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# NOTES ON GRAPHS WITH LEAST EIGENVALUE AT LEAST $-2$ [*]

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**Abstract.** A new proof concerning the determinant of the adjacency matrix of the line graph of a tree is presented and an invariant for line graphs, introduced by Cvetković and Lepović, with least eigenvalue at least $-2$ is revisited and given a new equivalent definition [D. Cvetković and M. Lepović. Cospectral graphs with least eigenvalue at least $-2$. *Publ. Inst. Math., Nouv. Sér.*, 78(92):51–63, 2005.]. Employing this invariant and other techniques, it is shown that the line graphs of double stars are determined by their adjacency spectra.

**Key words.** Adjacency matrix, Least eigenvalue $-2$, Spectral characterization, Generalized line graph, Double star.

**AMS subject classifications.** 05C50.

**1. Introduction.** In this paper, we are concerned only with undirected simple graphs (loops and multiple edges are not allowed). All notions on graphs that are used here can be found in [5]. For a graph $G = (V(G), E(G))$, let $n(G)$, $m(G)$, $\ell(G)$ and $A = A(G)$ be the order, size, line graph and adjacency matrix of $G$, respectively. For some vertex $v_i \in V(G)$, let $d_i = d(v_i)$ stand for the degree of $v_i$ and $\Delta(G)$ be the maximum degree of $G$. We denote the *characteristic polynomial* $\det(\lambda I - A)$ of $G$ by $\phi(G, \lambda)$ or simply $\phi(G)$. The *adjacency spectrum* of $G$, denoted by $\mathrm{Spec}(G)$, is the multiset of eigenvalues of $A(G)$. Since $A(G)$ is symmetric, its eigenvalues are real and we set $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_{n(G)}(G)$. The maximum eigenvalue $\lambda_1(G)$ of $G$ is called the *spectral radius* (or *index*) of $G$ and it is often denoted by $\rho(G)$. Additionally, let $\lambda(G)$ denote the least eigenvalue of $G$.

Two graphs $G$ and $H$ are said to be *A-cospectral* if the corresponding adjacency spectra are the same. A graph is said to be *determined by the A-spectrum* (or simply a

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DAS-graph) if there is no non-isomorphic graph $A$-cospectral to it, i.e., $\phi(G) = \phi(H)$ implies $G \cong H$. The background of the question "which graphs are determined by their spectrum?" originates from chemistry (in 1956, Günthadr and Primas [16] raised this question in the context of Hückel's theory). For additional remarks on the topic we refer the readers to [11, 12]. There are many papers on $A$-cospectral graphs and the methods used to construct them (see [14, 17, 18, 20] for example). By contrast, the question of determining what kinds of graphs are DAS-graphs is far from resolved. See [1, 22, 23, 24, 28, 29, 30, 31] for some recent results in this field.

Some other notation and terminology are also needed. Let $G_1 \cup G_2$ denote the *disjoint union* of graphs $G_1$ and $G_2$, and $kG_1$ the disjoint union of $k$ copies of $G_1$. Let $n_G(H)$ denote the number of the subgraphs $H$ in graph $G$. As usual, let $P_n$, $K_n$ and $K_{1,n-1}$ denote the *path*, the *complete graph* and the *star* of order $n$ respectively. Let $S_{a,b}$ be the *double star* obtained from the stars $K_{1,a}$ and $K_{1,b}$ by joining the vertex of degree $a$ in $K_{1,a}$ and the vertex of degree $b$ in $K_{1,b}$ (see Fig. 4.1). Note that $S_{0,b} = K_{1,b+1}$. The well-known *cocktail-party graph* $CP(k)$ is obtained from $K_{2k}$ by removing $k$ disjoint edges. Hoffman [19] introduced the *generalized line graph* as follows: for any graph $H$ with $n$ vertices $v_1, v_2, \ldots, v_n$ and any non-negative integers $a_1, a_2, \ldots, a_n$, then the generalized line graph $L(H; a_1, a_2, \ldots, a_n)$ is the graph consisting of disjoint copies of $\ell(H)$ and $CP(a_i)$ together with additional edges joining a vertex in $\ell(H)$ with a vertex in $CP(a_i)$ if the vertex in $\ell(H)$ corresponding to an edge in $H$ has $v_i$ as an end-vertex ($i = 1, 2, \ldots, n$). The following theorem is a well-known result relating to generalized line graphs.

THEOREM 1.1. (Cameron, Goethals, Seidel, and Shult [3]) *Let $G$ be a connected graph with least eigenvalue at least $-2$. Then either $G$ is a generalized line graph or $G$ can be represented by vectors in the root system $E_8$.*

Graphs with least eigenvalue at least $-2$ have been studied since the very beginnings of the theory of graph spectra. Much information on this field can be found in the books [2, 4, 9, 15].

It is an interesting problem to find which graphs with least eigenvalue at least $-2$ are $A$-cospectral graphs or DAS-graphs. Here we mention some known results. An *exceptional graph* is a connected graph with least eigenvalue at least $-2$ which is not a generalized line graph. Cvetković and Lepović [6, 7] studied the phenomenon of $A$-cospectrality in generalized line graphs and in exceptional graphs. For the regular DAS-graphs with least eigenvalue at least $-2$, van Dam and Haemers [11] gave an almost complete answer (see their Theorem 8). Further results on $A$-cospectral graphs may be found in Section 4.2 of [9]. However, for the non-regular case, van Dam and Haemers [11] stated that the following question remains open.

PROBLEM 1.2. Which non-regular graphs with least eigenvalue at least $-2$ are

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DAS-graphs?

This paper is organized as follows: In Section 2, we cite some useful results of graphs with least eigenvalue at least −2, and give a new proof of an important result about the determinant of adjacent matrix of line graph of a tree. In Section 3, we revisit an invariant, defined by Cvetković and Lepović [8], for the graphs with least eigenvalue at least −2, and give it a new equivalent definition. In Section 4, we prove that the line graphs of double stars are determined by their adjacency spectra.

**2. Some useful results and a new proof.** Doob and Cvetković [13] characterized all connected graphs with the least eigenvalue greater than −2 as follows:

THEOREM 2.1. *$G$ is a connected graph with $\lambda(G) > -2$ if and only if one of the following holds:*

(i) *$G \in \mathcal{G}_1 = \{\ell(T) \mid T$ is a tree\};*
(ii) *$G \in \mathcal{G}_2 = \{L(T; 1, 0, \ldots, 0) \mid T$ is a tree\};*
(iii) *$G \in \mathcal{G}_3 = \{\ell(H) \mid H$ is an odd-unicyclic graph\};*
(iv) *$G \in \mathcal{G}_4 = \{20$ graphs with order 6 that are represented in $E_6\};*
(v) *$G \in \mathcal{G}_5 = \{110$ graphs with order 7 that are represented in $E_7\};*
(vi) *$G \in \mathcal{G}_6 = \{443$ graphs with order 8 that are represented in $E_8\}.*

For convenience, set $\mathcal{L} = \{G \mid G$ is a connected graph and $\lambda(G) \geq -2\}$, $\mathcal{L}^+ = \{G \mid G$ is a connected graph and $\lambda(G) > -2\}$ and $\mathcal{L}^0 = \{G \mid G$ is a connected graph and $\lambda(G) = -2\}$. Clearly, $\mathcal{L} = \mathcal{L}^+ \cup \mathcal{L}^0$ and $\mathcal{L}^+ = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_4 \cup \mathcal{G}_5 \cup \mathcal{G}_6$. A graph in $\mathcal{L}$ ($\mathcal{L}^+$ or $\mathcal{L}^0$) is called an $\mathcal{L}$-graph ($\mathcal{L}^+$-graph or $\mathcal{L}^0$-graph) (see [8]). For the $\mathcal{L}^+$-graphs we have the following result.

THEOREM 2.2. (Brouwer, Cohen, and Neumaier [2]) *Let $G$ be an $\mathcal{L}^+$-graph with order $n$. Then*

$$
\det(2I + A(G)) = \begin{cases} n+1 & \text{if } G \in \mathcal{G}_1; \\ 4 & \text{if } G \in \mathcal{G}_2 \cup \mathcal{G}_3; \\ 3 & \text{if } G \in \mathcal{G}_4; \\ 2 & \text{if } G \in \mathcal{G}_5; \\ 1 & \text{if } G \in \mathcal{G}_6. \end{cases}
$$

The above theorem has been proved by a lattice-theoretic argument (see, for example, [2]). It is worth pointing out that Doob and Cvetković [13] gave a proof for the case $G \in \mathcal{G}_2 \cup \mathcal{G}_3$ by other techniques. Here, we also give a new proof for the case $G \in \mathcal{G}_1$ which is independent of lattice theory. Let $\psi(G, \lambda) = \det(\lambda I - (D - A))$ and $\varphi(G, \lambda) = \det(\lambda I - (D + A))$ be respectively the characteristic polynomials of

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*Laplacian matrix* $D-A$ and *signless Laplacian matrix* $D+A$ of $G$, where $D = D(G) = \mathrm{diag}(d_1, d_2, \ldots, d_{n(G)})$ is the degree matrix of $G$.

LEMMA 2.3. [10] *Let $G$ be a graph with order $n$ and size $m$. Then*

(i) $\varphi(G, \lambda) = \lambda^{n-m}\phi(\ell(G), \lambda - 2)$.
(ii) *If $G$ is a bipartite graph, then $\psi(G, \lambda) = \varphi(G, \lambda)$.*

THEOREM 2.4. *Let $G \in \mathcal{G}_1$ be a graph with order $n$. Then $\det(2I + A(G)) = n+1$.*

*Proof.* Since $G \in \mathcal{G}_1$, then there exists a tree $T$ such that $G = \ell(T)$ and $n(T) = n + 1$. Note that $A(G)$ is a matrix of order $n$. Set

$$\vartheta(\lambda) = |\lambda I - (2I + A(G))| = \lambda^n + a_1\lambda^n + \cdots + a_n.$$

Suppose that the roots of $\vartheta(\lambda)$ are $\lambda_1, \lambda_2, \ldots, \lambda_n$. Thus,

(2.1) $$\det(2I + A(G)) = \lambda_1\lambda_2\cdots\lambda_n.$$

Since $\prod_{i=1}^{n} \lambda_i = (-1)^n a_n$, by (2.1) we have that

$$\det(2I + A(G)) = (-1)^n a_n.$$

Since $T$ is bipartite, from Lemma 2.3 we get that $\psi(T, \lambda) = \varphi(T, \lambda) = \lambda\phi(G, \lambda - 2)$. Since $\phi(G, \lambda - 2) = |(\lambda - 2)I - A(G)| = \vartheta(\lambda)$, we have

$$\psi(T, \lambda) = \lambda\vartheta(\lambda) = \lambda^{n+1} + a_1\lambda^n + \cdots + a_n\lambda.$$

Note that $a_n = (-1)^n(n + 1)\tau(T)$, where $\tau(T)$ is the number of spanning trees in $T$ [21]. Consequently, $a_n = (-1)^n(n + 1)$, and therefore, $\det(2I + A(G)) = n + 1$. ◻

**3. An important graph invariant.** Cvetković and Lepović [8] adopted the nomenclature from lattice theory and defined

$$d_G = (-1)^n\phi(G, -2)$$

as the *discriminant* of an $\mathscr{L}$-graph $G$. Additionally, for an $\mathscr{L}$-graph $G$ they obtained an important graph invariant named *star value* and showed that its formula is

$$S = \frac{(-1)^n}{(n-k)!}\phi^{(n-k)}(G, \lambda - 2) = (-1)^n\Pi_G(0) = \prod_{i=1}^{k}(\lambda_i + 2),$$

where $\phi^{(p)}(x)$ denotes the $p$-th derivative function of $\phi(x)$, $\Pi_G(\lambda) = \prod_{i=1}^{k}(\lambda - (\lambda_i + 2))$ (it is called the *principal polynomial* of $G$ [8]) and $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the eigenvalues greater than $-2$ of $G$. For the discriminant and the star value of $G$, we have the

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following conclusion. If $G$ is an $\mathscr{L}^0$-graph, then $d_G = 0 < S$. On the other hand, it is easy to see that $\phi(G, \lambda - 2) = \lambda^{n-k}\Pi_G(\lambda)$ and then $d_G = S$ if $G$ is an $\mathscr{L}^+$-graph.

Now we synthesize the above facts into the following definition:

DEFINITION 3.1. Let $G$ be an $\mathscr{L}$-graph of order $n$ and $\Pi_G(\lambda)$ the principal polynomial of $G$. Then the *star value* of $G$ is defined as

$$\star(G) = \begin{cases} (-1)^n \Pi_G(0) & \text{if} \quad G \in \mathscr{L}^0; \\ (-1)^n \phi(G, -2) & \text{if} \quad G \in \mathscr{L}^+. \end{cases}$$

The following corollary is an immediate consequence of Definition 3.1:

COROLLARY 3.2. Let $G = \cup_{i=1}^k G_i$ and $H$ be two $\mathscr{L}$-graphs. Then

(i) $\star(G) = \prod_{i=1}^k \star(G_i)$;
(ii) if $G$ and $H$ are A-cospectral, then $\star(G) = \star(H)$.

Corollary 3.2(ii) indicates that the star value of a graph is a graph invariant determined by the spectrum. The readers will see that it will play an important role in studying the spectral characterization of graphs with least eigenvalue at least $-2$.

Note that $\phi(G, -2) = \det(-2I - A(G)) = (-1)^n \det(2I + A(G))$, where $n$ is the order of $G$. Hence, $\det(2I + A(G)) = (-1)^n \phi(G, -2)$ and so the following corollary follows from Theorem 2.2 and Definition 3.1:

COROLLARY 3.3. Let $G$ be an $\mathscr{L}^+$-graph with order $n$. Then

$$(3.1) \qquad \star(G) = \begin{cases} n+1 & \text{if} \ G \in \mathcal{G}_1; \\ 4 & \text{if} \ G \in \mathcal{G}_2 \cup \mathcal{G}_3; \\ 3 & \text{if} \ G \in \mathcal{G}_4; \\ 2 & \text{if} \ G \in \mathcal{G}_5; \\ 1 & \text{if} \ G \in \mathcal{G}_6. \end{cases}$$

For an $\mathscr{L}^0$-graph, we pose the following problem.

PROBLEM 3.4. Let $G$ be an $\mathscr{L}^0$-graph. Can we determine a formula of its star value $\star(G)$ by means of the parameters of $G$, such as the order, the size, etc?

**4. Spectral characterization of line graphs of double stars.** The graphs used in this section are shown in Figure 4.1. In this section, we investigate the spectral characterization of line graphs of double stars $S_{a,b}$ and set $b \geq a \geq 0$. Note $\ell(S_{0,b}) = K_{b+1}$ which has been shown to be a $DAS$-graph (see [11]). In what follows we will directly use a well-known fact that if $G$ and $H$ are A-cospectral graphs, then they respectively share the same order and size.
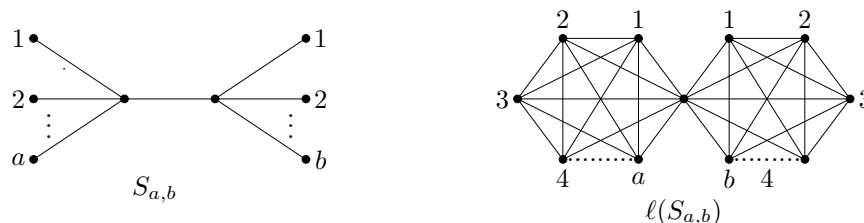
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FIG. 4.1. *Double star $S_{a,b}$ and its line graph.*

Schwenk [25] defined an operation on graphs named *generalized composition*. If a graph $H$ is labeled and has $s$ vertices, then the graph $G = H[H_1, H_2, \ldots, H_s]$ is formed by taking the disjoint graphs $H_1, H_2, \ldots, H_s$ and then joining every vertex of $H_i$ to every vertex of $H_j$ whenever $u_i$ is adjacent to $u_j$ in $H$ ($1 \le i, j \le s$). Assume that $H_i$ is an $r_i$-regular graph ($1 \le i \le s$). Then the partition $V(G) = V(H_1) \cup V(H_2) \cup \cdots V(H_s)$ is equitable. Let $T$ be the quotient matrix associated with this partition.

LEMMA 4.1. (Schwenk [25]) *Let $H_i$ be an $r_i$-regular graph ($1 \le i \le s$) and let $G = H[H_1, H_2, \ldots, H_s]$ defined above. Then*

$$\phi(G) = \phi(\lambda I - T) \prod_{i=1}^{s} \frac{\phi(H_i)}{\lambda - r_i}.$$

LEMMA 4.2. *Let $n(\ell(S_{a,b})) = n$. Then $\rho(\ell(S_{a,b})) > n-2$ if and only if $a \in \{0, 1\}$.*

*Proof.* If $b = 1$, then $a = 0$ or $1$. Thus, $\ell(S_{a,b}) = P_2$ or $P_3$ and so the lemma holds. Next set $b \ge 2$. Clearly, $\ell(S_{a,b}) = P_3[K_a, K_1, K_b]$. Since the partition $V(S_{a,b}) = V(K_a) \cup V(K_1) \cup V(K_b)$ is equitable, from Lemma 4.1 we get

$$\phi(\ell(S_{a,b})) = \phi(\lambda I - T) \cdot \frac{\phi(K_a)}{\lambda - (a-1)} \cdot \frac{\phi(K_1)}{\lambda} \cdot \frac{\phi(K_b)}{\lambda - (b-1)},$$

where $T = \begin{pmatrix} a-1 & 1 & 0 \\ a & 0 & b \\ 0 & 1 & b-1 \end{pmatrix}$. A direct calculation shows that

(4.1) $\phi(\ell(S_{a,b})) = (\lambda+1)^{a+b-2}[\lambda^3 - (a+b-2)\lambda^2 - (2a-ab+2b-1)\lambda - (a-2ab+b)]$.

Let $f(\lambda)$ denote the cubic factor. It is easy to see that $\rho(\ell(S_{a,b}))$ is the largest root of $f(\lambda)$. Note that $f(a+b) = ab(a+b+2) > 0$ and $f(a+b-1) = -a^2 - ab - b^2 + a^2b + ab^2$. Since $\frac{\partial f(a+b-1)}{\partial a} = (2a + b)(b - 1) > 0$, then for $a > 2$ we get that

$$f(a + b - 1) > f(2 + b - 1) = (b+1)^2 - 5 > 0$$

(4.2) $$> f(1 + b - 1) = -1 > f(0 + b - 1) = -b^2.$$

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Since $f(\lambda)$ is continuous in the interval $[a + b - 1, a + b]$, if $f(\lambda) > 0$ for any $\lambda \in [a + b - 1, a + b]$, then $f(\lambda)$ has no roots in $[a + b - 1, a + b]$. Note that $n - 1 = a + b$. Since $a + b - 1 = n - 2 < \rho(\ell(S_{a,b})) < n - 1 = a + b$, there exists at least one point $a_0 + b - 1 \in [a + b - 1, a + b]$ such that $f(a_0 + b - 1) < 0$. Since $a$ is a positive integer, it follows from (4.2) that $a = 1$ or $0$. This establishes the necessity.

Now we show the sufficiency. If $a = 0$, then $n = b + 1$ and $\ell(S_{a,b}) = K_n$ with $\rho(K_n) = n - 1 > n - 2$. If $a = 1$, then $n = b + 2$. from (4.1) we get that $\rho(S_{1,b})$ is the largest root of

$$f(\lambda) = \lambda^3 - (b - 1)\lambda^2 - (b + 1)\lambda + (b - 1) = 0.$$

Since $f(n-1) = f(b+1) = b(b+3) > 0$ and $f(n-2) = f(b) = -1$, $n-2 < \rho(\ell(S_{1,b})) < n - 1$. $\square$

LEMMA 4.3. *Let $b \geq a > 1$. Then $\ell(S_{a,b})$ has exactly four different eigenvalues $\rho(\ell(S_{a,b})), \lambda_2, \lambda_3 = -1, \lambda(\ell(S_{a,b}))$ such that $\lambda_2$ and $\lambda(\ell(S_{a,b}))$ are simple, the multiplicity of $-1$ is $a + b - 2$ and*

(4.3)          $b + 1 > \rho(\ell(S_{a,b})) > b > \lambda_2 > 0 > \lambda_3 = -1 > \lambda(\ell(S_{a,b})) > -2.$

*Proof.* We retain the notation of Lemma 4.2. From (4.1) it follows that $\rho(\ell(S_{a,b}))$ and $\lambda(\ell(S_{a,b}))$ are the roots of $f(\lambda)$. Since $f(-1) = ab > 0$, then $-1$ is not a root of $f(\lambda)$. Thus, $\lambda_2$ is also a root of $f(\lambda)$ and the multiplicity of $-1$ is $a+b-2$ by (4.1). A straightforward calculation shows that $f(b+1) = (b-a)(b+4) + 4 > 0, f(b) = -a < 0, f(0) = a(b-1) + b(a-1) > 0$ and $f(-2) = -a - b - 2 < 0$, and so the inequality (4.3) follows. $\square$

LEMMA 4.4. *No two non-isomorphic line graphs of double stars are A-cospectral.*

*Proof.* Let $\ell(S_{a,b})$ and $\ell(S_{a',b'})$ be A-cospectral. Then $a + b = a' + b'$. From (4.1) it follows that $ab = a'b'$ (since the constant terms in the cubic factors of $\phi(\ell(S_{a,b}))$ and $\phi(\ell(S_{a',b'}))$ are equal). Solving the equations $a + b = a' + b'$ and $ab = a'b'$ we get that $a = a', b = b'$ or $a = b', b = a'$. $\square$

THEOREM 4.5. *$\ell(S_{1,n-2})$ is a DAS-graph.*

*Proof.* Let $\ell(S_{1,n-2}) = G$ and let $H$ be any graph A-cosptral with $G$. Then $n(H) = n(G) = n$ and $\rho(H) = \rho(G)$. Since $\Delta(H) \geq \rho(H)$, by Lemma 4.2 we get that $\Delta(H) > n - 2$ and so $\Delta(H) = n - 1$. Hence, $H$ is a connected graph. Since $\lambda(H) = \lambda(G) > -2$, $H \in \bigcup_{i=1}^{6} \mathcal{G}_i$ by Theorem 2.1. From Corollary 3.3 it follows that $\star(H) = \star(G) = n + 1$. If $n = 2$ or $3$, then $G = P_2$ or $P_3$ which has been proved to be a DAS-graph. Now set $n \geq 4$; then $\star(H) \geq 5$, and therefore, $H \in \mathcal{G}_1$ by Corollary 3.3. Thus, $H = \ell(T)$, where $T$ is a tree. Since $e_1$ has degree $n - 1$ in $H$, the other

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edges of $T$ must be adjacent to $e_1$, and this implies that $T$ is a double star $S_{a,b}$. From Lemma 4.4 it follows that $H \cong G$. $\square$

LEMMA 4.6. [26] *The double star $S_{a,b}$ is determined by the Laplacian spectrum.*

LEMMA 4.7. [27] *A graph has exactly one positive eigenvalue if and only if its non-isolated vertices form a complete multipartite graph.*

THEOREM 4.8. *Let $n = a + b + 1$ and $b \geq a \geq 2$. Then $\ell(S_{a,b})$ is a DAS-graph.*

*Proof.* Let $\ell(S_{a,b}) = G$ and let $H$ be any graph $A$-cospectral with $G$. Then $n(H) = n(G) = n$. We first show the following consequence:

*Claim:* $H$ is a connected graph.

Assume by way of contradiction that $H = \bigcup_{i=1}^{k} H_i$, where $k > 1$ and $H_i$ $(1 \leq i \leq k)$ is a connected component of $H$. Since $\lambda(H) = \min\{\lambda(H_i) \mid i = 1, \ldots, k\} = \lambda(G) > -2$, by Theorem 2.1 we get that $H_i \in \bigcup_{j=1}^{6} \mathcal{G}_j$ $(i = 1, \ldots, k)$. Without loss of generality, set $\lambda(H) = \lambda(H_1) = \lambda(\ell(S_{a,b}))$. Note that $\ell(S_{a,b})$ has only one eigenvalue $\lambda(\ell(S_{a,b}))$ less than $-1$. Hence, $\lambda(H_1) < -1$ and $\lambda(H_i) \geq -1$ $(i = 2, \ldots, k)$. Since $H$ has no eigenvalue 0, then each $H_i$ is not an isolated vertex, and therefore, $H_i$ has at least one edge $(2 \leq i \leq k)$. Consequently, $\lambda(H_i) \leq -1$ and so $\lambda(H_i) = -1$ which implies that $H_i$ is a complete graph $K_{a_i}$ $(2 \leq i \leq k)$. Thus,

$$H = H_1 \cup K_{a_2} \cup \cdots \cup K_{a_k}, \quad \text{where } a_2 \geq a_3 \geq \cdots \geq a_k.$$

From (4.3) it follows that $\rho(H) = \rho(\ell(S_{a,b}))$ is not an integer, which, together with $\rho(H) = \max\{\rho(H_1), \rho(K_{a_2}) = a_2 - 1\}$, leads to $\rho(H) = \rho(H_1)$.

If $H_1$ has exactly two different eigenvalues $\rho(H_1)$ and $\lambda(H_1)$, then $H_1$ must be a complete graph whose spectral radius $\rho(H_1) = n(H_1) - 1$ is an integer, a contradiction. Hence, $H_1$ has at least three different eigenvalues $\rho(H_1), \lambda_2(H_1), \lambda(H_1)$.

Note that $\text{Spec}(K_{a_i}) = \{a_i - 1, -1^{a_i - 1}\}$ $(i = 2, \ldots, k)$. If $a_i \neq a_j$ $(2 \leq i \neq j \leq k)$, then $H$ has at least three positive eigenvalues $\rho(H), a_i - 1, a_j - 1$ which contradicts the fact that $H$ has exactly two positive eigenvalues (see (4.3)). Thus, $a_2 = \cdots = a_k$ which shows that the multiplicity of eigenvalue $a_2 - 1$ is $k - 1$.

Since the second largest positive eigenvalue of $H$ is simple (see Lemma 4.3), then $k - 1 = 1$ and so $k = 2$ and $H = H_1 \cup K_{a_2}$. Note again that $H$ has four different eigenvalues $\rho(H) = \rho(H_1) = \rho(\ell(S_{a,b})), \lambda_2(H) = \lambda_2(\ell(S_{a,b})), \lambda_3(H) = \lambda_3(\ell(S_{a,b})) = -1, \lambda(H) = \lambda(H_1) = \lambda(\ell(S_{a,b}))$. If $H_1$ has four different eigenvalues $\rho(H_1), \lambda_2(H_1), \lambda_3(H_1), \lambda(H_1)$, then $\lambda_2(H) = \lambda_2(H_1) = a_2 - 1$ and $\lambda_3(H) = \lambda_3(H_1) = -1$ which contradicts the fact that $\lambda_2(H)$ is simple. So, $H_1$ has exactly three different eigenvalues $\rho(H_1), \lambda_2(H_1), \lambda(H_1)$ and the possibly different eigenvalues of $H$ lie in $\{\rho(H_1), \lambda_2(H_1), \lambda(H_1)\} \cup \{a_2 - 1, -1\}$.

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If $\lambda_2(H_1) > 0$, then $\lambda_2(H_1) = a_2 - 1$ which contradicts that the second largest eigenvalue of $H$ is simple. If $\lambda_2(H_1) < 0$, then $\lambda_2(H_1) = -1$ and thus $H_1$ has only one positive eigenvalue. By Lemma 4.7 we get that $H_1$ is a complete multipartite graph $K_{c_1,c_2,\ldots,c_s}$. It has been proved that $H_1 = K_{c_1,c_2,\ldots,c_s}$ has eigenvalue 0 with multiplicity $n(H_1) - s$ (see [5] pp. 74). Since $H$ has no eigenvalue 0, then $n(H_1) = s$ and so $H_1$ is a complete graph $K_{n(H_1)}$ with integral spectral radius $n(H_1) - 1$, a contradiction. This finishes the proof of the Claim.

From Corollaries 3.2 and 3.3 we get that $\star(H) = \star(G) = n+1$. By the condition of the theorem, we have $\star(H) \geq 6$ and thus $H \in \mathcal{G}_1$ by Theorem 2.1. So, there exists a tree $T$ such that $H = \ell(T)$. From $\phi(\ell(S_{a,b}), \lambda) = \phi(\ell(T), \lambda)$ it follows that $\lambda\phi(\ell(S_{a,b}), \lambda-2) = \lambda\phi(\ell(T), \lambda-2)$ which by Lemma 2.3(i) yields $\varphi(T, \lambda) = \varphi(S_{a,b}, \lambda)$ and so $\psi(T, \lambda) = \psi(S_{a,b}, \lambda)$ by Lemma 2.3(ii). From Lemma 4.6 it follows that $T \cong S_{a,b}$ and so $H = \ell(T) \cong \ell(S_{a,b})$. $\square$

Combining the fact that $\ell(S_{0,b}) = K_{b+1}$ is a DAS-graph and Theorems 4.5 and 4.8 we immediately obtain the main result of this section:

THEOREM 4.9. *Let $b \geq a \geq 0$. Then $\ell(S_{a,b,})$ is a DAS-graph.*

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