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A FUNCTIONAL BLOCK POSITIVITY CHARACTERIZATION OF NORMAL MATRICES*

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Abstract. A functional version of the block positivity characterization of normal matrices is established, thereby answering in a corrected form, a question posed in [1]. In particular, if $A \in M_n$ and $\alpha \in \mathbb{R} \setminus \{1\}$, we show that

$$\begin{bmatrix} (A^*A)^\alpha & A^* \\ A & (A^*A)^{1-\alpha} \end{bmatrix},$$

is positive if and only if A is normal. An operator analog of this power-type criterion on infinite-dimensional Hilbert spaces is also proved; in this setting, the corresponding conclusion is formulated in terms of semi-hyponormality.

Key words. Positive block matrices, Normal matrices, Semi-hyponormal operators, Functional calculus.

AMS subject classifications. Primary 47A63; Secondary 47A08, 47B15, 47B20.

1. Introduction. Let $B(\mathcal{H})$ be the algebra of bounded linear operators on a complex Hilbert space $\mathcal{H}$ with identity operator I. When $\mathcal{H} = \mathbb{C}^n$, $B(\mathcal{H})$ is identified with M_n, the algebra of complex $n \times n$ matrices. For $A \in B(\mathcal{H})$, let A^* be the adjoint of A, and $|A| = (A^*A)^{1/2}$. We write $A \geq O$ if A is positive, that is, $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$. If A is positive and invertible, we write $A > O$. Here, O denotes the zero operator. For self-adjoint operators $A, B \in B(\mathcal{H})$, we write $A \leq B$ if $B - A \geq O$. We say that A is normal if $A^*A = AA^*$, and semi-hyponormal if $|A^*| \leq |A|$. In finite dimensions, semi-hyponormality is equivalent to normality.

The 2×2 block matrix of operators

$$\begin{bmatrix} A & C^* \\ C & B \end{bmatrix},$$

has long been a basic object in matrix analysis and operator theory, especially in the study of positivity criteria and structural properties of positive block matrices. We refer the reader to [2, 3, 4, 5, 6, 7] for representative contributions in these directions.

Let $A \in M_n$. In [8], Sababheh et al. used properties of the geometric mean to prove that

$$\begin{bmatrix} |A| & A^* \\ A & |A| \end{bmatrix} \geq O,$$

if and only if A is normal. In more recent work, Zhang and Jing [1] established a parameterized extension of this result by showing that for $0 \leq \alpha \leq \frac{1}{2}$,

$$\begin{bmatrix} |A|^{2\alpha} & A^* \\ A & |A|^{2(1-\alpha)} \end{bmatrix} \geq O,$$

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if and only if A is normal. Thus, the preceding criterion may be viewed as the special case $\alpha = \frac{1}{2}$.

Motivated by these results, it is natural to ask whether the diagonal entries $|A|^{2\alpha}$ and $|A|^{2(1-\alpha)}$ can be replaced by more general nonnegative functions of $|A|$. In this note, we show that

$$\begin{bmatrix} f(|A|) & A^* \\ A & g(|A|) \end{bmatrix} \geq O,$$

holds whenever A is normal and $f(t)g(t) \geq t^2$ for all $t \geq 0$; conversely, if $f(t)g(t) = t^2$ for all $t \geq 0$, $g(0) = 0$, and g is injective, then the positivity of the above block matrix implies that A is normal. We further investigate the power-function case for bounded linear operators on Hilbert spaces, obtaining an operator analogue in which normality is replaced by semi-hyponormality.

2. Main results. In what follows, $\sigma(A)$ denotes the spectrum of A.

THEOREM 2.1. *Let $A \in M_n$. Then, the following assertions hold.*

(i) *Let $f, g : [0, \infty) \to [0, \infty)$ satisfy*

$$f(t)g(t) \geq t^2 \qquad \text{for all } t \geq 0.$$

If A is normal, then

$$\begin{bmatrix} f(|A|) & A^* \\ A & g(|A|) \end{bmatrix} \geq O.$$

(ii) *Let $f, g : [0, \infty) \to [0, \infty)$ satisfy*

$$f(t)g(t) = t^2 \qquad \text{for all } t \geq 0.$$

Assume further that $g(0) = 0$ and that g is injective. If

$$\begin{bmatrix} f(|A|) & A^* \\ A & g(|A|) \end{bmatrix} \geq O,$$

then A is normal.

Proof. Write $A = U|A|$ as a polar decomposition of A, where $U \in M_n$ is unitary. Recall that A is normal if and only if $U|A| = |A|U$; see [9, Sec. 1.3, p. 12].

Assume first that $f, g : [0, \infty) \to [0, \infty)$ satisfy $f(t)g(t) \geq t^2$ for all $t \geq 0$ and A is normal. Since $|A| = V\Lambda V^*$ for some unitary matrix $V \in M_n$, where

$$\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n), \qquad \lambda_i \geq 0 \quad (i = 1, \ldots, n),$$

the functional calculus gives

$$f(|A|) = V \operatorname{diag}\big(f(\lambda_1), \ldots, f(\lambda_n)\big)V^*, \qquad g(|A|) = V \operatorname{diag}\big(g(\lambda_1), \ldots, g(\lambda_n)\big)V^*.$$

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Therefore,

$$\begin{bmatrix} f(|A|) & |A| \\ |A| & g(|A|) \end{bmatrix},$$

is unitarily similar to

$$\begin{bmatrix} \operatorname{diag}\big(f(\lambda_1),\dots,f(\lambda_n)\big) & \operatorname{diag}(\lambda_1,\dots,\lambda_n) \\ \operatorname{diag}(\lambda_1,\dots,\lambda_n) & \operatorname{diag}\big(g(\lambda_1),\dots,g(\lambda_n)\big) \end{bmatrix},$$

which is permutation similar to

$$\bigoplus_{i=1}^{n} \begin{bmatrix} f(\lambda_i) & \lambda_i \\ \lambda_i & g(\lambda_i) \end{bmatrix}.$$

For each $i = 1,\dots,n$, we have

$$f(\lambda_i) \geq 0, \qquad g(\lambda_i) \geq 0, \qquad f(\lambda_i)g(\lambda_i) - \lambda_i^2 \geq 0.$$

Hence,

$$\begin{bmatrix} f(\lambda_i) & \lambda_i \\ \lambda_i & g(\lambda_i) \end{bmatrix} \geq O \qquad (i = 1,\dots,n),$$

and so

$$\begin{bmatrix} f(|A|) & |A| \\ |A| & g(|A|) \end{bmatrix} \geq O.$$

Finally, since U commutes with $|A|$, it also commutes with $g(|A|)$. Hence,

$$\begin{bmatrix} f(|A|) & A^* \\ A & g(|A|) \end{bmatrix} = \begin{bmatrix} I & O \\ O & U \end{bmatrix} \begin{bmatrix} f(|A|) & |A| \\ |A| & g(|A|) \end{bmatrix} \begin{bmatrix} I & O \\ O & U^* \end{bmatrix} \geq O.$$

We now prove (ii). Suppose that $f, g : [0,\infty) \to [0,\infty)$ satisfy $f(t)g(t) = t^2$ for all $t \geq 0$, $g(0) = 0$, g is injective, and

$$\begin{bmatrix} f(|A|) & A^* \\ A & g(|A|) \end{bmatrix} \geq O.$$

Applying the unitary congruence associated with $\operatorname{diag}(I, U)$, we obtain

$$\begin{bmatrix} f(|A|) & |A| \\ |A| & U^*g(|A|)U \end{bmatrix} \geq O.$$

Hence, for every $\varepsilon > 0$,

$$\begin{bmatrix} f(|A|) + \varepsilon I & |A| \\ |A| & U^*g(|A|)U \end{bmatrix} \geq O.$$

Since $f(|A|) + \varepsilon I > O$, the Schur complement theorem (see, for example, [9, Theorem 1.3.3]) implies that

$$U^*g(|A|)U \geq |A|\big(f(|A|) + \varepsilon I\big)^{-1}|A|. \tag{2.1}$$

For each $\varepsilon > 0$, define $h_\varepsilon : [0,\infty) \to [0,\infty)$ by

$$h_\varepsilon(t) := \frac{t^2}{f(t) + \varepsilon}, \qquad t \geq 0.$$

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Then, (2.1) can be rewritten as

$$U^*g(|A|)U \geq h_\varepsilon(|A|), \qquad \varepsilon > 0. \tag{2.2}$$

Indeed, if $\lambda = 0$, then $h_\varepsilon(0) = 0 = g(0)$. If $\lambda > 0$, then $f(\lambda)g(\lambda) = \lambda^2 > 0$, and hence $f(\lambda) > 0$. Therefore, for every $\lambda \in \sigma(|A|)$,

$$h_\varepsilon(\lambda) = \frac{\lambda^2}{f(\lambda) + \varepsilon} \xrightarrow[\varepsilon \downarrow 0]{} g(\lambda).$$

Letting $\varepsilon \to 0$ in (2.2), we obtain

$$U^*g(|A|)U \geq g(|A|).$$

Note that $U^*g(|A|)U$ and $g(|A|)$ have the same trace. Therefore,

$$U^*g(|A|)U - g(|A|) \geq O \quad \text{and} \quad \operatorname{Tr}\big(U^*g(|A|)U - g(|A|)\big) = 0,$$

which implies

$$U^*g(|A|)U = g(|A|).$$

As g is injective on $\sigma(|A|)$, the inverse map $g^{-1} : g(\sigma(|A|)) \to \sigma(|A|)$ is well defined. By the spectral functional calculus,

$$U^*|A|U = U^*g^{-1}(g(|A|))U = g^{-1}\big(U^*g(|A|)U\big) = g^{-1}(g(|A|)) = |A|.$$

Then, we have $U|A| = |A|U$ and hence A is normal. ∎

In order to discuss zero and negative powers, we adopt the following convention. For a positive operator $A \in B(\mathcal{H})$, we set $A^0 := s(A)$, where $s(A)$ denotes the support projection of A. If $A \in M_n$ is positive and $r < 0$, we define

$$A^r := s(A)(E + A)^r,$$

where $E = I - s(A)$ is the orthogonal projection onto $\ker A$.

Corollary 2.2. *Let $A \in M_n$ and let $\alpha \in \mathbb{R} \setminus \{1\}$. Then,*

$$\begin{bmatrix} |A|^{2\alpha} & A^* \\ A & |A|^{2(1-\alpha)} \end{bmatrix} \geq O,$$

if and only if A is normal.

Proof. For each $r \in \mathbb{R}$, define $\varphi_r \colon [0,\infty) \to [0,\infty)$ by

$$\varphi_r(t) := \begin{cases} t^r, & t > 0, \\ 0, & t = 0. \end{cases}$$

Set

$$f(t) := \varphi_{2\alpha}(t), \qquad g(t) := \varphi_{2(1-\alpha)}(t), \qquad t \geq 0.$$

Then, $f(t)g(t) = t^2$ for all $t \geq 0$, and $g(0) = 0$. Since $\alpha \neq 1$, g is injective on $[0,\infty)$. Therefore, Theorem 2.1 applies. ∎

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Remark 2.3. The conjecture in Question 2.14 of [1] is false in its original form. Indeed, let

$$f(t) := t^2, \qquad g(t) := 1, \qquad t \geq 0.$$

Then, f and g satisfy the hypotheses stated there. For an arbitrary matrix $A \in M_n$,

$$\begin{bmatrix} f(|A|) & A^* \\ A & g(|A|) \end{bmatrix} = \begin{bmatrix} |A|^2 & A^* \\ A & I \end{bmatrix} = \begin{bmatrix} A^* \\ I \end{bmatrix} \begin{bmatrix} A & I \end{bmatrix} \geq O.$$

However, this positivity condition does not imply that A is normal. Therefore, the additional assumptions that $g(0) = 0$ and that g is injective are indispensable for the converse part of Theorem 2.1.

In what follows, we extend the power-type criterion to bounded linear operators on an infinite-dimensional Hilbert space. We shall use the following standard identity; see, for example, [10] for the relevant background. A proof is included for completeness.

LEMMA 2.4. *Let $A = W|A|$ be the polar decomposition of $A \in B(\mathcal{H})$. If $f : \sigma(|A|) \cup \{0\} \to \mathbb{R}$ is continuous and $f(0) = 0$, then*

$$f(|A^*|) = W f(|A|) W^*.$$

Proof. Since $|A^*| = W|A|W^*$ and $W^*W|A| = |A|$, it follows by induction that

$$|A^*|^n = W|A|^n W^* \qquad (n \geq 1).$$

Consequently,

$$p(|A^*|) = W p(|A|) W^*,$$

for every polynomial p satisfying $p(0) = 0$.

Let $K := \sigma(|A|) \cup \{0\} = \sigma(|A^*|) \cup \{0\}$, and let f be as in the statement. By the Stone–Weierstrass theorem, there exist polynomials q_n converging uniformly to f on K. Setting

$$p_n(t) := q_n(t) - q_n(0),$$

we have $p_n(0) = 0$ and $p_n \to f$ uniformly on K. Hence, by the continuous functional calculus,

$$f(|A^*|) = \lim_{n\to\infty} p_n(|A^*|) = \lim_{n\to\infty} W p_n(|A|) W^* = W f(|A|) W^*.$$

∎

THEOREM 2.5. *Let $A \in B(\mathcal{H})$. Then, the following statements hold.*

(i) *If $\frac{1}{2} \leq \alpha \leq 1$ and $|A^*| \leq |A|$, then*

$$\begin{bmatrix} |A|^{2\alpha} & A^* \\ A & |A|^{2(1-\alpha)} \end{bmatrix} \geq O.$$

(ii) *If $0 \leq \alpha \leq \frac{1}{2}$ and*

$$\begin{bmatrix} |A|^{2\alpha} & A^* \\ A & |A|^{2(1-\alpha)} \end{bmatrix} \geq O,$$

then $|A^*| \leq |A|$.

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Proof. Let $A = W|A|$ be the polar decomposition of A. Assume first that $\frac{1}{2} \le \alpha < 1$ and $|A^*| \le |A|$. Since $2(1-\alpha) > 0$, Lemma 2.4 yields

$$|A^*|^{2(1-\alpha)} = W|A|^{2(1-\alpha)}W^*.$$

Therefore,

$$\begin{bmatrix} |A|^{2\alpha} & A^* \\ A & |A^*|^{2(1-\alpha)} \end{bmatrix} = \begin{bmatrix} |A|^{\alpha} \\ W|A|^{1-\alpha} \end{bmatrix} \begin{bmatrix} |A|^{\alpha} & |A|^{1-\alpha}W^* \end{bmatrix} \ge O.$$

As $0 < 2(1-\alpha) \le 1$, the Löwner–Heinz inequality (see [11, Example 4.45]) implies

$$|A^*|^{2(1-\alpha)} \le |A|^{2(1-\alpha)}.$$

Consequently,

$$\begin{bmatrix} |A|^{2\alpha} & A^* \\ A & |A|^{2(1-\alpha)} \end{bmatrix} = \begin{bmatrix} |A|^{2\alpha} & A^* \\ A & |A^*|^{2(1-\alpha)} \end{bmatrix} + \begin{bmatrix} O & O \\ O & |A|^{2(1-\alpha)} - |A^*|^{2(1-\alpha)} \end{bmatrix} \ge O. \tag{2.3}$$

Taking the strong operator limit in (2.3) as $\alpha \uparrow 1$ yields the assertion for $\alpha = 1$, completing the proof of (i).

We next prove (ii). Let $0 \le \alpha \le \frac{1}{2}$ and assume that

$$\begin{bmatrix} |A|^{2\alpha} & A^* \\ A & |A|^{2(1-\alpha)} \end{bmatrix} \ge O.$$

Suppose first that $\alpha = 0$ and set $P = s(|A|)$. Then,

$$\begin{bmatrix} I & A^* \\ A & |A|^2 \end{bmatrix} = \begin{bmatrix} P & A^* \\ A & |A|^2 \end{bmatrix} + \begin{bmatrix} I-P & O \\ O & O \end{bmatrix} \ge O.$$

Taking the Schur complement of I, we obtain

$$|A^*|^2 = AA^* \le |A|^2.$$

Since $t \mapsto t^{1/2}$ is operator monotone on $[0,\infty)$, it follows that $|A^*| \le |A|$.

Now suppose that $0 < \alpha \le \frac{1}{2}$. Then, for every $\varepsilon > 0$,

$$\begin{bmatrix} |A|^{2\alpha} + \varepsilon I & A^* \\ A & |A|^{2(1-\alpha)} \end{bmatrix} \ge O.$$

Since $|A|^{2\alpha} + \varepsilon I > O$, the Schur complement theorem yields

$$A\big(|A|^{2\alpha} + \varepsilon I\big)^{-1} A^* \le |A|^{2(1-\alpha)}. \tag{2.4}$$

Define $h_\varepsilon : [0,\infty) \to [0,\infty)$ by

$$h_\varepsilon(t) := \frac{t^2}{t^{2\alpha} + \varepsilon}, \qquad t \ge 0.$$

For each $t \in [0, \|A\|]$, we have $h_\varepsilon(t) \to t^{2(1-\alpha)}$ as $\varepsilon \downarrow 0$ and $0 \le h_\varepsilon(t) \le \|A\|^{2(1-\alpha)}$. Thus, by the bounded convergence theorem for spectral measures

$$h_\varepsilon(|A^*|) \xrightarrow{\text{SOT}} |A^*|^{2(1-\alpha)}.$$

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Since h_ε is continuous and satisfies $h_\varepsilon(0) = 0$, Lemma 2.4 yields

$$\begin{aligned} A\big(|A|^{2\alpha} + \varepsilon I\big)^{-1} A^* &= W|A|\big(|A|^{2\alpha} + \varepsilon I\big)^{-1}|A|W^* \\ &= W h_\varepsilon(|A|)W^* = h_\varepsilon(|A^*|). \end{aligned}$$

Combining this with (2.4), we obtain

$$h_\varepsilon(|A^*|) \le |A|^{2(1-\alpha)}.$$

Letting $\varepsilon \downarrow 0$ in the preceding inequality, we obtain

$$|A^*|^{2(1-\alpha)} \le |A|^{2(1-\alpha)}.$$

Since $0 < \alpha \le \frac{1}{2}$, one has $\frac{1}{2(1-\alpha)} \in (\frac{1}{2}, 1]$. Another application of the Löwner–Heinz inequality gives $|A^*| \le |A|$. This completes the proof. ∎

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