the second inequality in (5.20).

Now, let

$$\alpha e^{i\phi} = \frac{r(\lambda - 1) + \omega}{\lambda - 1 + \omega}, \quad \beta e^{i\varphi} = \frac{\omega}{\lambda - 1 + \omega}.$$

Then it follows from (5.20) that we have

$$\alpha = |\alpha e^{i\phi}| = \frac{|r(\lambda - 1) + \omega|}{|\lambda - 1 + \omega|} \leq 1, \quad \beta = |\beta e^{i\varphi}| = \frac{|\omega|}{|\lambda - 1 + \omega|} \leq 1.$$

Thus, from (5.19), we have

$$Q = D - \alpha e^{i\phi} L - \beta e^{i\varphi} U,$$

where  $0 < \alpha \leq 1$  and  $0 < \beta \leq 1$ . It follows from Theorem 3.5 that  $Q + Q^H > 0$  and Hence,  $Q$  is nonsingular which contradicts (5.18), and consequently (5.17). Therefore,  $\rho(H_{AOR(r, \omega)}) < 1$ .  $\square$

In what follows, we will discuss the incomplete block LDU-factorization (IBLDU-factorization) for the matrices in  $C^{mk \times mk}$ . In a complete block LDU-factorization of a matrix  $A \in C^{mk \times mk}$ , we have

$$A = LDU,$$

where  $D$  is a block diagonal matrix,  $L$  a low block triangular matrix and  $U$  an upper block triangular matrix. In an incomplete block LDU-factorization of a matrix  $A \in C^{mk \times mk}$ , we have the form

$$A = LDU - N,$$

where  $D$  is a block diagonal matrix,  $L$  a low block triangular matrix but some zero block in low triangular part and  $U$  an upper block triangular matrix but some zero block in upper triangular part. In these block matrices  $L$  and  $U$ , zero blocks may occur in arbitrary off-diagonal places, which can be chosen in advance. Let

$$P \subseteq P(m) = \{(i, j) \mid 1 \leq i, j \leq m, i \neq j\},$$

where  $P$  gives positions of zero blocks in matrices  $L$  and  $U$ . Then the matrices  $L$ ,  $U$  and  $N$  in the incomplete block  $LDU$ -factorization have the following structure

$$\begin{cases} L_{ij} = U_{ij} = 0, & \text{if } (i, j) \in P, \\ N_{ij} = 0, & \text{if } (i, j) \notin P. \end{cases}$$

It follows that some results will be presented to determine whether a matrix  $A \in \Omega_m^k$  admits an incomplete block  $LDU$ -factorization and whether the related iterative method converges.

**THEOREM 5.5.** *Let  $A = [A_{ij}] \in \Omega_m^k$  be block acyclic with  $\eta(A) + \eta(A^H) \in M_m^k$ . Then,  $A$  admits an incomplete block  $LDU$ -factorization for all  $P \subseteq P(m)$  and  $\rho((LDU)^{-1}N) < 1$ .*

*Proof.* Consider the construction of the  $LDU$ -factorization for example in [13]. Obviously,  $A = [A_{ij}] \in \Omega_m^k$  with  $\eta(A) + \eta(A^H) \in M_m^k$  remains true even if we delete some off-diagonal blocks of  $A$ . With Theorem 4.2 the matrix  $\tilde{A}$ , which we obtain after one step block Gaussian elimination, is also block acyclic and we have  $\tilde{A} = [\tilde{A}_{ij}] \in \Omega_m^k$  with  $\eta(\tilde{A}) + \eta(\tilde{A}^H) \in M_m^k$ . Thus,  $A$  admits an incomplete block  $LDU$ -factorization, that is,  $A = LDU - N$ . Since  $A = [A_{ij}] \in \Omega_m^k$  with  $\eta(A) + \eta(A^H) \in M_m^k$ , it follows from Theorem 5.2 that  $\rho((LDU)^{-1}N) < 1$ .  $\square$

For a matrix that has a block graph which contains cycles, one step of Gaussian elimination can destroy the structure of the off-diagonal blocks if  $k \neq 1$ . However, for an arbitrary matrix  $A \in \Omega_m^k$  we can choose a subset  $E \subseteq P(m)$  such that  $M_E = [M_{ij}]$  is block acyclic. Here as in (2.2)

$$\begin{cases} M_{ij} = A_{ij}, & \text{if } (i, j) \in E \text{ or } i = j, \\ M_{ij} = 0, & \text{otherwise.} \end{cases}$$

Since  $M_E$  is block acyclic, we have with Theorem 5.3 the factorization

$$(5.26) \quad M_E = LDU,$$

where  $L$  is a strictly block lower triangular,  $U$  is a strictly block upper triangular, and  $D$  is block diagonal matrix. Furthermore, with Theorem 5.3 the splitting

$$A = LDU - N_E$$

yields a convergent iterative method. Thus, the following result follows directly from the facts mentioned above.

**THEOREM 5.6.** *Let  $A = [A_{ij}] \in \Omega_m^k$  with  $\eta(A) + \eta(A^H) \in M_m^k$  and let  $E \subseteq P(m)$  be such that  $M_E$  is block acyclic, where  $M_E$  is as in (2.2). Then  $A$  admits the incomplete block  $LDU$ -factorization*

$$A = LDU - N_E,$$

with  $M_E = LDU$  as in (5.26). Furthermore,  $\rho((LDU)^{-1}N) < 1$ .

Therefore, block methods, such as the block cyclic reduction method, can be used for the solution of the system  $Ax = b$ , and this is a very good method for preconditioning.

**6. Conclusions.** Following the results in [5, 14, 19], we have proposed a more general class of block matrices – the class of  $EH$ –matrices and established some results on the positive definiteness and the invariance under block Gaussian elimination for a subclass of  $EH$ –matrices. Furthermore, we have discussed convergence of block iterative methods for linear systems with such subclass of  $EH$ –matrices. For example, we have presented the convergence of the block Jacobi method, block Gauss-Seidel method, block JOR-method, block SOR-method and block AOR-method as well. In particular, we investigated the incomplete  $LDU$ –factorization of  $EH$ –matrices, which is applied to establish some results on the convergent iterative methods.

All results proposed in this paper generalize the corresponding ones of L. Elsner and R. Nabben and answer the open problem proposed by R. Nabben in [14].

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#### REFERENCES

- [1] A. Berman and R.J. Plemmons. *Nonnegative Matrices in the Mathematical Sciences*. Academic Press, New York, 1979. Reprinted by SIAM, Philadelphia, 1994.
- [2] E. Dick and J. Linden. A multigrid flux-difference splitting method for steady incompressible Navier-Stokes equations. *Proceedings of the GAMM Conference on Numerical Methods in Fluid Mechanics*, Delft, 1989.
- [3] L. Dieci and J. Lorenz. Block  $M$ –matrices and computation of invariant tori. *SIAM J. Sci. Statist. Comput.*, 13:885–903, 1992.
- [4] L. Dieci and G. Bader. Block iterations and compactification for periodic block dominant systems associated to invariant tori approximation. *Appl. Numer. Math.*, 17:251–274, 1995.
- [5] L. Elsner and V. Mehrmann. Convergence of block iterative methods for linear systems arising in the numerical solution of Euler equations. *Numer. Math.*, 59:541–559, 1991.
- [6] L. Elsner and V. Mehrmann. Convergence of block iterative methods for matrices arising in flow computations. *Proceedings of the IMACS International Symposium on Iterative Methods in Linear Algebra: Iterative Methods in Linear Algebra*, Brussels Belgium, 391–394, 1991.
- [7] A. Hadjidimos. Accelerated overrelaxation method. *Math. Comp.*, 141(32):149–157, 1978.
- [8] P.W. Hemker, W. Hoffmann, and M.H. van Raalte. Two-level fourier analysis of a multigrid approach for discontinuous Galerkin discretization. *SIAM J. Sci. Comput.*, 25(3):1018–1041, 2003.
- [9] P.W. Hemker and S.P. Spekreijse. Multiple grid and Osher’s scheme for the efficient solution of the steady Euler equations. *Appl. Numer. Math.*, 2:475–493, 1986.
- [10] P.W. Hemker and M.H. van Raalte. Fourier two-level analysis for higher dimensional discontinuous Galerkin discretisation. *Comput. Vis. Sci.*, 7:159–172, 2004.

- [11] R.A. Horn and C.R. Johnson. *Matrix Analysis*, Cambridge University Press, New York, 1985.
- [12] R.A. Horn and C.R. Johnson. *Topics in Matrix Analysis*. Cambridge University Press, New York, 1985.
- [13] T.-Z. Huang, S.-Q. Shen, and H.-B. Li. On generalized  $H$ -matrices. *Linear Algebra Appl.*, 396:81–90, 2005.
- [14] R. Nabben. On a class of matrices which arise in the numerical solution of Euler equations. *Numer. Math.*, 63:411–431, 1992.
- [15] R. Nabben. A new application for generalized  $M$ -matrices. In *Numerical Linear Algebra*, L. Reichel, A. Ruttan, and R.S. Varga, (editors), Walter de Gruyter, New York, 179–193, 1993.
- [16] R. Nabben. Matrices whose inverses are generalized  $M$ -matrices. *Linear Multilinear Algebra*, 44(1):67–84, 1998.
- [17] Y. Robert. Regular incomplete factorizations of real positive definite matrices. *Linear Algebra Appl.*, 48:105–117, 1982.
- [18] R.S. Varga. *Matrix Iterative Analysis*, second edition. Spriger-Verlag, Berlin/Heidelberg, 2000.
- [19] C.-Y. Zhang, C. Xu, and S. Luo. Convergence of block iterative methods for linear systems with generalized  $H$ -matrices. *J. Comput. Appl. Math.*, 229(1):70–84, 2009.