Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 295-305, March 2012

ELA

ORDERING TREES BY THE MINIMAL ENTRIES OF THEIR DOUBLY STOCHASTIC GRAPH MATRICES*

SHUCHAO LI† AND QIN ZHAO†

Abstract. Given an n-vertex graph G, the matrix $\Omega(G) = (I_n + L(G))^{-1} = (\omega_{ij})$ is called the doubly stochastic graph matrix of G, where I_n is the $n \times n$ identity matrix and $L(G)$ is the Laplacian matrix of G. Let $\underline{\omega}(G)$ be the smallest element of $\Omega(G)$. Zhang and Wu [X.D. Zhang and J.X. Wu. Doubly stochastic matrices of trees. *Appl. Math. Lett.*, 18:339–343, 2005.] determined the tree T with the minimum $\underline{\omega}(T)$ among all the n-vertex trees. In this paper, as a continuance of the Zhang and Wu's work, we determine the first $\lceil \frac{n-1}{2} \rceil$ trees $T_1, T_2, \ldots, T_{\lceil \frac{n-1}{2} \rceil}$ such that $\underline{\omega}(T_1) < \underline{\omega}(T_2) < \cdots < \underline{\omega}\left(T_{\lfloor \frac{n-1}{2} \rfloor}\right) \le \underline{\omega}\left(T_{\lceil \frac{n-1}{2} \rceil}\right) < \underline{\omega}(T)$, where T_i is obtained by attaching a pendant vertex to v_i of path $P_{n-1} = v_1 v_2 \cdots v_i \cdots v_{n-1}$ and T is an n-vertex tree different from the trees $T_1, T_2, \ldots, T_{\lceil \frac{n-1}{2} \rceil}$.

Key words. Tree, Doubly stochastic graph matrix, Pendant vertex.

AMS subject classifications. 05C50, 05C05.

1. Introduction. Let $G = (V_G, E_G)$ be a simple graph with vertex set V_G and the edge set E_G. Let $\deg_G(v_i)$ or d_i be the degree of vertex v_i and $D(G) = \operatorname{diag}(d_1, d_2, \ldots, d_n)$. Let $A(G)$ be the $n \times n$ *adjacency matrix* whose (i,j)-entry is 1 if $v_i v_j \in E_G$ and 0 otherwise. The matrix $L(G) = D(G) - A(G)$ is called the *Laplacian matrix* of G, which may be dated back to Kirchhoff's Matrix-Tree Theorem and has been extensively studied for the past fifty years (e.g., see [15, 19] and the references therein). It is well known that $L(G)$ is positive semidefinite and singular. Thus, its eigenvalues can be arranged as $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_{n-1} \ge \lambda_n = 0$. The second smallest eigenvalue λ_{n-1}, also denoted by $\alpha(G)$, is known as the *algebraic connectivity* of G; see [10, 15]. It follows from Kirchhoff's Matrix-Tree Theorem that $\lambda_{n-1} > 0$ if and only if G is connected. Since the algebraic connectivity is relevant to important problems regarding the diameter of graphs, the expanding properties of graphs, the combinatorial optimization problems, and the theory of elasticity, etc. (for example, see [18, 19]), it has received much more attention. Recently, there is an excellent survey on algebraic connectivity of graphs written by de Abreu [1], which is referred to the reader for further information.

*Received by the editors on October 3, 2011. Accepted for publication on March 10, 2012. Handling Editor: Michael Tsatsomeros.

†Faculty of Mathematics and Statistics, Central China Normal University, Wuhan 430079, P.R. China (lscmath@mail.ccnu.edu.cn).

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 295-305, March 2012

In the study of chemical information processing, Golender et al. [11] introduced another important matrix: doubly stochastic graph matrix associated with a graph, which may be used to describe some properties of the topological structure of chemical molecules. Let I_n be the $n \times n$ identity matrix and $\Omega(G) = (I_n + L(G))^{-1} = (\omega_{ij})$. It can be verified that $\Omega(G)$ is a doubly stochastic matrix (see, for example, [11]). Thus, $\Omega(G)$ is called the *doubly stochastic graph matrix.* In [20] Pereira and Vali studied the spectra of doubly stochastic matrices. On the other hand, Chebotarev and Shamis [6] and Chebotarev [5] pointed out that the doubly stochastic graph matrix may be used to measure the proximity among vertices and evaluate the group cohesion in the construction of sociometric indices and represent a random walk. In particular, the diagonal entries ω_{ii} of $\Omega(G)$ measure the peripherality (solitariness) of v_i [6]. An important result on the relationship between the entry of the doubly stochastic matrix of a graph and its number of spanning forests, which is called the matrix forest theorem (1995) (see Refs. [7, 8]), is due to Chebotarev and Shamis. Shamis presented the result at the 1995 International Linear Algebra Society (ILAS) Conference in Atlanta (see Ref. [9]), and a proof of an earlier version of this theorem can be found in [21]. Moreover, Merris [17] established the relationship between the entry ω_{ij} of $\Omega(G)$ and structure of the simple n-vertex graph G: If $\omega_{ij} < 4/(n^2+4n)$, then v_i is not adjacent to v_j. Moreover, ω_{ij} is also relative to the weights of routes of various lengths between v_i and v_j (see [6], Proposition 10). Zhang [23] determined a sharp lower bound for the smallest entries, among those corresponding to edges, of doubly stochastic matrices of trees. Zhang [24] also presented some relations between the diameter of a tree and the smallest entry $\underline{\omega}(G) = \min\{\omega_{ij} : 1 \le i, j \le n\}$. Recently, Zhang [25] studied the relationship between vertex degrees and entries of the doubly stochastic graph matrix.

Zhang and Wu [22] obtained sharp upper and lower bounds for the smallest entries of doubly stochastic matrices of trees and characterized all extreme graphs which attain the bounds. Motivated by these related results, we investigate the property of the smallest entries of doubly stochastic matrices of graphs. We identify the first $\lceil \frac{n-1}{2} \rceil$ n-vertex trees according to their smallest entries in the corresponding doubly stochastic matrices:

$$\underline{\omega}(T_1) < \underline{\omega}(T_2) < \cdots < \underline{\omega}\left(T_{\lfloor \frac{n-1}{2} \rfloor}\right) \le \underline{\omega}\left(T_{\lceil \frac{n-1}{2} \rceil}\right) < \underline{\omega}(T),$$

where T_i is obtained by attaching a pendant vertex to v_i of path $P_{n-1} = v_1 v_2 \cdots v_i \cdots v_{n-1}$ and T is an n-vertex tree different from the trees $T_1, T_2, \ldots, T_{\lceil \frac{n-1}{2} \rceil}$.

2. Preliminaries. We define $\overline{\omega}(G) = \max\{\omega_{ii} : 1 \le i \le n\}$. A *dominating vertex* of G is a vertex of degree $n-1$, i.e., a vertex adjacent to every other vertex. We call v a *pendant vertex*, if $\deg_G(v) = 1$. For convenience, let $PV(G)$ be the set of all pendant vertices of G, and we set $\mathscr{T}_{n,k} = \{T :\ T$ is an n-vertex tree with k pendant

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 295-305, March 2012

vertices}. The *distance* between vertices u and v in G is denoted by $d_G(u, v)$. Let $N_G(v) = \{w \in V_G | vw \in E_G\}$. Clearly, $\deg_G(v) = |N_G(v)|$.

We first give some lemmas that will be used in the proofs of our main results.

LEMMA 2.1 ([17]). *Let G be a graph with doubly stochastic graph matrix $\Omega(G) = (\omega_{ij})$. If $r \neq s$, then $\omega_{rr} \geq 2\omega_{rs}$, with equality if v_r is a dominating vertex of the connected component of G that contains v_s, or if $\deg_G(v_s) = 1$ and $v_r v_s \in E_G$.*

LEMMA 2.2 ([25]). *Let G be a simple connected graph on n vertices with doubly stochastic graph matrix $\Omega(G) = (\omega_{ij})$. Then*

$$\overline{\omega}(G) \leq \frac{\phi^{2n-1} + \phi^{1-2n}}{\phi^{2n} - \phi^{-2n}}$$

with equality if and only if G is a path on n vertices, where $\phi = \frac{\sqrt{5}+1}{2}$.

LEMMA 2.3. *Let P_n be a path on n vertices with doubly stochastic matrix $\Omega(P_n) = (\omega_{ij})$. Then*

$$\omega_{11} > \omega_{22} > \cdots > \omega_{\lfloor \frac{n}{2} \rfloor \lfloor \frac{n}{2} \rfloor} \geq \omega_{\lceil \frac{n}{2} \rceil \lceil \frac{n}{2} \rceil},$$

the last equality holds if and only if n is even.

Proof. This result immediately follows from Theorem 1 in [5] and the main theorem in [2]. ◻

LEMMA 2.4 ([24]). *Let T be a tree of order n with vertex set $V_T = \{v_1, v_2, \ldots, v_n\}$. If $\underline{\omega}(T) = \min\{\omega_{ij} : 1 \leq i, j \leq n\} = \omega_{kl}$, then the following hold:*

(i) *v_k and v_l are two pendant vertices.*

(ii) *If the diameter d of T is no more than 4, then $d_G(v_k, v_l) = d$.*

LEMMA 2.5 ([23]). *Let T be a tree of order $n \geq 4$ with at least three pendant vertices, say, v_1, v_2, v_n. Suppose that $v_3 \in N_T(v_2)$ and $T' = T - v_2v_3 + v_1v_2$. Let $\Omega(T) = (\omega_{ij})$ and $\Omega(T') = (\omega'_{ij})$. Then $\omega_{1n} > \omega'_{2n}$.*

COROLLARY 2.6. *Let $T \in \mathscr{T}_{n,k}$ be a tree such that $\underline{\omega}(T)$ is as small as possible, where $n \geq 4$ and $k \geq 3$ are fixed. Suppose that $\underline{\omega}(T) = \omega_{1n}$, v_2 is a pendant vertex of T different from v_1, v_n and $v_3 \in N_T(v_2)$, then we have $\deg_T(v_3) > 2$.*

Proof. Suppose to the contrary that $\deg_T(v_3) = 2$. Since $\underline{\omega}(T) = \omega_{1n}$ by Lemma 2.4(i), we have that $v_1, v_n \in PV(T)$. Let

$$T' = T - v_2v_3 + v_1v_2.$$

Then $T' \in \mathscr{T}_{n,k}$ with doubly stochastic graph matrix $\Omega(T') = (\omega'_{ij})$. By Lemma 2.5,

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 295-305, March 2012

ELA

we obtain

$$\underline{\omega}(T') \leq \omega'_{2n} < \omega_{1n} = \underline{\omega}(T).$$

A contradiction to the choice of T. □

COROLLARY 2.7. *For any tree $T \in \mathscr{T}_{n,k+1}$, there exists a tree $T' \in \mathscr{T}_{n,k}$ such that $\underline{\omega}(T) > \underline{\omega}(T')$, where $n \geq 4,\ k \geq 2$.*

Proof. Choose $T \in \mathscr{T}_{n,k+1}$ such that $\underline{\omega}(T)$ is as small as possible. Assume that $\underline{\omega}(T) = \omega_{1n}$, then $v_1, v_n \in PV(T)$. Note that $k \geq 2$. Let v_2 be a pendant vertex of T different from v_1, v_n and $v_3 \in N_T(v_2)$. Then, by Corollary 2.6, we have $\deg_T(v_3) > 2$. Let

$$T' = T - v_2v_3 + v_1v_2.$$

Then $T' \in \mathscr{T}_{n,k}$ with doubly stochastic graph matrix $\Omega(T') = (\omega'_{ij})$. By Lemma 2.5, we obtain

$$\underline{\omega}(T') \leq \omega'_{2n} < \omega_{1n} = \underline{\omega}(T).$$

Thus, for any tree $T \in \mathscr{T}_{n,k+1}$, there exists a tree $T' \in \mathscr{T}_{n,k}$ such that $\underline{\omega}(T) > \underline{\omega}(T')$, as desired. □

3. Main results. Let G be a simple graph obtained from graph G^* by attaching k pendant vertices to a vertex of G^*; see Fig. 3.1. Then we characterize the $\underline{\omega}(G)$ by the entries of $\Omega(G^*)$.

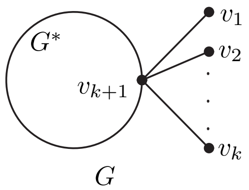

FIG. 3.1. *Graph G.*

THEOREM 3.1. *Let G be a graph of order n, which has $k \geq 1$ pendant vertices $v_1, \ldots, v_k$ attached to the same vertex v_{k+1} of G^* (see Fig. 3.1). Let $\Omega(G) = (\omega_{ij})$ for $1 \leq i, j \leq n$ and $\Omega(G^*) = (\omega^*_{ij})$ for $k+1 \leq i, j \leq n$. Then the smallest entry of $\Omega(G)$ satisfies*

$$\underline{\omega}(G) = \min\left\{\omega^*_{ij} - \frac{k\omega^*_{i,k+1}\omega^*_{k+1,j}}{2 + k\omega^*_{k+1,k+1}},\quad \frac{\omega^*_{k+1,j}}{2 + k\omega^*_{k+1,k+1}}\right\}.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 295-305, March 2012

Proof. Let

$$\mathbf{x} = \Bigg(\overbrace{-\frac{1}{\sqrt{k}} \ \cdots \ -\frac{1}{\sqrt{k}}}^{k} \ \sqrt{k} \ 0 \ \cdots \ 0 \Bigg)^{\mathrm{T}}.$$

Then $\Omega(G) = (I_n + L(G))^{-1} = (F + \mathbf{x}\mathbf{x}^{\mathrm{T}})^{-1}$ with

$$F = \begin{pmatrix} K & \mathbf{0} \\ \mathbf{0} & I_{n-k} + L(G^*) \end{pmatrix},$$

where

$$K = \begin{pmatrix} \frac{2k-1}{k} & -\frac{1}{k} & \cdots & -\frac{1}{k} \\ -\frac{1}{k} & \frac{2k-1}{k} & \cdots & -\frac{1}{k} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{k} & -\frac{1}{k} & \cdots & \frac{2k-1}{k} \end{pmatrix}.$$

Hence, we have

$$F^{-1} = \begin{pmatrix} K^{-1} & \mathbf{0} \\ \mathbf{0} & \Omega(G^*) \end{pmatrix} = \left(\begin{array}{cccc|ccc} \frac{k+1}{2k} & \frac{1}{2k} & \cdots & \frac{1}{2k} & & & \\ \frac{1}{2k} & \frac{k+1}{2k} & \cdots & \frac{1}{2k} & & \mathbf{0} & \\ \vdots & \vdots & \ddots & \vdots & & & \\ \frac{1}{2k} & \frac{1}{2k} & \cdots & \frac{k+1}{2k} & & & \\ \hline & & & & \omega^*_{k+1,k+1} & \cdots & \omega^*_{k+1,n} \\ & \mathbf{0} & & & \vdots & \ddots & \vdots \\ & & & & \omega^*_{n,k+1} & \cdots & \omega^*_{nn} \end{array}\right).$$

By the Sherman–Morrison formula (see [13, p 19]), we have

$$\Omega(G) = F^{-1} - \frac{F^{-1}\mathbf{x}\mathbf{x}^{\mathrm{T}}F^{-1}}{1 + \mathbf{x}^{\mathrm{T}}F^{-1}\mathbf{x}}.$$

Note that $1 + \mathbf{x}^{\mathrm{T}}F^{-1}\mathbf{x} = 2 + k\omega^*_{k+1,k+1}$ and

$$F^{-1}\mathbf{x}\mathbf{x}^{\mathrm{T}}F^{-1} = \Bigg(\overbrace{-\frac{1}{\sqrt{k}} \cdots -\frac{1}{\sqrt{k}}}^{k} \ \sqrt{k}\omega^*_{k+1,k+1} \cdots \sqrt{k}\omega^*_{n,k+1} \Bigg)^{\mathrm{T}}$$
$$\times \Bigg(\overbrace{-\frac{1}{\sqrt{k}} \cdots -\frac{1}{\sqrt{k}}}^{k} \ \sqrt{k}\omega^*_{k+1,k+1} \cdots \sqrt{k}\omega^*_{n,k+1} \Bigg).$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 295-305, March 2012

ELA

Hence, we have

$$\omega_{ij} = \begin{cases} \frac{1}{2k} - \frac{1}{k(2+k\omega^*_{k+1,k+1})}, & \text{if} \quad i \neq j \leq k; \\ \omega^*_{ij} - \frac{k\omega^*_{i,k+1}\omega^*_{k+1,j}}{2+k\omega^*_{k+1,k+1}}, & \text{if} \quad i \neq j \geq k+1; \\ \frac{\omega^*_{k+1,j}}{2+k\omega^*_{k+1,k+1}}, & \text{if} \quad i \leq k < j \end{cases} \tag{3.1}$$

and

$$\omega_{ii} = \begin{cases} \frac{k+1}{2k} - \frac{1}{k(2+k\omega^*_{k+1,k+1})}, & \text{if} \quad i \leq k; \\ \omega^*_{ii} - \frac{k\omega^{*2}_{i,k+1}}{2+k\omega^*_{k+1,k+1}}, & \text{if} \quad i \geq k+1. \end{cases} \tag{3.2}$$

By Lemma 2.1, it is straightforward to check that $\frac{1}{2k} - \frac{1}{k(2+k\omega^*_{k+1,k+1})} > \frac{\omega^*_{k+1,j}}{2+k\omega^*_{k+1,k+1}}$ for $k \geq 1$, and hence

$$\underline{\omega}(G) = \min\left\{\omega^*_{ij} - \frac{k\omega^*_{i,k+1}\omega^*_{k+1,j}}{2+k\omega^*_{k+1,k+1}}; \quad \frac{\omega^*_{k+1,j}}{2+k\omega^*_{k+1,k+1}}\right\}.$$

This completes the proof. ∎

Corollary 3.2. *Let G be a simple connected graph on n vertices with doubly stochastic graph matrix $\Omega(G) = (\omega_{ij})$. If v_1 is a pendant vertex of G and $v_1v_i \in E_G$, then*

$$\omega_{ii} \leq \frac{2\overline{\omega}(P_{n-1})}{2+\overline{\omega}(P_{n-1})}$$

with equality if and only if G is a path on n vertices.

Proof. Let $G^* = G - v_1v_i$ with doubly stochastic graph matrix $\Omega(G^*) = (\omega^*_{ij})$ for $2 \leq i, j \leq n$. By Lemma 2.2, we have $\omega^*_{ii} \leq \overline{\omega}(P_{n-1})$ with equality if and only if G^* is a path on $n-1$ vertices and v_i is a pendant vertex of G^*. Therefore, by Eq. (3.2) in Theorem 3.1, we have

$$\omega_{ii} = \omega^*_{ii} - \frac{{\omega^*_{ii}}^2}{2+\omega^*_{ii}} = \frac{1}{\frac{1}{\omega^*_{ii}} + \frac{1}{2}} \leq \frac{1}{\frac{1}{\overline{\omega}(P_{n-1})} + \frac{1}{2}} = \frac{2\overline{\omega}(P_{n-1})}{2+\overline{\omega}(P_{n-1})}$$

with equality if and only if G^* is a path on $n-1$ vertices and v_i is a pendant vertex of G^*, which is equivalent to that G is a path on n vertices. ∎

Corollary 3.3. *For any P_i, we have*

(i) $\underline{\omega}(P_{i+1}) = \frac{\underline{\omega}(P_i)}{2+\overline{\omega}(P_i)}$ *and* $\overline{\omega}(P_{i+1}) = \frac{1+\overline{\omega}(P_i)}{2+\overline{\omega}(P_i)}$, *where $i = 1, 2, \ldots, n-1$.*

(ii) $\overline{\omega}^2(P_i) + \overline{\omega}(P_i) - \underline{\omega}^2(P_i) - 1 = 0$, *where $i = 1, 2, \ldots, n-1$.*

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 295-305, March 2012

Proof. By Theorem 1 in [5], $\underline{\omega}(P_i) = \Phi_{2i}^{-1}$ and $\overline{\omega}(P_i) = \frac{\Phi_{2i-1}}{\Phi_{2i}}$, where Φ_k are the Fibonacci numbers. This yields (i) and, using Cassini's identity, (ii). □

Let T_i (see Fig. 3.2) be the tree on n vertices obtained from a path $P_{n-1} = v_1 v_2 \cdots v_i \cdots v_{n-1}$ by attaching a new pendant vertex v_n at v_i.

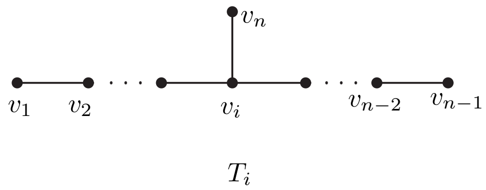

FIG. 3.2. *Graph T_i.*

THEOREM 3.4. *Let T_i $\left(i = 1, 2, \ldots, \lceil \frac{n-1}{2} \rceil\right)$ be a tree of order n, which is depicted in Fig. 3.2, then we have*

$$\underline{\omega}(T_1) < \underline{\omega}(T_2) < \cdots < \underline{\omega}\left(T_{\lfloor \frac{n-1}{2} \rfloor}\right) \leq \underline{\omega}\left(T_{\lceil \frac{n-1}{2} \rceil}\right)$$

with equality if and only if n is odd.

Proof. Let $\mathbf{x}_i$ be an n dimensional vector whose only nonzero component is 1 in the ith position and $\mathbf{x} = \mathbf{x}_n - \mathbf{x}_i$. Thus, $L(T_i) = L(F) + \mathbf{x}\mathbf{x}^{\mathrm{T}}$. Let $\Omega(T_i) = (\omega_{ij})$ and $\Omega(P_{n-1}) = (\omega'_{ij})$. By the Sherman-Morrison formula (see, e.g. [13, p 19]), we have

$$\Omega(T_i) = \Omega(F) - \frac{\Omega(F)\mathbf{x}\mathbf{x}^{\mathrm{T}}\Omega(F)}{1 + \mathbf{x}^{\mathrm{T}}\Omega(F)\mathbf{x}}.$$

By Lemma 2.4(i), we have $\underline{\omega}(T_i) = \min\{\omega_{1,n-1}, \omega_{1n}, \omega_{n-1,n}\}$. Note that

$$\omega_{1n} = \frac{\omega_{1i}\omega_{in}}{\omega_{ii}} = \frac{\omega_{1i}}{2} > \frac{\omega_{1i}\omega_{i,n-1}}{\omega_{ii}} = \omega_{1,n-1},$$
$$\omega_{n-1,n} = \frac{\omega_{n-1,i}\omega_{in}}{\omega_{ii}} = \frac{\omega_{n-1,i}}{2} > \frac{\omega_{1i}\omega_{i,n-1}}{\omega_{ii}} = \omega_{1,n-1}.$$

Hence, we obtain

$$\underline{\omega}(T_i) = \omega_{1,n-1} = \omega'_{1,n-1} - \frac{\omega'_{1i}\omega'_{i,n-1}}{2 + \omega'_{ii}} = \omega'_{1,n-1} - \frac{\omega'_{1,n-1}\omega'_{ii}}{2 + \omega'_{ii}} = \frac{2\omega'_{1,n-1}}{2 + \omega'_{ii}} = \frac{2\underline{\omega}(P_{n-1})}{2 + \omega'_{ii}}.$$

By Lemma 2.3, we have $\underline{\omega}(T_1) < \underline{\omega}(T_2) < \cdots < \underline{\omega}\left(T_{\lfloor \frac{n-1}{2} \rfloor}\right) \leq \underline{\omega}\left(T_{\lceil \frac{n-1}{2} \rceil}\right)$ with equality if and only if $n-1$ is even, which is equivalent to that n is odd. □

THEOREM 3.5. *Let $T \in \mathscr{T}_{n,4}$ and $T_i \in \mathscr{T}_{n,3}$ where T_i $\left(i = 1, 2, \ldots, \lceil \frac{n-1}{2} \rceil\right)$ is depicted in Fig. 3.2, then we have $\underline{\omega}(T_i) < \underline{\omega}(T)$.*

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 295-305, March 2012

Proof. Let $T \in \mathscr{T}_{n,4}$ with doubly stochastic graph matrix $\Omega(T) = (\omega_{ij})$, such that $\underline{\omega}(T)$ is as small as possible. By Corollary 2.6, we obtain that $T \in \{\hat{T}_1, \hat{T}_2, \hat{T}_3\}$; see Fig. 3.3. In what follows, if two graphs, say G and H, are *isomorphic* graphs, then we write $G \cong H$.

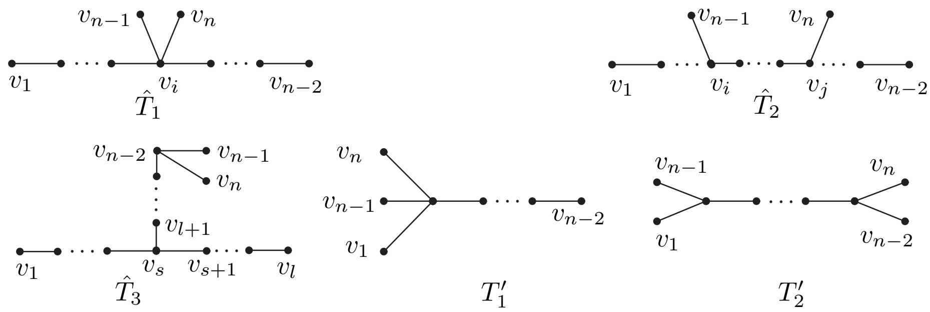

FIG. 3.3. *Graphs* $\hat{T}_1, \hat{T}_2, \hat{T}_3, T_1'$ *and* T_2'.

If $T \cong \hat{T}_1$, then $T - v_iv_{n-1} - v_iv_n \cong P_{n-2}$. Let $\Omega(P_{n-2}) = (\omega'_{pq})$. By Theorem 3.1, [14, Theorem 3.2], and Lemma 2.3, we have

$$\begin{aligned}\underline{\omega}(T) = \omega_{1,n-2} &= \omega'_{1,n-2} - \frac{2\omega'_{1i}\omega'_{i,n-2}}{2+2\omega'_{ii}} = \omega'_{1,n-2} - \frac{\omega'_{ii}\omega'_{1,n-2}}{1+\omega'_{ii}}\\ &= \frac{\omega'_{1,n-2}}{1+\omega'_{ii}} \geq \frac{\omega'_{1,n-2}}{1+\omega'_{22}} = \underline{\omega}(T_1').\end{aligned}$$

If $T \cong \hat{T}_2$, then $T - v_jv_n \cong T_i$. Let $\Omega(T_i) = (\omega''_{ij})$. By Theorem 3.1 and [14, Theorem 3.2], we have

$$\underline{\omega}(T) = \omega_{1,n-2} = \omega''_{1,n-2} - \frac{\omega''_{1i}\omega''_{i,n-2}}{2+\omega''_{ii}} = \omega''_{1,n-2} - \frac{\omega''_{ii}\omega''_{1,n-2}}{2+\omega''_{ii}} = \frac{2\omega''_{1,n-2}}{2+\omega''_{ii}}.$$

Since $\omega''_{ii} = \omega'_{ii} - \frac{\omega'^2_{ij}}{2+\omega'_{jj}}$, $\omega''_{1,n-2} = \frac{2\omega'_{1,n-2}}{2+\omega'_{jj}}$ and $\omega'_{ij} = \frac{\omega'_{ii}\omega'_{jj}\omega'_{1,n-2}}{\omega'_{1i}\omega'_{j,n-2}}$, together with Lemma 2.1, [14, Theorem 3.2], and Lemma 2.3, we obtain

$$\begin{aligned}\underline{\omega}(T) &= \frac{4\omega'_{1,n-2}}{4+2\omega'_{ii}+2\omega'_{jj}+\omega'_{ii}\omega'_{jj}-\omega'^2_{ij}} = \frac{4\omega'_{1,n-2}}{4+2\omega'_{ii}+2\omega'_{jj}+\omega'_{ii}\omega'_{jj}-\left(\frac{\omega'_{ii}\omega'_{jj}\omega'_{1,n-2}}{\omega'_{1i}\omega'_{j,n-2}}\right)^2}\\ &\geq \frac{4\omega'_{1,n-2}}{4+2\omega'_{ii}+2\omega'_{jj}+\omega'_{ii}\omega'_{jj}-\left(\frac{4\omega'_{1i}\omega'_{j,n-2}\omega'_{1,n-2}}{\omega'_{1i}\omega'_{j,n-2}}\right)^2}\\ &= \frac{4\omega'_{1,n-2}}{4+2\omega'_{ii}+2\omega'_{jj}+\omega'_{ii}\omega'_{jj}-16\omega'^2_{1,n-2}}\\ &\geq \frac{4\omega'_{1,n-2}}{4+2\omega'_{22}+2\omega'_{n-3,n-3}+\omega'_{22}\omega'_{n-3,n-3}-16\omega'^2_{1,n-2}} = \underline{\omega}(T_2').\end{aligned}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 295-305, March 2012

If $T \cong \hat{T}_3$, let $T_s^* = T - v_i v_{n-1} - v_i v_n$ and $\Omega(T_s^*) = (\omega_{pq}^*)$. Then $T_s^* \in \mathscr{T}_{n-2,3}$ and $\underline{\omega}(P_{n-2}) < \omega_{pq}^* \le 1$ for $1 \le p, q \le n-2$. By Theorem 3.1, [14, Theorem 3.2], Lemma 2.1, and Corollary 3.2, we have

$$\begin{aligned}
\underline{\omega}(T) &= \omega_{1l} = \omega_{1l}^* - \frac{2\omega_{1,n-2}^*\omega_{n-2,l}^*}{2 + 2\omega_{n-2,n-2}^*} \\
&= \omega_{1l}^* - \frac{\frac{\omega_{1s}^*\omega_{s,n-2}^*}{\omega_{ss}^*}\frac{\omega_{s,n-2}^*\omega_{sl}^*}{\omega_{ss}^*}}{1 + \omega_{n-2,n-2}^*} = \omega_{1l}^*\left[1 - \frac{\omega_{s,n-2}^{*2}}{(1+\omega_{n-2,n-2}^*)\omega_{ss}^*}\right] \\
&\ge \omega_{1l}^*\left[1 - \frac{\omega_{s,n-2}^*}{2(1+\omega_{n-2,n-2}^*)}\right] > \omega_{1l}^*\left(1 - \frac{\omega_{n-2,n-2}^*}{1+\omega_{n-2,n-2}^*}\right) = \frac{\omega_{1l}^*}{1+\omega_{n-2,n-2}^*} \\
&> \frac{\underline{\omega}(P_{n-2})}{1+\omega_{n-2,n-2}^*} \ge \frac{\underline{\omega}(P_{n-2})}{2}.
\end{aligned}$$

For $1 \le i \le \lceil \frac{n-1}{2} \rceil$, by Theorem 3.1 and Corollary 3.3, we get

$$\underline{\omega}(T_i) = \frac{2\omega_{1,n-1}'}{2+\omega_{ii}'} = \frac{2\underline{\omega}(P_{n-1})}{2+\omega_{ii}'} = \frac{2}{2+\omega_{ii}'}\,\frac{\underline{\omega}(P_{n-2})}{2+\overline{\omega}(P_{n-2})} < \frac{\underline{\omega}(P_{n-2})}{2}.$$

Therefore, $\underline{\omega}(T_i) < \underline{\omega}(T)$ for any tree $T \cong \hat{T}_3$. Furthermore, note that

$$\begin{aligned}
\underline{\omega}(T_i) &= \frac{2}{2+\omega_{ii}'}\,\frac{\underline{\omega}(P_{n-2})}{2+\overline{\omega}(P_{n-2})} = \frac{2}{2+\omega_{ii}'}\,\frac{\underline{\omega}(P_{n-3})}{5+3\overline{\omega}(P_{n-3})} \\
&= \frac{2}{2+\omega_{ii}'}\,\frac{\underline{\omega}(P_{n-4})}{13+8\overline{\omega}(P_{n-4})}, \\
\underline{\omega}(T_1') &= \frac{\omega_{2,n-2}'}{2+3\omega_{22}'} = \frac{\underline{\omega}(P_{n-3})}{2+3\overline{\omega}(P_{n-3})} = \frac{\underline{\omega}(P_{n-4})}{7+5\overline{\omega}(P_{n-4})}, \\
\underline{\omega}(T_2') &= \frac{\omega_{2,n-2}'}{2+2\omega_{22}'} = \frac{\frac{\underline{\omega}(P_{n-4})}{2+2\overline{\omega}(P_{n-4})}}{2+2\left[\overline{\omega}(P_{n-4}) - \frac{2\underline{\omega}^2(P_{n-4})}{2+2\overline{\omega}(P_{n-4})}\right]} \\
&= \frac{\underline{\omega}(P_{n-4})}{4[\overline{\omega}(P_{n-4}) + \underline{\omega}(P_{n-4}) + 1][\overline{\omega}(P_{n-4}) - \underline{\omega}(P_{n-4}) + 1]} \\
&= \frac{\underline{\omega}(P_{n-4})}{8+4\overline{\omega}(P_{n-4})},
\end{aligned}$$

and hence it is easy to obtain $\underline{\omega}(T_i) < \underline{\omega}(T_2') < \underline{\omega}(T_1')$ for $\underline{\omega}(P_{n-4}) < \overline{\omega}(P_{n-4}) < 1$. Thus, $\underline{\omega}(T_i) < \underline{\omega}(T)$, where $T \in \mathscr{T}_{n,4}$ and $i = 1, 2, \ldots, \lceil \frac{n-1}{2} \rceil$. ∎

Our last main result of this paper is to generalize [22, Theorem 2.1]. We identify the first $\lceil \frac{n-1}{2} \rceil$ trees on n vertices according to their smallest entries in the corresponding doubly stochastic matrices.

THEOREM 3.6. *Let T be a tree of order n. If $T \ncong T_i$, where $T_i\,(i = 1, 2, \ldots,$*

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 295-305, March 2012

$\lceil \frac{n-1}{2} \rceil$) *is depicted in Fig. 3.2, then we have*

$$\underline{\omega}(T_1) < \underline{\omega}(T_2) < \cdots < \underline{\omega}\left(T_{\lfloor \frac{n-1}{2} \rfloor}\right) \leq \underline{\omega}\left(T_{\lceil \frac{n-1}{2} \rceil}\right) < \underline{\omega}(T)$$

with equality if and only n is odd.

Proof. Theorem 3.6 follows directly from Corollary 2.7, Theorem 3.4, and Theorem 3.5. This completes the proof. □

Acknowledgment. This work is financially supported by the National Natural Science Foundation of China (grant no. 11071096) and the Special Fund for Basic Scientific Research of Central Colleges (CCNU11A02015). The authors would like to express their sincere gratitude to the referees for their careful reading of the paper and for all their insightful comments and valuable suggestions, which led to a number of improvements in this paper.

REFERENCES

[1] N.M.M. de Abreu. Old and new results on algebraic connectivity of graphs. *Linear Algebra Appl.*, 423:53–73, 2007.

[2] K.T. Atanassov, R. Knott, K. Ozeki, A.G. Shannon, and L. Szalay. Inequalities among related pairs of Fibonacci numbers. *Fibonacci Quart.*, 41(1):20–22, 2003.

[3] A. Berman and R.J. Plemmons. Nonnegative Matrices in the Mathematical Sciences. Academic Press, SIAM, 1994.

[4] A. Berman, X.D. Zhang. A note on the degree antiregular graphs. *Linear Multilinear Algebra*, 47:307–311, 2000.

[5] P.Y. Chebotarev. Spanning forests and the golden ratio. *Discrete Appl. Math.*, 156(5):813–821, 2008.

[6] P.Y. Chebotarev and E. V. Shamis. A martix-forest theorem and the measurement of relations in small social groups. *Automat. Remote Control*, 58(9):1505–1514, 1997.

[7] P.Y. Chebotarev and E.V. Shamis. On the proximity measure for graph vertices provided by the inverse Laplacian characteristic matrix. Abstracts of the Conference *Linear Algebra and its Applications*, University of Manchester, 6–7, 10–12 July, 1995. Available at `http://www.maths.manchester.ac.uk/ higham/laa95/abstracts.ps`.

[8] P.Y. Chebotarev and E.V. Shamis. Matrix-forest theorems. Preprint, 1995. Available at `http://arxiv.org/abs/math.co/0602575`.

[9] P.Y. Chebotarev and E.V. Shamis. On the proximity measure for graph vertices provided by the inverse Laplacian characteristic matrix. *Fifth Conference of the International Linear Algebra Society*, 30–31, 1995.

[10] M. Fiedler. Algebraic connectivity of graphs. *Czechoslovak Math. J.*, 23(98):298–305, 1973.

[11] V.E. Golender, V.V. Drboglav, and A.B. Rosenblit. Graph potentails method and its application for chemical information processing. *J. Chem. Inf. Comput. Sci.*, 21:196–204, 1981.

[12] R. Grone and R. Merris. Ordering trees by algebraic connectivity. *Graphs Combin.*, 6:229–237, 1990.

[13] R.A. Horn and C.R. Johnson. *Matrix Analysis.* Cambridge University Press, London, 1985.

[14] J.J. McDonald, M. Neumann, H. Schneider, and M.J. Tsatsomeros. Inverses of unipathic M-matrices. *SIAM J. Matrix Anal. Appl.*, 17:1025–1036, 1996.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 295-305, March 2012

ELA

[15] R. Merris. Laplacian matrices of graphs: a survey. *Linear Algebra Appl.*, 197/198:143–176, 1994.

[16] R. Merris. Doubly stochastic graph matrices. *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat.*, 8:64–71, 1997.

[17] R. Merris. Doubly stochastic graph matrices II. *Linear Multilinear Algebra*, 45:275–285, 1998.

[18] B. Mohar. Eigenvalues, diameter, and mean distance in graphs. *Graphs Combin.*, 7:53–64, 1991.

[19] B. Mohar. Some applications of Laplace eigenvalues of graphs. In G. Hahn and G. Sabidussi (editors), *Graph Symmetry*, Kluwer Academic Publishers, Dordrecht, 225–275, 1997.

[20] R. Pereira and M.A. Vali. Inequalities for the spectra of symmetric doubly stochastic matrices. *Linear Algebra Appl.*, 419:643–647, 2006.

[21] E. Shamis. Graph-theoretic interpretation of the generalized row sum method, *Math. Social Sci.*, 27:321–333, 1994.

[22] X.D. Zhang and J.X. Wu. Doubly stochastic matrices of trees. *Appl. Math. Lett.*, 18:339–343, 2005.

[23] X.D. Zhang. A note on doubly stochastic graph matrices. *Linear Algebra Appl.*, 407:196–200, 2005.

[24] X.D. Zhang. Algebraic connectivity and doubly stochastic tree matrices. *Linear Agebra Appl.*, 430:1656–1664, 2009.

[25] X.D. Zhang. Vertex degrees and doubly stochastic graph matrices. *J. Graph Theory*, 66:104–114, 2011.