

A CLASS OF UNICYCLIC GRAPHS DETERMINED BY THEIR LAPLACIAN SPECTRUM*

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Abstract. Let $G_{r,p}$ be a graph obtained from a path by adjoining a cycle C_r of length r to one end and the central vertex of a star S_p on p vertices to the other end. In this paper, it is proven that unicyclic graph $G_{r,p}$ with r even is determined by its Laplacian spectrum except for $n = p + 4$.

Key words. Adjacency spectrum, Laplacian spectrum, Cospectral graph, Unicyclic graph.

AMS subject classifications. 05C05, 05C50.

1. Introduction. Let G be a simple graph on n vertices and $A(G)$ be its adjacency matrix. Let $d_G(v)$ be the degree of vertex v in G , and $D(G)$ be the diagonal matrix with the degrees of the corresponding vertices of G on the diagonal and zero elsewhere. Matrix $Q(G) = D(G) - A(G)$ is called the Laplacian matrix of G . The eigenvalues of $A(G)$ (resp., $Q(G)$) and the spectrum (which consists of eigenvalues) of $A(G)$ (resp., $Q(G)$) are also called the adjacency (resp., Laplacian) eigenvalues of G and the adjacency (resp., Laplacian) spectrum of G . Since both matrices $A(G)$ and $Q(G)$ are real symmetric matrices, their eigenvalues are all real numbers. So we can assume that $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ and $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G) = 0$ are the adjacency eigenvalues and the Laplacian eigenvalues of G , respectively.

Two graphs are adjacency (resp., Laplacian) cospectral if they have the same adjacency (resp., Laplacian) spectrum. Denote by $\phi(G) = \phi(G; \lambda) = \det(\lambda I - A(G))$ and $\chi(G; \mu) = \det(\mu I - Q(G))$ the characteristic polynomial of adjacency matrix and Laplacian matrix of G , respectively. A graph is said to be determined by the adjacency (resp., Laplacian) spectrum if there is no non-isomorphic graph with the same adjacency (resp., Laplacian) spectrum.

In general, the spectrum of a graph does not determine the graph and the question “Which graphs are determined by their spectrum?” ([3]) remains a difficult problem. For the background and some known results about this problem and related topics, we refer the readers to [4] and references therein. For the unicyclic graphs, Haemers

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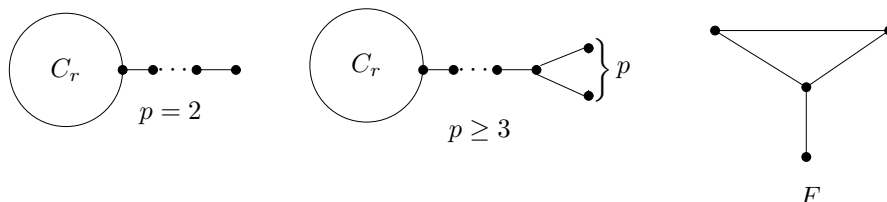


FIG. 1.1. Graphs $G_{r,p}$ and F .

et al. [5] showed that lollipop graphs H with p odd are determined by the adjacency spectrum. Boulet and Jouve proved in [1] that the remaining lollipop graphs are also determined by their adjacency spectrum. Haemers et al. showed that lollipop graphs are determined by their Laplacian spectrum as well. Let $U_{n,r}$ be the graph obtained by attaching $n - r$ pendent edges to a vertex of cycle C_r . Zhang et al. proved in [13] that $U_{n,r}$ is determined by its Laplacian spectrum. We shall prove a class of unicyclic graphs determined by their Laplacian spectra in this paper.

Let $G_{r,p}$ (see Fig. 1.1) be a graph obtained from a path by adjoining a cycle C_r of length r to one end and the central vertex of a star S_p on p vertices to the other end. For $p = 2$, $G_{r,p}$ is a lollipop graph, which is determined by its adjacency spectrum and Laplacian spectrum respectively. Without loss of generality, we assume that $p \geq 3$ and n is the order of $G_{r,p}$. In this paper, we prove that $G_{r,p}$ with r even is determined by its Laplacian spectrum except for $n = p + 4$, which extends the known families of unicyclic graphs determined by their Laplacian spectrum.

2. Preliminaries. The following lemmas will be used in the next section.

LEMMA 2.1. ([3]) For $n \times n$ matrices A and B , the following are equivalent:

- (i) A and B are cospectral;
- (ii) A and B have the same characteristic polynomial;
- (iii) $\text{tr}(A^i) = \text{tr}(B^i)$ for $i = 1, 2, \dots, n$.

If A is the adjacency matrix of a graph, then $\text{tr}(A^i)$ gives the total number of closed walks of length i . So cospectral graphs have the same number of closed walks of each given length i . In particular, they have the same number of edges (taking $i = 2$) and triangles (taking $i = 3$).

LEMMA 2.2. ([2]) Let G be a connected graph, and H a proper subgraph of G . Then $\lambda_1(H) < \lambda_1(G)$.

LEMMA 2.3. ([2]) Let G be the graph obtained from the disjoint union $H_1 \cup H_2$

by adding an edge v_1v_2 joining the v_1 of H_1 and v_2 of H_2 , then $\phi(G) = \phi(H_1)\phi(H_2) - \phi(H_1 - v_1)\phi(H_2 - v_2)$, where $H_i - v_i$ denote the graph obtained from H_i by deleting the vertex v_i and the edges incident to v_i .

Hoffman and Smith defined an internal path [6] of a graph as a walk $v_0, v_1, \dots, v_k$ ($k \geq 1$) such that $v_1, \dots, v_k$ are distinct (v_0, v_k need not be distinct), $d_{v_0} > 2, d_{v_k} > 2$ and $d_{v_i} = 2, 0 < i < k$.

LEMMA 2.4. ([6]) Let G be a connected graph that is not isomorphic to W_n , where W_n is a graph obtained from the path P_{n-2} (indexed in natural order $1, 2, \dots, n-2$) by adding two pendant edges at vertices 2 and $n-3$. Let G_{uv} be the graph obtained from G by subdividing the edge uv of G . If uv lies on an internal path of G , then $\lambda_1(G_{uv}) \leq \lambda_1(G)$.

LEMMA 2.5. ([2]) Let the eigenvalues of graphs G and $G-v$ be $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ and $\lambda'_1 \geq \lambda'_2 \geq \dots \geq \lambda'_{n-1}$, respectively. Then $\lambda_1 \geq \lambda'_1 \geq \lambda_2 \geq \lambda'_2 \geq \dots \geq \lambda'_{n-1} \geq \lambda_n$.

LEMMA 2.6. ([2]) Let C_n, P_n be the cycle and path on n vertices respectively. Then

$$\phi(C_n) = \prod_{j=1}^n \left(\lambda - 2 \cos \frac{2\pi j}{n} \right) = \lambda \phi(P_{n-1}) - 2\phi(P_{n-2}) - 2;$$

$$\phi(P_n) = \prod_{j=1}^n \left(\lambda - 2 \cos \frac{\pi j}{n} \right) = \lambda \phi(P_{n-1}) - \phi(P_{n-2}).$$

We write the Laplacian characteristic polynomial as $\chi(G; \mu) = q_0\mu^n + q_1\mu^{n-1} + \dots + q_{n-1}\mu + q_n$.

LEMMA 2.7. ([3, 11]) Let G be a graph with n vertices and m edges and $d = (d_1, \dots, d_n)$ be its non-increasing degree sequence. Then

$$q_0 = 1; \quad q_1 = -2m; \quad q_2 = 2m^2 - m - \frac{1}{2} \sum_{i=1}^n d_i^2; \quad q_{n-1} = (-1)^{n-1} nt(G); \quad q_n = 0;$$

where $t(G)$ is the number of spanning trees in G .

Part (i) and (ii) of the following are given in [10] and [9], respectively.

LEMMA 2.8. Let G be a graph with $V(G) \neq \emptyset$ and $E(G) \neq \emptyset$.

(i) Then $\Delta(G) + 1 \leq \mu_1 \leq \max \left\{ \frac{d_u(d_u + m_u) + d_v(d_v + m_v)}{d_u + d_v}, uv \in E(G) \right\}$, where $\Delta(G)$ denotes the maximum vertex degree of G , u_1 is the largest Laplacian eigenvalue of G , $d_v m_v$ means the sum of degrees of vertices adjacent to v in G .

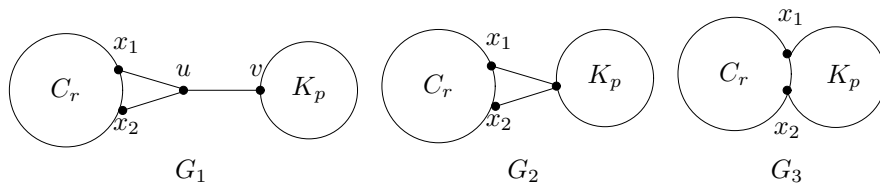


FIG. 3.1. Graphs G_1 , G_2 and G_3 .

(ii) If G is a connected graph with at least 2 vertices, then $\mu_1 = \Delta(G) + 1$ if and only if $|V(G)| = \Delta(G) + 1$.

LEMMA 2.9. ([7, 8]) Let G be a graph with n vertices and $\overline{G}$ its complement, then $\mu_i(G) = n - \mu_{n-i}(\overline{G})$ for $1 \leq i \leq n - 1$.

LEMMA 2.10. ([12]) Let F be the graph in Fig. 1.1, $N_G(F)$ the number of subgraphs F of a graph G , and $N_G(i)$ the number of closed walks of length i in G . Then $N_G(5) = 30N_G(K_3) + 10N_G(C_5) + 10N_G(F)$, where K_3 is the complete graph of order 3, C_5 is the circle of length 5.

For a bipartite graph G with n vertices and m edges, the Laplacian matrix $Q(G) = D - A$ and signless Laplacian matrix $|Q(G)| = D + A$ are similar by a diagonal matrix with diagonal entries ± 1 , hence they have the same spectrum. Let N be the vertex-edge incidence matrix of G and B the adjacency matrix of the line graph $L(G)$ of G . Since $|Q(G)| = NN^T$, $N^TN = 2I + B$, NN^T and N^TN have the same non-zero eigenvalues, for $\mu \neq 0$, μ is an eigenvalue of $|Q(G)|$ with multiplicity a if and only if $\mu - 2$ is an eigenvalue of B with multiplicity a , and the multiplicity of the eigenvalue -2 equals $m - n + 1$ ([3]). For a unicyclic connected bipartite graph G , $Q(G)$ has one eigenvalue 0, since $m = n$, the multiplicity of eigenvalue -2 of B is 1. Thus, we have the following lemma.

LEMMA 2.11. Let G be a connected unicyclic bipartite graph with n vertices and $L(G)$ its line graph. Then $\mu_i(G) = \lambda_i(L(G)) + 2$ for $i = 1, 2, \dots, n - 1$, where $\lambda_i(L(G))$ is the i -th largest adjacency eigenvalue of $L(G)$.

3. Main results. We need the following key lemmas to prove our results. Let K_p be a complete graph on p vertices, and G_i a graph depicted in Fig. 3.1, x_1x_2 an edge of G_i ($i = 1, 2, 3$).

LEMMA 3.1. $\lambda_1(G_1) < \min\{\lambda_1(G_2), \lambda_1(G_3)\}$ for $p > 3$.

Proof. By Lemma 2.3 and direct calculation, we obtain the characteristic polynomial of G_i ($i = 1, 2, 3$):

$$\begin{aligned}\phi(G_1) &= (\lambda + 1)^{p-2}((\lambda(\lambda + 1)(\lambda - p + 1) - (\lambda - p + 2))\phi(C_r) \\ &\quad - 2(\lambda + 1)(\lambda - p + 1)(\phi(P_{r-1}) + \phi(P_{r-2}) + 1)), \\ \phi(G_2) &= (\lambda + 1)^{p-2}((\lambda(\lambda - p + 2) - (p - 1))\phi(C_r) - 2(\lambda - p + 2)(\phi(P_{r-1}) \\ &\quad + \phi(P_{r-2}) + 1)), \\ \phi(G_3) &= (\lambda + 1)^{p-3}((\lambda(\lambda - p + 3) - 2(p - 2))\phi(P_{r-1}) - 2(\lambda + 1)(\phi(P_{r-2}) + 1)).\end{aligned}$$

Let

$$\begin{aligned}\phi^*(G_1) &= (\lambda(\lambda + 1)(\lambda - p + 1) - (\lambda - p + 2))\phi(C_r) - 2(\lambda + 1)(\lambda - p + 1)(\phi(P_{r-1}) \\ &\quad + \phi(P_{r-2}) + 1), \\ \phi^*(G_2) &= (\lambda(\lambda - p + 2) - (p - 1))\phi(C_r) - 2(\lambda - p + 2)(\phi(P_{r-1}) + \phi(P_{r-2}) + 1), \\ \phi^*(G_3) &= (\lambda(\lambda - p + 3) - 2(p - 2))\phi(P_{r-1}) - 2(\lambda + 1)(\phi(P_{r-2}) + 1).\end{aligned}$$

Obviously, $\lambda_1(G_i)$ is also the largest root of $\phi^*(G_i)$ ($i = 1, 2, 3$). Since $\phi^*(G_1; p - 1) = -\phi(C_r, p - 1)$ and $p > 3$, $\phi^*(G_1; p - 1) < 0$ by Lemma 2.6. By the intermediate value theorem, $\lambda_1(G_1) > p - 1$. As G_1 is not regular, $\lambda_1(G_1) < \Delta(G_1)$, where $\Delta(G_1)$ is the maximum degree of G_1 . Hence $\lambda_1(G_1) < p$. By Lemma 2.6, $\lambda\phi(P_{r-i}) = \phi(P_{r-i+1}) + \phi(P_{r-i-1})$, $i = 1, \dots, r - 1$.

$$\begin{aligned}\phi^*(G_1) - \lambda\phi^*(G_2) &= (p - 2 - \lambda)\phi(C_r) + 2(p - 1)(\phi(P_{r-1}) + \phi(P_{r-2}) + 1) \\ &= (p - 2 - \lambda)(\lambda\phi(P_{r-1}) - 2\phi(P_{r-2}) - 2) + 2(p - 1)(\phi(P_{r-1}) + \phi(P_{r-2}) + 1) \\ &= (\lambda(p - 2 - \lambda) + 2(p - 1))\phi(P_{r-1}) + 2(\lambda + 1)(\phi(P_{r-2}) + 1) \\ &= (\lambda(p - 2 - \lambda) + 2(p - 1))\phi(P_{r-1}) + 2(\phi(P_{r-1}) + \phi(P_{r-3})) + 2\phi(P_{r-2}) + 2(\lambda + 1) \\ &= (\lambda(p - 2 - \lambda) + 2p)\phi(P_{r-1}) + 2(\phi(P_{r-2}) + \phi(P_{r-3})) + 2(\lambda + 1).\end{aligned}$$

Thus, we have

$$\begin{aligned}\phi^*(G_1; \lambda_1(G_1)) - \lambda_1(G_1)\phi^*(G_2; \lambda_1(G_1)) &> (\lambda_1(G_1)(p - 2 - p) + 2p)\phi(P_{r-1}, \lambda_1(G_1)) + 2(\phi(P_{r-2}, \lambda_1(G_1)) \\ &\quad + \phi(P_{r-3}, \lambda_1(G_1)) + 2(\lambda + 1)) \\ &> 0.\end{aligned}$$

Since $p > \lambda_1(G_1) > p - 1$, $\phi(P_{r-1}, \lambda_1(G_1)), \phi(P_{r-2}, \lambda_1(G_1)), \phi(P_{r-3}, \lambda_1(G_1))$ are all positive for $p > 3$. Thus, $\phi^*(G_2; \lambda_1(G_1)) < 0$. By the intermediate value theorem the largest root of $\phi^*(G_2)$ exceeds $\lambda_1(G_1)$. So, $\lambda_1(G_1) < \lambda_1(G_2)$. Similarly, by Lemma

2.6, we have

$$\begin{aligned}
 & \phi^*(G_1) - \lambda^2(\lambda - 2)\phi^*(G_3) \\
 &= (2\lambda^4 - (2p - 2)\lambda^3 - 2\lambda^2p + (5p - 8)\lambda + 2p - 2)\phi(P_{r-1}) \\
 & \quad + ((2p - 10)\lambda^3 + (6p - 14)\lambda^2 + (4p - 2)\lambda + 2)(\phi(P_{r-2}) + 1) \\
 &= (2\lambda^4 - 2(p - 1)\lambda^3 - 10\lambda^2 + (11p - 22)\lambda + 8p - 14)\phi(P_{r-1}) + (6p - 12)\phi(P_{r-2}) \\
 & \quad + (6p - 12)\phi(P_{r-3}) + ((2p - 10)\lambda + 6p - 14)\phi(P_{r-4}) + (2p - 10)\lambda^3 + (6p - 14)\lambda^2 \\
 & \quad + (4p - 2)\lambda + 2.
 \end{aligned}$$

For convenience, we set $\alpha = \lambda_1(G_1)$. Then

$$\begin{aligned}
 & \phi^*(G_1; \alpha) - \alpha^2(\alpha - 2)\phi^*(G_3; \alpha) \\
 &= (2\alpha^4 - 2(p - 1)\alpha^3 - 10\alpha^2 + (11p - 22)\alpha + 8p - 14)\phi(P_{r-1}, \alpha) \\
 & \quad + (6p - 12)\phi(P_{r-2}, \alpha) + (6p - 12)\phi(P_{r-3}, \alpha) + ((2p - 10)\alpha + 6p - 14)\phi(P_{r-4}, \alpha) \\
 & \quad + (2p - 10)\alpha^3 + (6p - 14)\alpha^2 + (4p - 2)\alpha + 2.
 \end{aligned}$$

Let

$$b = 2\alpha^4 - 2(p - 1)\alpha^3 - 10\alpha^2 + (11p - 22)\alpha + 8p - 14,$$

$$c = (2p - 10)\alpha^3 + (6p - 14)\alpha^2 + (4p - 2)\alpha + 2.$$

Obviously, $c > 0$ for $p \geq 5$, and

$$\begin{aligned}
 b &= (\alpha - p + 1)(\alpha - 3)(2(\alpha - 3)^2 + 18(\alpha - 3) + 43) + \alpha^2 + 10\alpha - 13p + 7 \\
 &> (\alpha - p + 1)(\alpha - 3)(2(\alpha - 3)^2 + 18(\alpha - 3) + 43) + (p - 1)^2 - 3p - 3 \\
 & \quad + 10(\alpha - p + 1) \\
 &> 0
 \end{aligned}$$

for $p \geq 6$. If $p = 5$, then $5 > \alpha > 4$, $c = 16\alpha^2 + 18\alpha + 2 > 0$. Using

$$5\phi(P_{r-i}, \alpha) > \alpha\phi(P_{r-i}, \alpha) = \phi(P_{r-i+1}, \alpha) + \phi(P_{r-i-1}, \alpha),$$

we have

$$\begin{aligned}
 & \phi^*(G_1; \alpha) - \alpha^2(\alpha - 2)\phi^*(G_3; \alpha) \\
 &= ((\alpha - 4)(\alpha - 3)(2(\alpha - 3)^2 + 18(\alpha - 3) + 43) + \alpha^2 + 10\alpha - 58)\phi(P_{r-1}, \alpha) \\
 & \quad + 18\phi(P_{r-2}, \alpha) + 18\phi(P_{r-3}, \alpha) + 16\phi(P_{r-4}, \alpha) + c \\
 &> ((\alpha - 4)(\alpha - 3)(2(\alpha - 3)^2 + 18(\alpha - 3) + 43) + \alpha^2 + 10\alpha - 54)\phi(P_{r-1}, \alpha) \\
 & \quad + 2\phi(P_{r-2}, \alpha) + \phi(P_{r-3}, \alpha) + 20\phi(P_{r-4}, \alpha) + c.
 \end{aligned}$$

Since $\alpha^2 + 10\alpha - 54 = (\alpha - 4)(\alpha + 14) + 2 > 0$, $-\alpha^2(\alpha - 2)\phi^*(G_3; \alpha) > 0$. This implies that $\phi^*(G_3; \alpha) < 0$.

Similarly, for $p = 4$, $4 > \alpha > 3$, $c = -2\alpha^3 + 10\alpha^2 + 8\alpha + 2 = -2\alpha^2(\alpha - 5) + 8\alpha + 2 > 0$. Then

$$\begin{aligned} & \phi^*(G_1; \alpha) - \alpha^2(\alpha - 2)\phi^*(G_3; \alpha) \\ & > ((\alpha - 3)^2(2(\alpha - 3)^2 + 18(\alpha - 3) + 43) + \alpha^2 + 10\alpha - 39)\phi(P_{r-1}, \alpha) \\ & \quad + 2\phi(P_{r-2}, \alpha) + 5\phi(P_{r-3}, \alpha) + 12\phi(P_{r-4}, \alpha) + c \\ & = ((\alpha - 3)^2(2(\alpha - 3)^2 + 18(\alpha - 3) + 43) + (\alpha - 3)(\alpha + 13))\phi(P_{r-1}, \alpha) \\ & \quad + 2\phi(P_{r-2}, \alpha) + 5\phi(P_{r-3}, \alpha) + 12\phi(P_{r-4}, \alpha) + c > 0, \end{aligned}$$

which implies that $\phi^*(G_3; \alpha) < 0$. Hence, by the intermediate value theorem, the largest root of $\phi^*(G_3)$ exceeds $\lambda_1(G_1)$. Thus, $\lambda_1(G_1) < \lambda_1(G_3)$. $\square$

LEMMA 3.2. *Let graphs G and $G_{r,p}$ be Laplacian cospectral. Then G is a connected unicyclic graph with circle length r and the same degree sequence with $G_{r,p}$.*

Proof. By Lemma 2.8(i), the largest eigenvalue of $G_{r,p}$ satisfies $p+1 \leq \mu_1 < p+2$. Suppose that graph G is Laplacian cospectral to $G_{r,p}$. By Lemma 2.8, the largest vertex degree of G is at most p . By Lemma 2.7, G and $G_{r,p}$ have the same number of vertices, edges, spanning trees. So G is a connected unicyclic graph with n vertices. Since $G_{r,p}$ has r spanning trees, the length of cycle in G is also r . Assume that G has n_i vertices of degree i , for $i = 1, \dots, p$. By Lemma 2.7, we have

$$(3.1) \quad \sum_{i=1}^p n_i = n, \sum_{i=1}^p i n_i = 2n, \sum_{i=1}^p i^2 n_i = p^2 + 3^2 + 2^2(n - p - 1) + p - 1.$$

This gives

$$(3.2) \quad \sum_{i=3}^p (i-1)(i-2)n_i = p^2 - 3p + 4.$$

By Lemma 2.11, $L(G)$ and $L(G_{r,p})$ are adjacency cospectral, so they have the same number of triangles. This gives

$$(3.3) \quad \sum_{i=3}^p \binom{i}{3} n_i = \binom{p}{3} + 1.$$

Obviously, $n_p \leq 1$ for $p > 3$. We assert that $n_p = 1$, $n_3 = 1$. Assume that $n_p = 0$. Combining equations (3.2) and (3.3), we have

$$\begin{aligned} p(p-1)(p-2) + 6 &= \sum_{i=3}^p (i(i-1)(i-2))n_i \leq (p-1) \left(\sum_{i=3}^{p-1} (i-1)(i-2)n_i \right) \\ &= (p-1)(p^2 - 3p + 4). \end{aligned}$$

This gives $p^2 - 5p + 10 \leq 0$, which is a contradiction. It is easy to obtain $n_3 = 1$, and $n_i = 0$, $i = 4, \dots, p-1$ from equation (3.3). By equation (3.1), we easily get that $n_2 = n - p - 1$, $n_1 = p - 1$. For $p = 3$, by equation (3.1), we have

$$n_1 + n_2 + n_3 = n; n_1 + 2n_2 + 3n_3 = 2n; n_1 + 4n_2 + 9n_3 = 4 + 4n.$$

Solving these equations gives that $n_1 = 2$, $n_2 = n - 4$, $n_3 = 2$, which is the same degree sequence with $G_{r,3}$. $\square$

LEMMA 3.3. *If r is even, $n > p + r$, $p > 3$, then $G_{r,p}$ is determined by its Laplacian spectrum.*

Proof. Assume that G and $G_{r,p}$ are Laplacian cospectral. By Lemma 3.2, G is a connected unicyclic graph with circle length r and has the same degree sequence as $G_{r,p}$. Since r is even, G and $G_{r,p}$ are bipartite graphs. By Lemma 2.11, their line graphs are adjacency cospectral. Since G and $G_{r,p}$ have the same degree sequence, the line graph $L(G)$ is a connected graph with n vertices and contains a subgraph G_i ($i = 1, 2, 3$) or a subgraph obtained by subdividing edge uv of G_1 several times. For $n = p + r + 1$, the line graph of $G_{r,p}$ is G_1 . By Lemma 3.1, $L(G) \cong G_1$. For $n > p + r + 1$, by Lemma 2.4, $\lambda_1(L(G_{r,p})) \leq \lambda_1(G_1)$. Since $L(G)$ and $L(G_{r,p})$ are adjacency cospectral, neither G_2 nor G_3 is a subgraph of $L(G)$ by Lemma 3.1. Since $n > p + r + 1$, G_1 is not a subgraph of $L(G)$. Thus, $L(G)$ contains a subgraph obtained by subdividing edge uv of G_1 several times. By Lemmas 2.4 and 2.2, $L(G) \cong L(G_{r,p})$. $\square$

For $n > p + r$, $p = 3$, we also have the following.

LEMMA 3.4. *$G_{r,3}$ is determined by its Laplacian spectrum for $n > 3 + r$.*

Proof. Let G and $G_{r,3}$ be Laplacian cospectral. By Lemma 3.2, G is a unicyclic graph with circle length r and has the same degree sequence as $G_{r,3}$. Then G is either G_4 or G_5 depicted in Fig. 3.2. Let a be the length of path from vertex u to v , b the length of path from u' to v' , c the length of path from z to w and d the length of path from z' to w' in Fig. 3.2. Note that x is not necessarily adjacent to y in G_5 , $L(G_{r,3})$ is G_6 with $a = b = 0$.

By Lemmas 2.1 and 2.11, $L(G)$ and $L(G_{r,3})$ are adjacency cospectral, so they have the same number of closed walks of length i for each i . Consider the closed walks of length 5. Since the line graphs of $G_{r,3}$ and G have the same number of triangles and C_5 's, we only need to enumerate $N(F)$ in G_i ($i = 6, 7$) by Lemma 2.10. Clearly, $N_{L(G_{r,3})}(F) = 4$.

If there is a path with length no less than 1 between two triangles, then

$$N_{G_6}(F) = \begin{cases} 6, & a \neq 0, b \neq 0; \\ 5, & \text{either } a \text{ or } b \text{ is } 0. \end{cases}$$

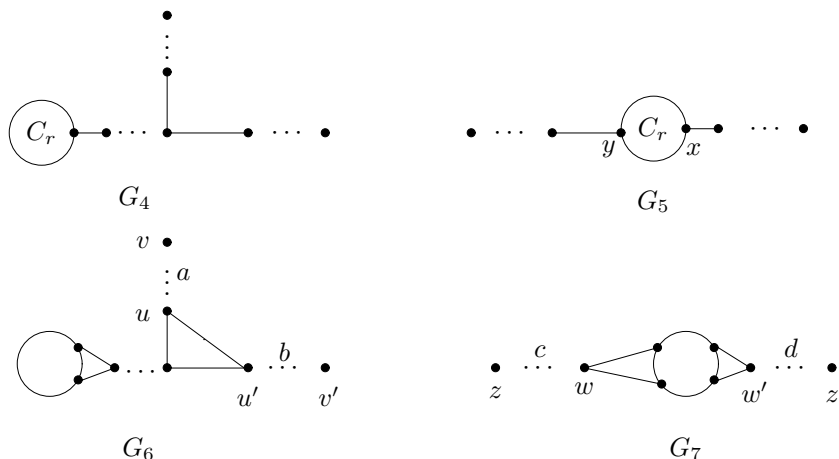


FIG. 3.2. Graphs G_4 , G_5 and the corresponding line graphs G_6 , G_7 , respectively.

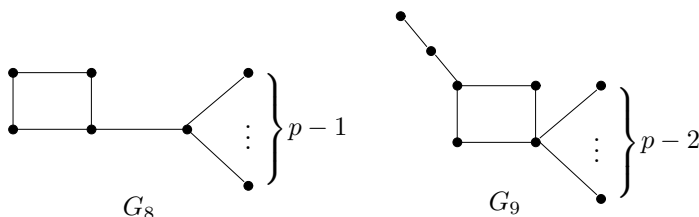


FIG. 3.3. A family of non-isomorphic but Laplacian cospectral graphs.

If two triangles share a common vertex, then

$$N_{G_6}(F) = \begin{cases} 8, & a \neq 0, b \neq 0; \\ 7, & \text{either } a \text{ or } b \text{ is } 0. \end{cases}$$

If $c = 0$ (resp., $d = 0$), then $d \neq 0$ (resp., $c \neq 0$) for $n > 3 + r$.

$$N_{G_7}(F) = \begin{cases} 5, & \text{either } c \text{ or } d \text{ is } 0, x \text{ is not adjacent to } y, \\ 7, & \text{either } c \text{ or } d \text{ is } 0, x \text{ is adjacent to } y, \\ 6, & c \neq 0, d \neq 0, x \text{ is not adjacent to } y, \\ 8, & c \neq 0, d \neq 0, x \text{ is adjacent to } y. \end{cases}$$

Thus, the number of closed walks of length 5 in $L(G_{r,3})$ is different to G_i ($i = 6, 7$) if $G_i \not\cong L(G_{r,3})$. Hence G is isomorphic to $G_{r,3}$ for $n > 3 + r$. $\square$

Let $n = p + r$. We determine a family of non-isomorphic Laplacian cospectral graphs for $r = 4$, see Fig. 3.3. Since the line graph of G_8 is isomorphic to G_2 in Fig.

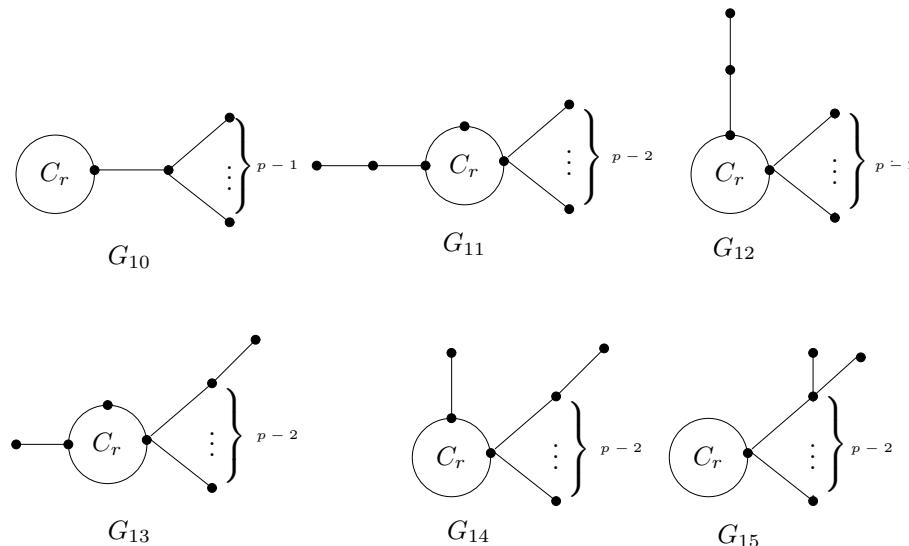


FIG. 3.4. Graphs G_j ($j = 10, \dots, 15$).

3.1, it is easy to check that the line graphs of G_8 and G_9 have the same adjacency characteristic polynomial: $\lambda(\lambda+1)^{p-2}(\lambda+2)(\lambda^4 - p\lambda^3 + (p-5)\lambda^2 + 4(p-1)\lambda + 4 - 2p)$.

For $n = p + r, r \neq 4$, we have:

LEMMA 3.5. $G_{r,p}$ is also determined by its Laplacian spectrum if $n = p + r$, $r \neq 4$.

Proof. Let graphs G and $G_{r,p}$ be Laplacian cospectral. By Lemma 3.2, G is a connected unicyclic graph with the same degree sequence as $G_{r,p}$. Then G is just one of these graphs depicted in Fig. 3.4, here G_{10} is $G_{r,p}$ for $n = p + r$.

By Lemma 2.11, their line graphs have the same adjacency spectrum, thus the closed walks of length i in these line graphs are the same by Lemma 2.1. The line graph of G_j ($j = 10, \dots, 15$) is depicted in Fig. 3.5, here x is adjacent to y in G_k ($k = 16, \dots, 21$).

Consider the closed walks of length 5 in G_k ($k = 16, \dots, 21$). By Lemma 2.10, since there are the same number of triangles and C_5 's respectively in these graphs, we only need to enumerate the number of subgraphs F in G_k . It is easy to get the

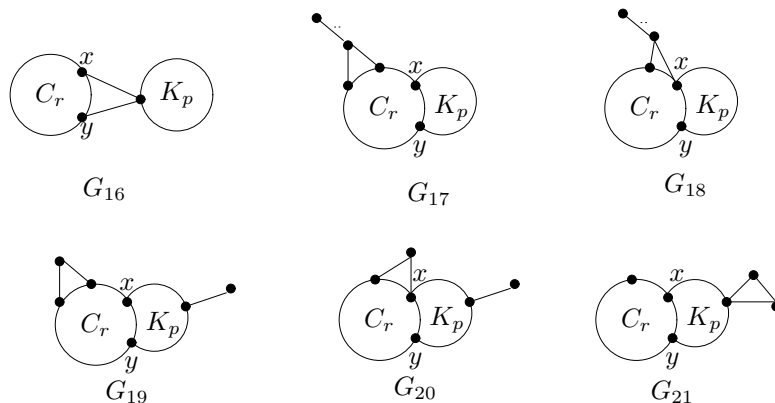


FIG. 3.5. Graph G_k , the corresponding line graphs of G_j , $j = 10, \dots, 15$.

following:

$$N_{G_{16}}(F) = p + 1 + 2 \binom{p-1}{2} + N_{K_p}(F); N_{G_{17}}(F) = 3 + 2 \binom{p-1}{2} + N_{K_p}(F);$$

$$N_{G_{18}}(F) = p + 1 + 3 \binom{p-1}{2} + N_{K_p}(F); N_{G_{19}}(F) = 2 + 3 \binom{p-1}{2} + N_{K_p}(F);$$

$$N_{G_{20}}(F) = p + 4 \binom{p-1}{2} + N_{K_p}(F); N_{G_{21}}(F) = p - 1 + 4 \binom{p-1}{2} + N_{K_p}(F);$$

Obviously, $N_{G_k}(F) \neq N_{G_{16}}(F)$ ($k = 17, \dots, 21$) except for $N_{G_{19}}(F)$ for $p = 4$. For $p = 4$, by Lemmas 2.5 and 2.2, we have $\lambda_2(G_{16}) \leq 2$ and $\lambda_2(G_{19}) > 2$. So if G is not isomorphic to $G_{r,p}$, then their line graphs are not adjacency cospectral. Hence, G is isomorphic to $G_{r,p}$ for $r \neq 4$ and $n = p + r$. $\square$

From Lemmas 3.3, 3.4 and 3.5, we obtain our main result.

THEOREM 3.6. *Unicyclic graph $G_{r,p}$ with r even is determined by its Laplacian spectrum except for $n = p + 4$.*

By Lemma 2.9, the complement of $G_{r,p}$ ($n \neq p + 4$) with r even is also determined by its Laplacian spectrum.

For r odd, a family of non-isomorphic but Laplacian cospectral graphs is given in Fig. 3.6.

If r is odd, since $G_{r,p}$ is not a bipartite graph, $u_i(G_{r,p}) \neq \lambda_i(L(G_{r,p})) + 2$ for $i = 1, \dots, n$ in general, and hence we cannot use line graph to characterize the spectrum

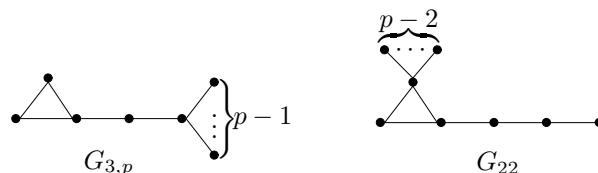


FIG. 3.6. Graphs $G_{3,p}$ and its Laplacian cospectral graph.

of $G_{r,p}$. The methods used here are invalid if r is odd. Some new techniques are needed to prove whether $G_{r,p}$ with r odd is determined by its Laplacian spectrum.

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