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THE LARGEST INTEGER CP-RANK OF 2 × 2 MATRICES — VIA GPT-5.5*

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Abstract. Each completely positive 2×2 matrix over $\mathbb{Z}$ is a sum of 9 matrices $(u_i\ v_i)^\top(u_i\ v_i)$ with nonnegative u_i, $v_i\in\mathbb{Z}$.

Key words. Completely positive matrices, Waring problem, Mordell's theorem.

AMS subject classifications. 11E25, 15B36, 15B48.

We address a notable recent question in [1, 2, 3, 4, 5]. It mirrors a classical theorem of Mordell [6], who shows $\mathcal{T}=5$ if u_i and v_i are allowed to have arbitrary signs, in the formulation below.

PROBLEM 1. *Let a, b, $c\geqslant 0$ be integers with $b^2\leqslant ac$. What is the least integer $\mathcal{T}$ for which each form $ax^2+2bxy+cy^2$ can be written as $(u_1x+v_1y)^2+\ldots+(u_{\mathcal{T}}x+v_{\mathcal{T}}y)^2$ with nonnegative u_i and v_i in $\mathbb{Z}$?*

Some bounds on Problem 1 were known, and an early reference is [1, page 12]. Laffey and Šmigoc prove $\mathcal{T}\leqslant 11$ in [5], and Dutour Sikirić, Schürmann and Vallentin [4] give an alternative argument without explicit bounds on $\mathcal{T}$. A recent article of Berman and Levy [2] shows $\mathcal{T}\leqslant 10$, and several authors repeat the question of whether $\mathcal{T}=9$ or $\mathcal{T}=10$, see [3, page 4]. One has $\mathcal{T}\geqslant 9$ by the example $8x^2+2xy+8y^2$ in [5]. Now, GPT-5.5 proves $\mathcal{T}=9$, in a single turn, with no human assistance (https://perma.cc/S52K-6Y5J).

THEOREM 2. *One has $\mathcal{T}=9$.*

Proof. We write $ax^2+2bxy+cy^2=(\lceil b^2/c\rceil x^2+2bxy+cy^2)+(a-\lceil b^2/c\rceil)x^2$ and note that the second summand is a sum of four squares, by *Lagrange's theorem*. By Mordell's theorem, we find $u,v\in\mathbb{Z}^5$ with $u\cdot v=b$, $\|u\|^2=\lceil b^2/c\rceil$ and $\|v\|^2=c$. As $\|u-bv/c\|^2=\|u\|^2+(b^2/c^2)\,\|v\|^2-(2b/c)\,(u\cdot v)=\lceil b^2/c\rceil-b^2/c$ is in $[0,1)$, each entry of $u-bv/c$ is in $(-1,\,1)$, which means $|u_i-bv_i/c|<1$. As u_i, $v_i\in\mathbb{Z}$, this implies $u_iv_i\geqslant 0$, and we can take $(u_i,\,v_i)\to(-u_i,\,-v_i)$ to make u_i and v_i nonnegative as $(u_ix+v_iy)^2$ is preserved. ∎

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