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BICYCLIC DIGRAPHS WITH EXTREMAL SKEW ENERGY*

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Abstract. Let $\overrightarrow{G}$ be a digraph and $S(\overrightarrow{G})$ be the skew-adjacency matrix of $\overrightarrow{G}$. The skew energy of $\overrightarrow{G}$ is the sum of the absolute values of eigenvalues of $S(\overrightarrow{G})$. In this paper, the bicyclic digraphs with minimal and maximal skew energy are determined.

Key words. Bicyclic digraph, Skew-adjacency matrix, Extremal skew energy.

AMS subject classifications. 05C05, 05C50, 15A03.

1. Introduction. Let G be a simple undirected graph of order n with vertex set $V(G) = \{1, \ldots, n\}$ and $\overrightarrow{G}$ be an orientation of G. The skew-adjacency matrix of $\overrightarrow{G}$ is the $n \times n$ matrix $S(\overrightarrow{G}) = [s_{i,j}]$, $s_{i,j} = 1$ and $s_{j,i} = -1$ if $i \to j$ is an arc of $\overrightarrow{G}$, and $s_{i,j} = s_{j,i} = 0$ otherwise. Since $S(\overrightarrow{G})$ is a real skew symmetric matrix, all eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ of $S(\overrightarrow{G})$ are pure imaginary numbers, the singular values of $S(\overrightarrow{G})$ are just the absolute values $\{|\lambda_1|, \ldots, |\lambda_n|\}$. So the energy of $S(\overrightarrow{G})$ defined as the sum of singular values of $S(\overrightarrow{G})$ [12] is the sum of the absolute values of its eigenvalues. For convenience, the energy of $S(\overrightarrow{G})$ is called *skew energy* of G ([1]), and denoted by $\mathcal{E}_s(\overrightarrow{G})$.

Energy has close links to chemistry (see, for instance, [6]). Since the concept of the energy of simple undirected graphs was introduced by Gutman in [5], there has been lots of research papers on this topic. For a survey, we refer to Section 7 in [3] and references therein. Denote, as usual, the n-vertex path and cycle by P_n and C_n, respectively. For the extremal energy of bicyclic graphs, let $G(n)$ be the class of bicyclic graphs with n vertices and containing no disjoint odd cycles of lengths k and ℓ with $k + \ell = 2 \pmod 4$. Let S_n^{ℓ} be the graph obtained by connecting $n - \ell$ pendant vertices to a vertex of C_ℓ, $S_n^{3,3}$ be the graph formed by joining $n-4$ pendant vertices to a vertex of degree three of the $K_4 - e$ (see Fig. 1.1), and $S_n^{4,4}$ be the graph formed by joining $n - 5$ pendant vertices to a vertex of degree three of the complete bipartite graph $K_{2,3}$. Let $\mathcal{B}_n$ be the class of all bipartite bicyclic graphs of order n that are not the graph obtained from two cycles C_a and C_b ($a, b \geq 10$ and $a = b = 2 \pmod 4$) joined by an edge. Zhang and Zhou [14] showed that $S_n^{3,3}$ is the graph with

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minimal energy in $G(n)$. In [10], Li et al. showed that $P_n^{6,6}$ is the graph with maximal energy in $\mathcal{B}_n$. Additional results on the energy of bicyclic graphs can be found in [4, 7, 11, 13]. The skew energy was first introduced by C. Adiga et al. in [1]. Some properties of the skew energy of a digraph are given in [1]. A connected graph with n vertices and $n+1$ edges is called a bicyclic graph. In this paper, we are interested in studying the bicyclic graphs with extremal skew energy.

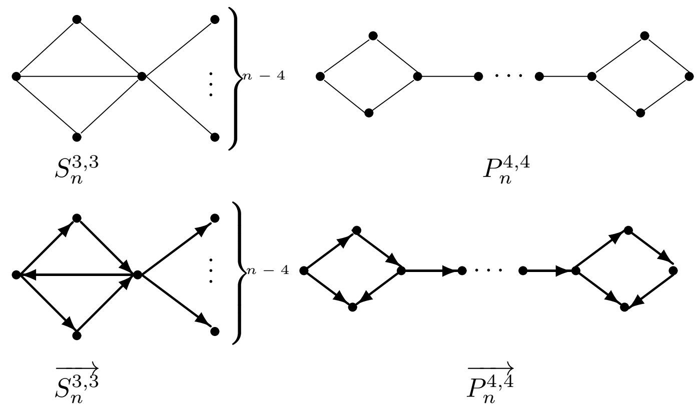

FIG. 1.1. *Graphs* $S_n^{3,3}$, $P_n^{4,4}$ *and their orientations.*

The rest of this paper is organized as follows. In Section 2, the bicyclic digraphs of each order n with minimal skew energy are determined. In Section 3, the bicyclic digraphs of each order n with maximal skew energy are determined.

2. Bicyclic graphs with minimal skew energy. Let G be a graph. A *linear subgraph* L of G is a disjoint union of some edges and some cycles in G ([2]). A k*-matching* M in G is a disjoint union of k-edges. If $2k$ is the order of G, then a k-matching of G is called a *perfect matching* of G. The number of k-matchings of graph G is denoted by $m(G,k)$. If C is an even cycle of G, then we say C is *evenly oriented* relative to an orientation $\overrightarrow{G}$ of G if it has an even number of edges oriented in the direction of the routing. Otherwise C is *oddly oriented*. We call a linear subgraph L of G *evenly linear* if L contains no cycle with odd length and denote by $\mathcal{EL}_i(G)$ (or $\mathcal{EL}_i$ for short) the set of all evenly linear subgraphs of G with i vertices. For a linear subgraph $L \in \mathcal{EL}_i$, denote by $p_e(L)$ (resp., $p_o(L)$) the number of evenly (resp., oddly) oriented cycles in L relative to $\overrightarrow{G}$. Denote the characteristic polynomial of $S(\overrightarrow{G})$ by

$$P_S(\overrightarrow{G};x) = \det(xI - S(\overrightarrow{G})) = \sum_{i=0}^{n} b_i x^{n-i}.$$

Then $b_0 = 1$, b_2 is the number of edges of G, all $b_i \geq 0$ and $b_i = 0$ for all odd i.

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We have the following results.

LEMMA 2.1. ([8]) *Let $\overrightarrow{G}$ be an orientation of a graph G. Then*

$$b_i(\overrightarrow{G}) = \sum_{L\in\mathcal{EL}_i} (-2)^{p_e(L)} 2^{p_o(L)}.$$

LEMMA 2.2. ([8]) *Let $e = uv$ be an edge of G that is on no even cycle of G. Then*

$$P_S(\overrightarrow{G};x) = P_S(\overrightarrow{G}-e;x) + P_S(\overrightarrow{G}-u-v;x). \tag{2.1}$$

By equating the coefficients of polynomials in Eq. (2.1), we have

$$b_{2k}(\overrightarrow{G}) = b_{2k}(\overrightarrow{G}-e) + b_{2k-2}(\overrightarrow{G}-u-v). \tag{2.2}$$

Furthermore, if $e = uv$ is a pendant edge with pendant vertex v, then

$$b_{2k}(\overrightarrow{G}) = b_{2k}(\overrightarrow{G}-v) + b_{2k-2}(\overrightarrow{G}-u-v). \tag{2.3}$$

For any orientation of a graph that does not contain any even cycle (in particular, a tree or a unicyclic non-bipartite graph), $b_{2k}(\overrightarrow{G}) = m(\overrightarrow{G},k)$ by Lemma 2.1.

In [9], the skew energy of $\overrightarrow{G}$ is expressed as the following integral formula:

$$\mathcal{E}_S(\overrightarrow{G}) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{t^2} \ln(1+\sum_{k=1}^{\lfloor \frac{n}{2}\rfloor} b_{2k}t^{2k})dt.$$

Thus $\mathcal{E}_s(\overrightarrow{G})$ is an increasing function of $b_{2k}(\overrightarrow{G})$, $k = 0,1,\ldots,\lfloor\frac{n}{2}\rfloor$. Consequently, if $\overrightarrow{G_1}$ and $\overrightarrow{G_2}$ are oriented graphs of G_1 and G_2, respectively, for which

$$b_{2k}(\overrightarrow{G_1}) \geq b_{2k}(\overrightarrow{G_2}) \tag{2.4}$$

for all $\lfloor\frac{n}{2}\rfloor \geq k \geq 0$, then

$$\mathcal{E}_s(\overrightarrow{G_1}) \geq \mathcal{E}_s(\overrightarrow{G_2}). \tag{2.5}$$

Equality in (2.5) is attained only if (2.4) is an equality for all $\lfloor\frac{n}{2}\rfloor \geq k \geq 0$. If the inequalities (2.4) hold for all k, then we write $G_1 \succeq G_2$ or $G_2 \preceq G_1$. If $G_1 \succeq G_2$, but not $G_2 \succeq G_1$, then we write $G_1 \succ G_2$.

Let $\overrightarrow{G}$ be an orientation of a graph G. Let W be a subset of $V(G)$ and $\overline{W} = V(G)\setminus W$. The orientation $\overrightarrow{G'}$ of G obtained from $\overrightarrow{G}$ by reversing the orientations

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of all arcs between $\overline{W}$ and W is said to be obtained from $\overrightarrow{G}$ by a switching with respect to W. Moreover, two orientations $\overrightarrow{G}$ and $\overrightarrow{G'}$ of a graph G are said to be switching-equivalent if $\overrightarrow{G'}$ can be obtained from $\overrightarrow{G}$ by a sequence of switchings. As noted in [1], since the skew adjacency matrices obtained by a switching are similar, their spectra and hence skew energies are equal.

It is easy to verify that up to switching equivalence there are just two orientations of a cycle C: (1) Just one edge on the cycle has the opposite orientation to that of others, we call it orientation $+$. (2) All edges on the cycle C have the same orientation, we denote this orientation by $-$. So if a cycle is of even length and oddly oriented, then it is equivalent to the orientation $+$; if a cycle is of even length and evenly oriented, then it is equivalent to the orientation $-$. The skew energy of a directed tree is the same as the energy of its underlying tree ([1]). So by switching equivalence, for a unicyclic digraph or bicyclic digraph, we only need to consider the orientations of cycles.

Let C_x, C_y be two cycles in bicyclic graph G with t ($t \geq 0$) common vertices. If $t \leq 1$, then G contains exactly two cycles. If $t \geq 2$, then G contains exactly three cycles. The third cycle is denoted by G_z, where $z = x + y - 2t + 2$. Without loss of generality, assume that $x \leq y \leq z$.

For convenience, we denote by G^+ (resp., G^-) the unicyclic graph on which the orientation of a cycle is of orientation $+$ (resp., $-$), and denote by G^* the unicyclic graph on which the orientation of a cycle is of arbitrary orientation $*$. If $t \leq 1$, we denote by $G^{a,b}$ the bicyclic graph on which cycle C_x is of orientation a and cycle C_y is of orientation b, where $a,\ b \in \{+, -, *\}$. If $t \geq 2$, we denote by $G^{a,b,c}$ the bicyclic graph on which C_x is of orientation a, C_y is of orientation b, C_z is of orientation c, where $a,\ b,\ c \in \{+, -, *\}$. See examples in Fig. 2.2.

For the k-matching number of a graph G, we have the following.

Lemma 2.3. *Let $e = uv$ be an edge of G. Then*

(i) $m(G, k) = m(G - e, k) + m(G - u - v, k - 1)$.
(ii) If G is a forest, then $m(G, k) \leq m(P_n, k)$, $k \geq 1$.
(iii) If H is a subgraph of G, then $m(H, k) \leq m(G, k)$, $k \geq 1$. Moreover, if H is a proper subgraph of G, then the inequality is strict.

We define $m(G, 0) = 1$ and $m(G, k) = 0$ for $k > \frac{n}{2}$.

In [9], the authors discussed the unicyclic digraph with extremal skew energy and established the following.

Lemma 2.4. ([9]) *(1) Among all unicyclic digraphs on n vertices, $\overrightarrow{S_n^3}$ has the*

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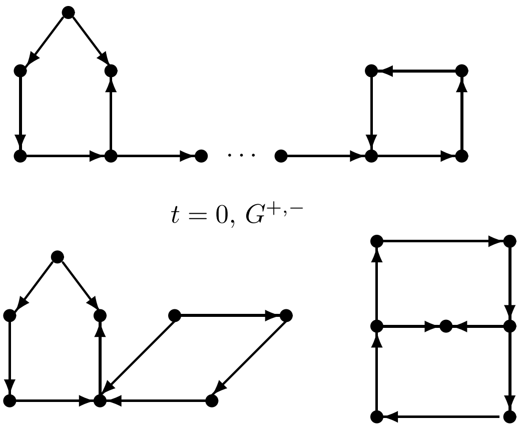

$t = 0$, $G^{+,-}$

$t = 1$, $G^{+,+}$ $t = 2$, $G^{+,+,-}$

FIG. 2.1. *Examples for orientation representations of bicyclic digraphs.*

minimal skew energy and $\overrightarrow{S_n^4}^{-}$ *has the second minimal skew energy for* $n \geq 6$*; both* $\overrightarrow{S_5^3}$ *and* $\overrightarrow{S_5^4}^{-}$ *have the minimal skew energy,* $\overrightarrow{S_5^4}^{+}$ *has the second minimal skew energy for* $n = 5$*;* $\overrightarrow{C^4}^{-}$ *has the minimal skew energy,* $\overrightarrow{S_4^3}$ *has the second minimal skew energy for* $n = 4$.

(2) Among all orientations of unicyclic graphs, $\overrightarrow{P_n^4}^{+}$ *is the unique directed graph with maximal skew energy.*

For bicyclic digraphs, we have the following.

LEMMA 2.5. *Let* $\overrightarrow{G}$ *be a bicyclic digraph of order* $n \geq 8$, $G \neq S_n^{3,3}$. *Then* $\overrightarrow{G} \succ (S_n^{3,3})^{*,*,-}$.

Proof. We prove the statement by induction on n. By Lemma 2.1, the characteristic polynomials of $S((S_n^{3,3})^{*,*,-})$, $S(\overrightarrow{S_n^3})$ and $S(\overrightarrow{S_n^4}^{-})$ are:

$$P_S((S_n^{3,3})^{*,*,-}) = x^{n-4}(x^4 + (n+1)x^2 + 2(n-4)),$$

$$P_S(\overrightarrow{S_n^3}) = x^{n-4}(x^4 + nx^2 + (n-3)),$$

$$P_S(\overrightarrow{S_n^4}^{-}) = x^{n-4}(x^4 + nx^2 + 2(n-4)).$$

It suffices to prove that $b_4(\overrightarrow{G}) > 2(n-4)$ for $\overrightarrow{G} \neq (S_n^{3,3})^{*,*,-}$.

Let $n = 8$.

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Case 1.1. $t \leq 1$.

Subcase 1.1.1. $x = y = 4$. Then we can choose an edge $e = uv$ on some C_4 such that $G - u - v$ is connected. By Lemma 2.1, we have

$$\begin{aligned} b_4(G^{*,*}) &\geq m(G,2) - 4 \\ &= m(G-e,2) + m(G-u-v,1) - 4 \\ &\geq m(G-e,2) + 6 - 4 \\ &> m(P_6,2) + 2 \qquad (P_6 \ \textit{is a proper subgraph of} \ G-e) \\ &= 8 = b_4((S_8^{3,3})^{*,*,-}). \end{aligned}$$

Subcase 1.1.2. Either x or y is 4. Without loss of generality, suppose that $x = 4$. Chose an edge $e = uv$ on C_y such that $G - u - v$ has at least 4 edges. By Lemma 2.1,

$$\begin{aligned} b_4(G^{*,*}) &\geq m(G,2) - 2 \\ &= m(G-e,2) + m(G-u-v,1) - 2 \\ &\geq m(G-e,2) + 4 - 2 \\ &> m(S_6^4,2) + 2 \qquad (S_6^4 \ \textit{is a proper subgraph of} \ G-e) \\ &> 8 = b_4((S_8^{3,3})^{*,*,-}). \end{aligned}$$

Subcase 1.1.3. Neither x nor y is 4. Then $x = y = 3$ or $x = 3$, $y = 5$. We can chose an edge $e = uv$ on any cycle such that $G - u - v$ contains at least 3 edges. By Lemma 2.1, we get

$$\begin{aligned} b_4(G^{*,*}) &= m(G,2) \\ &= m(G-e,2) + m(G-u-v,1) \\ &\geq m(G-e,2) + 3 \\ &= b_4(G-e) + 3 \quad (G-e \ \textit{is a unicyclic graph without cycle of length} \ 4) \\ &> b_4(\overrightarrow{S_8^3}) + 3 \qquad (\textit{by Lemma} \ 2.4) \\ &= 8 = b_4((S_8^{3,3})^{*,*,-}). \end{aligned}$$

Case 1.2. $t \geq 2$.

Subcase 1.2.1. Each cycle is of length 4. Then $t = 3$ and there are 3 vertices outside of those cycles, say v_1, v_2, v_3. Let v_1 be a pendant vertex of G and u_1 be the adjacent vertex of v_1, v_2 be a pendant vertex of $G - v_1$ and u_2 be the adjacent vertex of v_2, v_3 be a pendant vertex of $G - v_1 - v_2$ and u_3 be the adjacent vertex of v_3. By Eq. (2.3), we have

$$b_4(G^{*,*}) = b_4(G^{*,*} - v_1) + b_2(G^{*,*} - u_1 - v_1)$$

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$$
\begin{aligned}
&\geq b_4(G^{*,*}-v_1)+3\\
&\geq b_4(G^{*,*}-v_1-v_2)+6\\
&\geq b_4(G^{*,*}-v_1-v_2-v_3)+9\\
&> 8 = b_4((S_8^{3,3})^{*,*,-}).
\end{aligned}
$$

Subcase 1.2.2. There are two cycles of length 4 in G, say, C_x, C_y, then $t=2$ and $z=x+y-2t+2=6$. There are two vertices outside of G. Similar to the proof of subcase 1.2.1, we can obtain $b_4(G^{*,*}) > b_4((S_8^{3,3})^{*,*,-})$.

Subcase 1.2.3. There is just one cycle of length 4 in G, say, C_x. If $C_y = C_3$, then $t=2$ and there are 3 vertices outside of G. Similar to the proof of subcase 1.2.1, we get $b_4(G^{*,*}) > b_4((S_8^{3,3})^{*,*,-})$. If $y \geq 5$, we can chose an edge $e = uv$ on C_y such that $m(G-u-v,1) \geq 6$. Then similar to the proof of subcase 1.1.2, we get that $b_4(G^{*,*}) > b_4((S_8^{3,3})^{*,*,-})$.

Subcase 1.2.4. G contains no cycle of length 4. Similar to the proof of subcase 1.1.3, the result holds for $n=8$.

Suppose $n > 8$ and $\overrightarrow{G'} \succ (S_{n'}^{3,3})^{*,*,-}$ for any bicyclic digraph G' of order n', $n' < n$. Denote by p the number of pendant vertices in G.

If $p=0$, then $\overrightarrow{G}$ has no pendant vertex. Three cases are considered in the following.

Case 2.1. $t=1$. Let $e=uv$ be an edge on C_x and u is the common vertex of C_x and C_y. By Lemmas 2.1 and 2.3, and $m(P_n,2) = \frac{(n-2)(n-3)}{2}$, we have

$$
\begin{aligned}
b_4(G^{*,*}) &\geq m(G,2)-4\\
&= m(G-e,2)+m(G-u-v,1)-4\\
&= m(P_n^y,2)+m(P_{x-2}\bigcup P_{n-x},1)-4\\
&= m(P_n,2)+m(P_{y-2}\bigcup P_{n-y},1)+n-8\\
&= \frac{(n-2)(n-3)}{2}+n-4+n-8 > 2(n-4)
\end{aligned}
$$

since $\frac{(n-2)(n-3)}{2}-4 = \frac{(n-6)(n+1)+4}{2} > 0$ for $n > 7$.

Case 2.2. $t \geq 2$. Suppose $e=uv$ is an edge on C_x and u is the common vertex of C_x and C_y. By Lemmas 2.1 and 2.3

$$
\begin{aligned}
b_4(G^{*,*}) &\geq m(G,2)-6\\
&= m(G-e,2)+m(G-u-v,1)-6\\
&= m(P_n^y,2)+n-3-6
\end{aligned}
$$

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$$= m(P_n, 2) + m(P_{y-2} \bigcup P_{n-y}, 1) + n - 9$$
$$= \frac{(n-2)(n-3)}{2} + n - 4 + n - 9 > 2(n-4)$$

since $\frac{(n-2)(n-3)}{2} - 5 = \frac{(n-6)(n+1)+2}{2} > 0$ for $n > 7$.

Case 2.3. $t = 0$. Suppose that C_x and C_y are joined by a path of length a, $n-8 \geq a \geq 0$. Let $e = uv$ be an edge on C_x, where u is the vertex of degree 3. Similar to the proof of case 2.1, we obtain $b_4(G^{*,*}) > 2(n-4)$. Therefore, $b_4(G^{*,*}) \geq b_4((S_n^{3,3})^{*,*,-})$ for $p = 0$.

Let $p \geq 1$ and v be a pendant vertex of G with corresponding unique edge uv. Since $G^{*,*} - u - v$ has at least 3 edges, by Eq. (2.2) and the induction hypothesis,

$$\begin{aligned} b_4(G^{*,*}) &= b_4(G^{*,*} - v) + b_2(G^{*,*} - u - v) \\ &> b_4((S_{n-1}^{3,3})^{-,*,*}) + 3 \\ &= 2(n-1-4) + 3 > 2(n-4) = b_4((S_n^{3,3})^{*,*,-}). \quad \square \end{aligned}$$

For $n = 7$, similar to the proof of Lemma 2.5 for $n = 8$, both $(S_7^{4,4})^{-,-,-}$ and $(S_n^{3,3})^{*,*,-}$ have the minimal skew energy. Since

$$P_S((S_n^{4,4})^{-,-,-}) = x^{n-4}(x^4 + (n+1)x^2 + 3(n-5)),$$

$b_4((S_6^{4,4})^{-,-,-}) = 3$ and $b_4((S_5^{4,4})^{-,-,-}) = 0$. In a similar way to the proof of Lemma 2.5 for $n = 8$, we can get that $(S_6^{4,4})^{-,-,-}$ has the minimal skew energy for $n = 6$ and $(S_5^{4,4})^{-,-,-}$ has the minimal skew energy for $n = 5$.

By Lemma 2.5 and the above statements, we obtain the following.

THEOREM 2.6. *Among all bicyclic digraphs of order n, $(S_n^{3,3})^{*,*,-}$ has the minimal skew energy for $n \geq 8$; both $(S_7^{3,3})^{*,*,-}$ and $(S_7^{4,4})^{-,-,-}$ have the minimal skew energy for $n = 7$; $(S_n^{4,4})^{-,-,-}$ has the minimal skew energy for $n = 5, 6$.*

3. Bicyclic digraphs with maximal skew energy. For the path, by Lemmas 2.1 and 2.3, we can easily get the following statements.

LEMMA 3.1. *Let $\overrightarrow{F}_n$ be a forest of order n. Then $\overrightarrow{F}_n \preceq \overrightarrow{P}_n$. Equality holds if and only if $F_n = P_n$.*

Since the skew energy of a directed forest is the same as the energy of its underlying forest, by [6], we have the following.

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LEMMA 3.2.

$$\overrightarrow{P}_n \succ \overrightarrow{P}_2 \bigcup \overrightarrow{P}_{n-2} \succ \overrightarrow{P}_4 \bigcup \overrightarrow{P}_{n-4} \succ \cdots \succ \overrightarrow{P}_{2k} \bigcup \overrightarrow{P}_{n-2k} \succ \overrightarrow{P}_{2k+1} \bigcup \overrightarrow{P}_{n-2k-1}$$
$$\succ \overrightarrow{P}_{2k-1} \bigcup \overrightarrow{P}_{n-2k+1} \succ \cdots \succ \overrightarrow{P}_3 \bigcup \overrightarrow{P}_{n-3} \succ \overrightarrow{P}_1 \bigcup \overrightarrow{P}_{n-1}.$$

LEMMA 3.3. *For any bicyclic graph G with $t \leq 1$, $G^{*,*} \preceq G^{+,+}$.*

Proof. If two cycles are of odd length, then by Lemma 2.1, for any orientation of G, $b_{2k}(\overrightarrow{G}) = m(G,k)$, for all $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$. Thus $G^{*,*} = G^{+,+}$. If there is exactly one cycle of even length in G, say, C_x, then

$$b_{2k}(G^{-,*}) = m(G,k) - 2m(G - C_x, k - \frac{x}{2}) \leq b_{2k}(G^{+,+}) = m(G,k) + 2m(G - C_x, k - \frac{x}{2}).$$

If both x and y are even, then

$$\begin{aligned} b_{2k}(G^{-,-}) &= m(G,k) - 2m(G - C_x, k - \frac{x}{2}) - 2m(G - C_y, k - \frac{y}{2}) \\ &+ 4m(G - C_x - C_y, k - \frac{x+y}{2}) \leq b_{2k}(G^{\pm,\mp}) = m(G,k) \\ &\pm 2m(G - C_x, k - \frac{x}{2}) \mp 2m(G - C_y, k - \frac{y}{2}) \\ &- 4m(G - C_x - C_y, k - \frac{x+y}{2}) \leq b_{2k}(G^{+,+}) = m(G,k) \\ &+ 2m(G - C_x, k - \frac{x}{2}) + 2m(G - C_y, k - \frac{y}{2}) \\ &+ 4m(G - C_x - C_y, k - \frac{x+y}{2}). \quad \square \end{aligned}$$

LEMMA 3.4. *Let $\overrightarrow{G}$ be a bicyclic digraph of order n with $t \leq 1$, $\overrightarrow{G} \neq (P_n^{4,4})^{+,+}$. Then $\overrightarrow{G} \prec (P_n^{4,4})^{+,+}$ for $n \geq 8$.*

Proof. We divide the proof into two cases.

Case 1. There is at least one cycle of length odd, say, C_x.

(i) $t = 1$, we can choose an edge $e = uv$ on C_x such that u is the common vertex of two cycles . Obviously, $\overrightarrow{G} - e$ is a unicyclic graph and $\overrightarrow{G} - u - v$ is a forest.

By Eq. (2.2), Lemmas 2.3 and 2.4, we have

$$\begin{aligned} b_{2k}(\overrightarrow{G}) &= b_{2k}(\overrightarrow{G} - e) + b_{2k-2}(\overrightarrow{G} - u - v) \\ &< b_{2k}(\overrightarrow{P}_n^{4^+}) + b_{2k-2}(\overrightarrow{P}_2 \bigcup \overrightarrow{P}_{n-4}) \qquad (\textit{by Lemmas } 2.4 \textit{ and } 3.2) \\ &= m(P_n^4, k) + 3m(P_{n-4}, k-2) + m(P_{n-4}, k-1) \\ &< m(P_n^4, k) + m(P_{n-4}^4, k-1) + 5m(P_{n-4}^4, k-2) + 4m(P_{n-8}, k-4) \end{aligned}$$

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$$= b_{2k}((P_n^{4,4})^{+,+}).$$

(ii) $t = 0$. We can choose an edge $e = uv$ on C_x such that u is a vertex in a path which connects C_x and C_y. Obviously, $\overrightarrow{G} - e$ is a unicyclic graph and $\overrightarrow{G} - u - v$ is the disjoint union of a forest and a unicyclic graph.

Claim 1. $P_a \bigcup \overrightarrow{P_{n-a}^b}^+ \prec P_2 \bigcup \overrightarrow{P_{n-2}^4}^+$, $a \neq 2$.

Proof. By Lemmas 2.4 and 3.2, we have

$$\begin{aligned}
b_{2k}(P_a \bigcup \overrightarrow{P_{n-a}^b}^+) &< b_{2k}(P_a \bigcup \overrightarrow{P_{n-a}^4}^+) \\
&= m(P_a \bigcup P_{n-a}^4, k) + 2m(P_a \bigcup P_{n-a-4}, k-2) \\
&< m(P_a \bigcup P_{n-a}, k) + m(P_a \bigcup P_2 \bigcup P_{n-a-4}, k-1) \\
&\quad +2m(P_2 \bigcup P_{n-6}, k-2) \\
&< m(P_2 \bigcup P_{n-2}, k) + m(P_2 \bigcup P_2 \bigcup P_{n-6}, k-1) \\
&\quad +2m(P_2 \bigcup P_{n-6}, k-2) \\
&= b_{2k}(P_2 \bigcup \overrightarrow{P_{n-2}^4}^+). \quad \square
\end{aligned}$$

By Eq. (2.2), Lemmas 2.1, 2.3 and 2.4, we have

$$\begin{aligned}
b_{2k}(\overrightarrow{G}) &= b_{2k}(\overrightarrow{G} - e) + b_{2k-2}(\overrightarrow{G} - u - v) \\
&< b_{2k}(\overrightarrow{P_n^4}^+) + b_{2k-2}(P_2 \bigcup \overrightarrow{P_{n-4}^4}^+) \qquad (\textit{by Claim } 1) \\
&= m(P_n^4, k) + 2m(P_{n-4}, k-2) + m(P_2 \bigcup P_{n-4}^4, k-1) \\
&\quad +2m(P_2 \bigcup P_{n-8}, k-2) \\
&\leq m(P_n^4, k) + m(P_2 \bigcup P_{n-4}^4, k-1) + 4m(P_{n-4}, k-2) \\
&\leq m(P_n^4, k) + m(P_2 \bigcup P_{n-4}^4, k-1) + 4m(P_{n-4}^4, k-2) + 4m(P_{n-8}, k-4) \\
&= b_{2k}((P_n^{4,4})^{+,+}).
\end{aligned}$$

Case 2. Two cycles are of even lengths.

By Lemma 3.3, we only need to consider $G^{+,+}$.

(1) $t = 1$. We can choose an edge $e = uv$ in cycle C_x, u is the common vertex of two cycles. By Lemma 2.1, we have

$$b_{2k}(G^{+,+}) = m(G, k) + 2m(G - C_x, k - \frac{x}{2}) + 2m(G - C_y, k - \frac{y}{2})$$

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$$
\begin{aligned}
&\leq m(G-e,k)+m(G-u-v,k-1)+2m(P_{n-4},k-2)\\
&\quad +2m(G-C_y,k-\frac{y}{2})\\
&= b_{2k}(\overrightarrow{G-e}^{+})+m(G-u-v,k-1)+2m(P_{n-4},k-2)\\
&\leq b_{2k}(\overrightarrow{G-e}^{+})+m(P_2\bigcup P_{n-4},k-1)+2m(P_{n-4},k-2))\\
&< b_{2k}(\overrightarrow{P_n^4}^{+})+m(P_2\bigcup P_{n-4},k-1)+2m(P_{n-4},k-2)\\
&= m(P_n^4,k)+m(P_2\bigcup P_{n-4},k-1)+4m(P_{n-4},k-2)\\
&\leq m(P_n^4,k)+m(P_2\bigcup P_{n-4}^4,k-1)+4m(P_{n-4}^4,k-2)+4m(P_{n-8},k-4)\\
&= b_{2k}((P_n^{4,4})^{+,+}).
\end{aligned}
$$

(ii) $t=0$. Choose an edge $e=uv$ on the path which connects C_x and C_y. Assume that one of the two components of $G-e$ is of order j. By Eq. (2.2), we get

$$
\begin{aligned}
b_{2k}(G^{+,+}) &= b_{2k}(G^{+,+}-e)+b_{2k-2}(G^{+,+}-u-v)\\
&< b_{2k}(\overrightarrow{P_j^4}^{+}\bigcup \overrightarrow{P_{n-j}^4}^{+})+b_{2k-2}(\overrightarrow{P_{j-1}^4}^{+}\bigcup \overrightarrow{P_{n-j-1}^4}^{+})\\
&\quad (\ or < b_{2k}(\overrightarrow{P_j^4}^{+}\bigcup \overrightarrow{P_{n-j}^4}^{+})+b_{2k-2}(P_3\bigcup \overrightarrow{P_{n-5}^4}^{+}))\\
&= b_{2k}((P_n^{4,4})^{+,+}). \quad \square
\end{aligned}
$$

Lemma 3.5. *Let $\overrightarrow{G}$ be a bicyclic digraph of order n with $t\geq 2$. Then $\overrightarrow{G}\prec (P_n^{4,4})^{+,+}$ for $n\geq 8$.*

Proof. We prove statement by dividing three cases.

Case 1. $x=y=z=4$. Then $t=3$. If both C_x and C_y are oddly oriented, then C_z must be evenly oriented. We can choose an edge $e=uv$ such that $G-u-v$ is disconnected. Without loss of generality, we assume that e is on C_y.

$$
\begin{aligned}
b_{2k}(G^{+,+,-}) &= m(G,k)+2m(G-C_x,k-2)+2m(G-C_y,k-2)\\
&\quad -2m(G-C_z,k-2)\\
&\leq m(G-e,k)+m(G-u-v,k-1)+2m(G-C_x,k-2)\\
&\quad +2m(G-C_y,k-2)\\
&\leq b_{2k}(\overrightarrow{G}-e)+m(P_2\bigcup P_{n-4},k-1)+2m(P_{n-4},k-2)\\
&< b_{2k}(\overrightarrow{P_n^4}^{+})+m(P_2\bigcup P_{n-4},k-1)+2m(P_{n-4},k-2)\\
&= m(P_n^4,k)+m(P_2\bigcup P_{n-4},k-1)+4m(P_{n-4},k-2)\\
&\leq m(P_n^4,k)+m(P_2\bigcup P_{n-4}^4,k-1)+4m(P_{n-4}^4,k-2)
\end{aligned}
$$

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$$\begin{aligned}
&+4m(P_{n-8}, k-4)\\
&= b_{2k}((P_n^{4,4})^{+,+}).
\end{aligned}$$

If either C_x or C_y is oddly oriented, then C_z must be oddly oriented. Similarly, we can prove that $b_{2k}(G^{+,-,+}) < b_{2k}((P_n^{4,4})^{+,+})$ or $b_{2k}(G^{-,+,+}) < b_{2k}((P_n^{4,4})^{+,+})$.

If both C_x and C_y are evenly oriented, then C_z is also evenly oriented.

$$\begin{aligned}
b_{2k}(G^{-,-,-}) &= m(G,k) - 2m(G-C_x, k-2) - 2m(G-C_y, k-2)\\
&\quad -2m(G-C_z, k-2)\\
&\leq m(G,k) + 2m(G-C_x, k-2) + 2m(G-C_y, k-2)\\
&\quad -2m(G-C_z, k-2)\\
&= b_{2k}(G^{+,-,+}) < b_{2k}((P_n^{4,4})^{+,+}).
\end{aligned}$$

Case 2. $x = y = 4$, $z \neq 4$. Then $t = 2$ and $z = 6$. If both C_x and C_y are oddly oriented, then C_z is oddly oriented. Since $n \geq 8$, we can choose an edge $e = uv$ such that $G - u - v$ is disconnected and u is one of the common vertices between C_x and C_y. Without loss of generality, we suppose that e is on C_y. Then both $G - C_y$ and $G - C_z$ are also disconnected. Note that $G - C_x = G - e - C_x$, by Lemma 2.1, we get

$$\begin{aligned}
b_{2k}(G^{+,+,+}) &= m(G,k) + 2m(G-C_x, k-2) + 2m(G-C_y, k-2)\\
&\quad +2m(G-C_z, k-3)\\
&\leq m(G-e,k) + m(G-u-v, k-1) + 2m(G-e-C_x, k-2)\\
&\quad +2m(P_2 \bigcup P_{n-6}, k-2) + 2m(P_2 \bigcup P_{n-8}, k-3)\\
&\leq b_{2k}(\overrightarrow{G-e}^{+}) + m(P_2 \bigcup P_{n-4}, k-1)\\
&\quad +2m(P_2 \bigcup P_{n-6}, k-2) + 2m(P_2 \bigcup P_{n-8}, k-3)\\
&< b_{2k}(\overrightarrow{P_n^4}^{+}) + m(P_2 \bigcup P_{n-4}, k-1)\\
&\quad +2m(P_2 \bigcup P_{n-6}, k-2) + 2m(P_2 \bigcup P_{n-8}, k-3)\\
&= m(P_n^4, k) + m(P_2 \bigcup P_{n-4}, k-1) + 2m(P_{n-4}, k-2)\\
&\quad +2m(P_2 \bigcup P_{n-6}, k-2) + 2m(P_2 \bigcup P_{n-8}, k-3)\\
&\leq m(P_n^4, k) + m(P_2 \bigcup P_{n-4}^4, k-1) + 4m(P_{n-4}^4, k-2)\\
&\quad +4m(P_{n-8}, k-4)\\
&= b_{2k}((P_n^{4,4})^{+,+}).
\end{aligned}$$

If either C_x or C_y is oddly oriented, then C_z is evenly oriented. By Lemma

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2.1, $b_{2k}(G^{+,-,-}) \leq b_{2k}(G^{+,+,+}) < b_{2k}((P_n^{4,4})^{+,+})$, or $b_{2k}(G^{-,+,-}) \leq b_{2k}(G^{+,+,+}) < b_{2k}((P_n^{4,4})^{+,+})$.

If both C_x and C_y are evenly oriented, then C_z is oddly oriented and

$$b_{2k}(G^{-,-,+}) \leq b_{2k}(G^{+,+,+}) < b_{2k}((P_n^{4,4})^{+,+}).$$

Case 3. There aren't two cycles with length 4. Since there is at least one cycle of even length in $\overrightarrow{G}$, without loss of generality, we assume that C_x is a cycle of minimal even length.

Subcase 3.1. t is even.

Subcase 3.1.1. y is even and both C_x and C_y are oddly oriented. Then $y, z > x \geq 4$ and C_z is oddly oriented. Let $e = uv$ be an edge on C_y and u is the common vertex of C_x and C_y.

Claim 2. $m(P_{n-2}, k-1) \geq m(P_{n-4}, k-2) \geq \cdots \geq m(P_{n-2\ell}, k-\ell)$.

Proof. By Lemma 2.3, we get

$$\begin{aligned} m(P_{n-2}, k-1) &= m(P_{n-3}, k-1) + m(P_{n-4}, k-2) \\ &= m(P_{n-3}, k-1) + m(P_{n-5}, k-2) + m(P_{n-6}, k-3) \\ &= \sum_{i=1}^{\ell-1} m(P_{n-(2i+1)}, k-i) + m(P_{n-2\ell}, k-\ell). \end{aligned}$$

Claim 2 follows immediately. ☐

By Lemma 2.1 and Claim 2, we get

$$\begin{aligned} b_{2k}(G^{+,+,+}) &= m(G, k) + 2m(G - C_x, k-2) + 2m(G - C_y, k - \frac{y}{2}) \\ &\quad +2m(G - C_z, k - \frac{z}{2}) \\ &\leq m(G-e, k) + m(G-u-v, k-1) + 2m(G-e-C_x, k-2) \\ &\quad +4m(P_{n-6}, k-3) \\ &\leq b_{2k}(\overrightarrow{G-e}^{+}) + m(P_{n-2}, k-1) + 4m(P_{n-6}, k-3) \\ &< b_{2k}(\overrightarrow{P_n^4}^{+}) + m(P_{n-2}, k-1) + 4m(P_{n-8} \bigcup P_2, k-3) \\ &\quad +4m(P_{n-9}, k-4) \\ &= m(P_n^4, k) + 2m(P_{n-4}, k-2) + m(P_2 \bigcup P_{n-4}, k-1) + m(P_{n-5}, k-2) \\ &\quad +4m(P_{n-8} \bigcup P_2, k-3) + 4m(P_{n-9}, k-4) \\ &\leq m(P_n^4, k) + m(P_2 \bigcup P_{n-4}^4, k-1) + 4m(P_{n-4}, k-2) \end{aligned}$$

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$$
\begin{aligned}
&+4m(P_{n-8}\bigcup P_2, k-3)+4m(P_{n-8},k-4)\\
&= b_{2k}((P_n^{4,4})^{+,+}).
\end{aligned}
$$

If either C_x or C_y is oddly oriented, then C_z is evenly oriented. By Lemma 2.1, $b_{2k}(G^{+,-,-}) \leq b_{2k}(G^{+,+,+}) < b_{2k}((P_n^{4,4})^{+,+})$, or $b_{2k}(G^{-,+,-}) \leq b_{2k}(G^{+,+,+}) < b_{2k}((P_n^{4,4})^{+,+})$.

If both C_x and C_y are evenly oriented, then C_z is oddly oriented and

$$b_{2k}(G^{-,-,+}) \leq b_{2k}(G^{+,+,+}) < b_{2k}((P_n^{4,4})^{+,+}).$$

Subcase 3.1.2. y is odd. Then z is also odd. We can choose an edge $e = uv$ on C_y such that u is the common vertex and v is not the common vertex. By Lemma 2.1, we get

$$
\begin{aligned}
b_{2k}(G^{-,*,*}) &\leq b_{2k}(G^{+,*,*}) = b_{2k}(\overrightarrow{G-e}^{+}) + b_{2k-2}(\overrightarrow{G}-u-v)\\
&< b_{2k}(\overrightarrow{P_n^4}^{+}) + b_{2k-2}(\overrightarrow{P}_{n-2})\\
&= m(P_n^4,k)+2m(P_{n-4},k-2)+m(P_{n-2},k-1)\\
&< m(P_n^4,k)+m(P_2\bigcup P_{n-4},k-1)+m(P_{n-5},k-2)+2m(P_{n-4},k-2)\\
&\leq m(P_n^4,k)+m(P_2\bigcup P_{n-4}^4,k-1)+4m(P_{n-4}^4,k-2)\\
&\quad +4m(P_{n-8},k-4)\\
&= b_{2k}((P_n^{4,4})^{+,+}).
\end{aligned}
$$

Subcase 3.2. t is odd.

Subcase 3.2.1. y is even. Then $y > 4$. If both C_x and C_y are oddly oriented , then z is even and C_z is evenly oriented. Let $e = uv$ be an edge on C_y and u is the common vertex between C_x and C_y. Then

$$
\begin{aligned}
b_{2k}(G^{+,+,-}) &= m(G,k)+2m(G-C_x,k-2)+2m(G-C_y,k-\frac{y}{2})\\
&\quad -2m(G-C_z,k-\frac{z}{2})\\
&\leq m(G-e,k)+m(G-u-v,k-1)+2m(G-C_x,k-2)\\
&\quad +2m(P_{n-6},k-3)\\
&\leq b_{2k}(\overrightarrow{G-e}^{+})+m(P_{n-2},k-1)+2m(P_{n-6},k-3)\\
&< b_{2k}(\overrightarrow{P_n^4}^{+})+m(P_{n-2},k-1)+2m(P_{n-6},k-3)\\
&= m(P_n^4,k)+2m(P_{n-4},k-2)+m(P_2\bigcup P_{n-4},k-1)+m(P_{n-5},k-2)\\
&\quad +2m(P_{n-6},k-3)
\end{aligned}
$$

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$$\begin{aligned}
&\le m(P_n^4, k) + m(P_2 \bigcup P_{n-4}^4, k-1) + 4m(P_{n-4}^4, k-2) \\
&\quad +4m(P_{n-8}, k-4) \\
&= b_{2k}((P_n^{4,4})^{+,+}).
\end{aligned}$$

If either C_x or C_y is oddly oriented, then C_z is oddly oriented. Similar to the above proof, $b_{2k}(G^{+,-,+}) < b_{2k}((P_n^{4,4})^{+,+})$ or $b_{2k}(G^{-,+,+}) < b_{2k}((P_n^{4,4})^{+,+})$.

If both C_x and C_y are evenly oriented, then C_z is evenly oriented, so

$$b_{2k}(G^{-,-,-}) \le b_{2k}(G^{+,+,-}) < b_{2k}((P_n^{4,4})^{+,+}).$$

Subcase 3.2.2. y is odd. Then z is odd too. Similar to the proof of subcase 3.1.2, we obtain $b_{2k}(G^{-,*,*}) \le b_{2k}(G^{+,*,*}) < b_{2k}((P_n^{4,4})^{+,+})$.

Combining all those cases above, we complete the proof. □

By identifying two vertices of two cycles with length 4, we get a graph $G_7^{4,4}$. For $n = 6$, 7, similar to the proofs of Lemmas 3.4, 3.5, we obtain that the following graphs have the maximal skew energy.

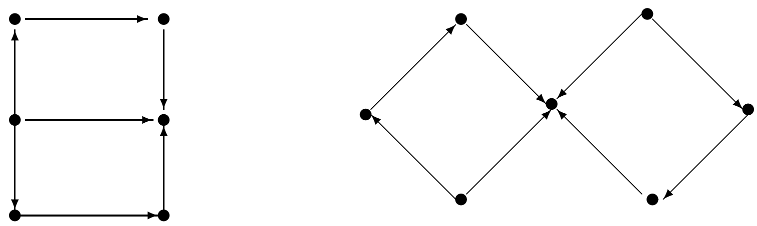

FIG. 3.1. *The maximal skew energy graph* $(P_6^{4,4})^{+,+,+}$ *for* $n = 6$ *and* $(G_7^{4,4})^{+,+}$ *for* $n = 7$.

By Lemmas 3.4 and 3.5, we obtain the following statement.

THEOREM 3.6. *Among all bicyclic digraphs with order* $n \ge 8$, $(P_n^{4,4})^{+,+}$ *has the maximal skew energy;* $(G_7^{4,4})^{+,+}$ *has the the maximal skew energy for* $n = 7$*;* $(P_6^{4,4})^{+,+,+}$ *has the the maximal skew energy for* $n = 6$.

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