Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 275-286, March 2012

ELA

THE PRODUCT DISTANCE MATRIX OF A TREE AND A BIVARIATE ZETA FUNCTION OF A GRAPH*

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Abstract. In this paper, the product distance matrix of a tree is defined and formulas for its determinant and inverse are obtained. The results generalize known formulas for the exponential distance matrix. When the number of variables are restricted to two, the bivariate analogue of the laplacian matrix of an arbitrary graph is defined. Also defined in this paper is a bivariate analogue of the Ihara-Selberg zeta function and its connection with the bivariate laplacian is shown. Finally, for connected graphs, there is a result connecting a partial derivative of the determinant of the bivariate laplacian and its number of spanning trees.

Key words. Laplacian, Ihara-Selberg zeta function, Trees.

AMS subject classifications. 15A15, 05C05.

1. Introduction. Let T be a tree with vertex set $[n] = \{1, 2, \ldots, n\}$. Let $d_{i,j}$ be the distance between vertex i and vertex j in T, which is defined as the length (the number of edges) in the unique path from i to j. Let q be an indeterminate with $q^0 = 1$. The exponential distance matrix of the tree T is defined to be the $n \times n$ matrix $E_T = (e_{i,j})_{1 \le i,j \le n}$ where $e_{i,j} = q^{d_{i,j}}$. In the context of investigation related to the distance matrix of a tree, the following result was obtained by Bapat, Lal and Pati in [1].

THEOREM 1.1. *Let E_T be the exponential distance matrix of a tree T on n vertices. Then,* $\det(E_T) = (1 - q^2)^{n-1}$.

Thus, $\det(E_T)$ is independent of the structure of the tree and is only dependent on n, the number of vertices of T. For a tree T, a formula for the inverse of E_T has been found in [1]. Define $\mathcal{L}_q$, the q-analogue of T's laplacian as

$$\mathcal{L}_q = I - qA + q^2(D - I), \tag{1.1}$$

where A is the adjacency matrix of T and $D = (d_{i,j})_{1 \le i,j \le n}$ is a diagonal matrix

*Received by the editors on July 25, 2011. Accepted for publication on March 7, 2012. Handling Editor: Leslie Hogben.

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Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 275-286, March 2012

with $d_{i,i} = \deg(i)$ where $\deg(i)$ is the degree of vertex i in T. The q-analogue of the laplacian has occurred in work of other authors in different contexts as we indicate below. Note that on setting $q = 1$, we get $\mathcal{L}_q = L$, where L is the laplacian matrix of T. For trees, the following (see [1, Proposition 3.3]) is known.

THEOREM 1.2. *Let T be a tree and let E_T and $\mathcal{L}_q$ be its exponential distance matrix and the q-analogue of its laplacian respectively. Then,* $E_T^{-1} = \frac{1}{1-q^2}\mathcal{L}_q$.

The matrix $\mathcal{L}_q$ is also known to have the following tree independent property (see [1, Proposition 3.4]), which we generalize in Lemma 3.3.

LEMMA 1.3. *Let T be a tree on n vertices. Then* $\det(\mathcal{L}_q) = 1 - q^2$.

The definition of $\mathcal{L}_q$ can be extended to graphs that are not trees in a straightforward manner using (1.1). When the graph G is connected, but not necessarily a tree, $\mathcal{L}_q$ has connections to the number of spanning trees of G. Northshield [8] showed the following about the derivative of the determinant of $\mathcal{L}_q$.

THEOREM 1.4. *Let G be a connected graph with m edges, n vertices and κ spanning trees. Let $\mathcal{L}_q$ be the q-analogue of its laplacian matrix and let $f(q) = \det(\mathcal{L}_q)$. Then $f'(1) = 2(m-n)\kappa$.*

The polynomial $\det(\mathcal{L}_q)$ has also occurred in connection with the Ihara-Selberg zeta function of G, see Bass [3]. Foata and Zeilberger [5] have given combinatorial proofs of the results of Bass. We elaborate on this result below.

Let G be a connected graph. Transform G into a directed graph G_d by replacing each edge $e = \{u, v\} \in E(G)$ by two directed arcs (u, v) and (v, u) (i.e., one in each direction). If e is a loop edge around vertex v, we get two directed loops around v in G_d. Henceforth, we will work exclusively with G_d. Duplication of edges into arcs in this manner gives for each directed edge a, a unique reverse edge a_{rev}, which we also denote as $J(a)$. As G_d is a directed graph, we use directed graph terminology like "start vertex" and "end vertex" of a directed edge.

A directed edge e is said to be a *successor* of a directed edge e' if the end vertex of e' coincides with the start vertex of e. A *directed path* from vertex i to vertex j is a sequence $e_1, e_2, \ldots, e_\ell$ of directed edges such that vertex i is the start-vertex of e_1, vertex j is the end-vertex of e_ℓ, and for each $k = 2, 3, \ldots, \ell$, e_k is a successor of e_{k-1}. The above path is said to be of *l*ength ℓ. When $i = j$, such directed paths are termed *directed cycles.*

A directed cycle $e_1, e_2, \ldots, e_\ell$ is said to be *reduced* if $J(e_i) \neq e_{i+1}$ for all $1 \le i < \ell$ and $J(e_\ell) \neq e_1$. A directed cycle C is said to be *prime* if C is not the power of a smaller oriented cycle, i.e., if there does not exist a directed cycle C' and a positive integer $r > 1$ such that $C = (C')^r$ where $(C')^r$ is the directed cycle obtained by repeating

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 275-286, March 2012

the directed cycle C', r-times. Two directed cycles C, C' are said to be *cyclically equivalent* if one is a cyclic rearrangement of the other. That is if $C = e_1, e_2, \ldots, e_\ell$ and $C' = e_k, e_{k+1}, \ldots, e_\ell, e_1, \ldots, e_{k-1}$ for some $1 \leq k \leq \ell$. Each equivalence class is called a *cycle* and let $\mathcal{C}$ be the set of prime and reduced cycles of G_d.

For $C \in \mathcal{C}$, let $|C|$ be the length of C (i.e., the number of edges in C). Consider

$$\eta(q) = \prod_{C \in \mathcal{C}} (1 - q^{|C|}).$$

Then $\eta(q)$ is called the Ihara-Selberg zeta function of the graph G. Bass [3] showed the following:

THEOREM 1.5. *Let G be a graph with n vertices and m undirected edges. Then, $\eta(q)$ is a polynomial in q and can be expressed in two ways as follows. There exists a $2m \times 2m$ matrix S such that*

$$\eta(q) = \det(I - qS), \tag{1.2}$$

$$\eta(q) = (1 - q^2)^{m-n} \det(\mathcal{L}_q). \tag{1.3}$$

In the first part of the present paper, we define a multivariate analogue of the exponential distance matrix of a tree, which we call the product distance matrix, and we explicitly determine its determinant and inverse. When we restrict the number of variables to two, we get a q, t-exponential distance matrix whose inverse, in analogy to Theorem 1.2, motivates us to define the q, t-laplacian of T which we denote as $\mathcal{L}_{q,t}$. To the best of our knowledge, the matrix $\mathcal{L}_{q,t}$ does not seem to have been considered before.

In the last section of the paper, we consider connected graphs which are not necessarily trees and imitating the definition of $\mathcal{L}_{q,t}$, we get a bivariate laplacian matrix $\mathcal{L}_{q,t}$ for such graphs. Using this matrix, we obtain a bivariate analogue of (1.3). We also obtain a connection between κ, the number of spanning trees of G and a partial derivative of $\det(\mathcal{L}_{q,t})$ with respect to either q or t, inspired by the result of Northshield [8] (see Theorem 1.4).

2. Product distance matrix of a tree. We begin with the definition of the product distance matrix of a tree. Let $T = (V, E(T))$ be a tree on the vertex set $V = [n]$. Replace each edge e with two arcs, one in each direction, and label the two arcs with "weight" q_e and t_e in an arbitrary manner. Define the arc-set $\mathcal{A}$ of T as the set of $2(n-1)$ directed arcs. If $e = \{u, v\} \in E(T)$, we denote the directed arc from u to v as (u, v) and the directed arc from v to u as (v, u). Let $Q = \{q_e : e \in E(T)\} \cup \{t_e : e \in E(T)\}$. We think of the labels q_e, t_e as a weight

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A publication of the International Linear Algebra Society
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function $w : \mathcal{A} \to Q$. For each pair of vertices i, j, $i \neq j$ let $p_{i,j}$ be the unique directed path between the vertices i, j in T. For $i \neq j$, define

$$d_{i,j} = \prod_{a \in p_{i,j}} w(a). \tag{2.1}$$

When $i = j$, define $d_{i,j} = 1$. Let $M_T = (d_{i,j})_{1 \leq i < j \leq n}$, be defined as the product distance matrix of T. Here, we suppress the underlying weights w though M_T depends on w. The underlying weight function will be clear from the context. For example, the edge labelled tree of Figure 2.1 has

$$M_T = \begin{bmatrix} 1 & t_2 & q_1 & t_3 \\ q_2 & 1 & q_1 q_2 & q_2 t_3 \\ t_1 & t_1 t_2 & 1 & t_1 t_3 \\ q_3 & t_2 q_3 & q_1 q_3 & 1 \end{bmatrix}.$$

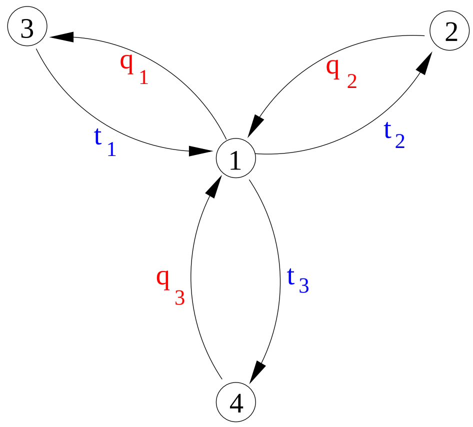

FIG. 2.1. *A bi-directed tree with labels on arcs.*

We note that setting $q_e = t_e = q$ for all $e \in E(T)$ results in $M_T = E_T$. For a tree T on n vertices, if the edges are labelled $e_1, e_2, \ldots, e_{n-1}$ in some order, we denote the arc labels as $q_1, q_2, \ldots, q_{n-1}$ and $t_1, t_2, \ldots, t_{n-1}$ respectively. The following result is a sharpening of Theorem 1.1.

LEMMA 2.1. *Let M_T be the product distance matrix of a tree T on n vertices. Then,* $\det(M_T) = \prod_{i=1}^{n-1}(1 - q_i t_i)$. *Thus,* $\det(M_T)$ *is independent of the structure of the tree and only depends on n and the* $2(n-1)$ *variables: the q_i's and the t_i's.*

Proof. By induction on n, the number of vertices. The statement is clearly true for $n = 2$. Assume that the statement is true for trees with $n - 1$ vertices and let $T = (V, E(T))$ be a tree with n vertices. Let $V = \{1, 2, \ldots, n\}$ and let n be

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A publication of the International Linear Algebra Society
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a leaf vertex adjacent to vertex $n-1$. Let the arc $e=(n,n-1)$ be labelled as q_{n-1} (the other case, when e is labelled t_{n-1} is identically proved). If we denote the ith column of M_T as Col_i, for $1 \le i \le n$, then it is clear that the elementary column operation $\mathsf{Col}_n := \mathsf{Col}_n - q_{n-1}\mathsf{Col}_{n-1}$ yields a matrix whose nth column is $(0,0,\ldots,0,1-q_{n-1}t_{n-1})^T$ where v^T is the transpose of vector v. If we denote as $T' = T - \{n\}$, the smaller tree obtained by deleting the leaf vertex n, then we have $\det(M_T) = (1-q_{n-1}t_{n-1})\det(M_{T'})$. The proof is complete by induction on the number of vertices of T. ☐

We now give an explicit formula for the inverse of M_T. To describe this, we need two $n\times n$ matrices, B and D described below: $B=(a_{u,v})_{1\le u,v\le n}$ where $a_{u,v}=0$ if there is no edge between vertices u and v. Set $a_{u,v}=q_i/(1-q_it_i)$ and $a_{v,u}=t_i/(1-q_it_i)$, if $e_i=\{u,v\}$ with $w(u,v)=q_i$ and $w(v,u)=t_i$ respectively. Note that when $q_i=t_i=q$ for all $1\le i<n$, then $B=\dfrac{q}{1-q^2}A$, where A is the adjacency matrix of T. Let D be a diagonal matrix with $d_{u,u}=\displaystyle\sum_{i:u\in e_i}\frac{q_it_i}{1-q_it_i}$. If $F=(f_{i,j})$ is the diagonal matrix with $f_{i,i}=\deg(i)$, the degree of vertex i, then if $q_i=t_i=q$ for all $1\le i<n$, we get $D=\dfrac{q^2}{1-q^2}F$. The following is a generalisation of [1, Proposition 3.3].

THEOREM 2.2. *For any tree T on n vertices, $M_T^{-1}=I-B+D$.*

Proof. We induct on n, the number of vertices of T. The base case when $n=2$ can be easily checked. Let T be a tree on n vertices with vertex n being a leaf vertex connected to vertex $n-1$. Let $T'=T-\{n\}$. Let M',B' and D' be the matrices analogous to M_T, B and D respectively for T'.

Let f_{n-1} be the edge $\{n,n-1\}$ and let $(n,n-1)$ be assigned weight q_{n-1} and $(n-1,n)$ be assigned weight t_{n-1} (the case when weights are assigned otherwise is identical to prove). Let $\mathbf{e_n}$ be the $n\times 1$ column vector with a 1 in position n and zeroes elsewhere. Define the $n\times 1$ column vector $\mathbf{v}$, and the $1\times n$ row vector $\mathbf{u}$, by

$$\mathbf{v}=M'\times\mathbf{e_n} \qquad \mathbf{u}=\mathbf{e_n}^T\times M'. \tag{2.2}$$

It is clear that $M_T=\left[\begin{array}{c|c} M' & q_{n-1}\mathbf{v} \\ \hline t_{n-1}\mathbf{u} & 1\end{array}\right]$. By induction, M' is invertible and it is easy to see that $q_{n-1}t_{n-1}\mathbf{u}\times(M')^{-1}\times\mathbf{v}\neq 1$. Thus, by the Sherman-Morrison formula (see [7, p. 124]), $M'-q_{n-1}t_{n-1}\mathbf{v}\times\mathbf{u}$ is invertible and we set $P=(M'-q_{n-1}t_{n-1}\mathbf{v}\times\mathbf{u})^{-1}$. Similarly, set $Q=q_{n-1}P\times\mathbf{v}$, $R=t_{n-1}\mathbf{u}\times P$, and $S=1-t_{n-1}\mathbf{u}\times Q$. With these, it is easy to see that the block partitioned matrix $\left[\begin{array}{c|c} P & Q \\ \hline R & S\end{array}\right]$ is the inverse of M_T.

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A publication of the International Linear Algebra Society
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Thus, we only need to show that P, Q, R, and S are in the form specified in the statement of Theorem 2.2. For this, it suffices to show that $P = (M')^{-1} + \dfrac{q_{n-1}t_{n-1}}{1-q_{n-1}t_{n-1}}\mathbf{e_n} \times \mathbf{e_n}^T$, $Q = \dfrac{q_{n-1}}{1-q_{n-1}t_{n-1}}\mathbf{e_n}$, $R = \dfrac{t_{n-1}}{1-q_{n-1}t_{n-1}}\mathbf{e_n}^T$, and $S = \dfrac{q_{n-1}t_{n-1}}{1-q_{n-1}t_{n-1}}$.

First, we consider

$$\alpha = 1 - q_{n-1}t_{n-1}\mathbf{u} \times (M^{-1}) \times \mathbf{v} = 1 - q_{n-1}t_{n-1}u \times \mathbf{e_n} = 1 - q_{n-1}t_{n-1},$$

where the second equality follows from (2.2) and the last equality follows since $u_n(= d_{n,n}) = 1$.

As $P = (M' - q_{n-1}t_{n-1}\mathbf{v} \times \mathbf{u})^{-1}$, by the Sherman-Morrison formula, we get

$$\begin{aligned}
P &= (M')^{-1} + \frac{q_{n-1}t_{n-1}M' \times \mathbf{v} \times \mathbf{u} \times (M')^{-1}}{1 - q_{n-1}t_{n-1}\mathbf{u} \times (M')^{-1} \times \mathbf{v}} \\
&= (M')^{-1} + \frac{q_{n-1}t_{n-1}}{\alpha}\mathbf{e_n} \times \mathbf{e_n}^T, \\
Q &= q_{n-1}\left[(M')^{-1} + \frac{q_{n-1}t_{n-1}}{\alpha}\mathbf{e_n} \times \mathbf{e_n}^T\right] \times M' \times \mathbf{e_n} \\
&= q_{n-1}\left[I + \frac{q_{n-1}t_{n-1}}{\alpha}\mathbf{e_n} \times \mathbf{u}\right] \times \mathbf{e_n} = q_{n-1}\left[\mathbf{e_n} + \frac{q_{n-1}t_{n-1}}{\alpha}\mathbf{e_n}\right] \\
&= \frac{q_{n-1}}{\alpha}\mathbf{e_n}, \\
R &= t_{n-1}\mathbf{e_n}^T \times M' \times \left[(M')^{-1} + \frac{q_{n-1}t_{n-1}}{\alpha}\mathbf{e_n} \times \mathbf{e_n}^T\right] \\
&= t_{n-1}\mathbf{e_n}^T \times \left[I + \frac{q_{n-1}t_{n-1}}{\alpha}\mathbf{v} \times \mathbf{e_n}^T\right] = t_{n-1}\left[\mathbf{e_n}^T + \frac{q_{n-1}t_{n-1}}{\alpha}\mathbf{e_n}^T\right] \\
&= \frac{t_{n-1}}{\alpha}\mathbf{e_n}^T, \\
S &= 1 - t_{n-1}\mathbf{u} \times \left[\frac{q_{n-1}}{\alpha}\mathbf{e_n}\right] = 1 - \frac{t_{n-1}q_{n-1}}{\alpha}\mathbf{u} \times \mathbf{e_n} = \frac{1}{\alpha}.
\end{aligned}$$

Thus, all matrices are in the same form as claimed, completing the proof. □

We note that result [1, Proposition 3.3] follows from Theorem 2.2. It may be remarked that an additive analogue of Theorem 2.2 has been considered by Bapat, Lal and Pati (see [2]). For this, suppose we bidirect the edges of a tree and have weights $w(a)$ for each arc $a \in A$. Define the distance between vertices i, j of the tree by replacing the product in (2.1) by the sum:

$$d_{i,j} = \sum_{a \in p_{i,j}} w(a)$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
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and set $d_{i,i} = 0$ for all i. Let us denote the resulting distance matrix with its (i,j)-element $d_{i,j}$ as Q_T. Then a formula for the inverse of Q_T has been provided in Theorem 3.1 of [2]. The formula is fairly complicated, but when $q_e = q, t_e = t$ for all $e \in E(T)$, it gets considerably simplified.

3. Bivariate exponential distance matrix of a tree. Let $T = (V, E(T))$ be a tree with $V = [n]$. Form the $n \times n$ matrix $E_{q,t}$ as follows. In M_T, set $q_e = q$ for all $e \in E(T)$ and $t_e = t$ for all $e \in E(T)$. With these specializations, the matrix M_T gives the matrix $E_{q,t}$. Further, setting $q = t$ in $E_{q,t}$ gives the exponential distance matrix $E_T = (x_{i,j})$ with $x_{i,j} = q^{d_{i,j}}$ where $d_{i,j}$ is the distance between vertices i and j in T. Hence, we call $E_{q,t}$ the bivariate exponential distance matrix of T. The following two results are obtained from Lemma 2.1 and Theorem 2.2.

COROLLARY 3.1. *Let T be a tree on n vertices and $E_{q,t}$ be its bivariate exponential distance matrix. Then,* $\det(E_{q,t}) = (1-qt)^{n-1}$. *Thus,* $\det(E_{q,t})$ *is independent of the tree structure and the manner of labelling its arcs.*

Let K be a diagonal matrix with the (i,i)th entry being $(\deg(i) - 1)qt$. The orientation of the edges of T into two directed arcs gives the following $n \times n$ "weights" matrix $W := (w_{i,j})$ with $w_{i,j} = 0$ if $\{i,j\} \notin E(T)$ and if $\{i,j\} \in E(T)$, then either $w_{i,j} = q$ and $w_{j,i} = t$, or vice versa according to whether the arc (i,j) is labelled q or t. It is easy to see that when $q = t$, W reduces to qA, where A is the adjacency matrix. The next theorem follows from Theorem 2.2.

THEOREM 3.2. *Let $E_{q,t}$ be the bivariate analog of the exponential distance matrix of a tree T with edge orientation matrix W. Then,* $E_{q,t}^{-1} = \dfrac{1}{1-qt}\left(I - W + K\right)$.

We recall that for a tree T, the inverse of its exponential distance matrix is $E_T^{-1} = \dfrac{1}{1-q^2}\mathcal{L}_q$, where $\mathcal{L}_q = I - qA + q^2(D-I)$ is the q-analogue of the laplacian matrix. Analogously, we define

$$\mathcal{L}_{q,t} = I - W + K \tag{3.1}$$

as the q,t-analogue of the laplacian. Thus, for a tree T, we have

$$\mathcal{L}_{q,t}^{-1} = \frac{1}{1-qt}E_{q,t}.$$

We next show a refinement of Lemma 1.3 for the matrix $\mathcal{L}_{q,t}$.

LEMMA 3.3. *Let T be a tree and let $\mathcal{L}_{q,t}$ be the q,t-analog of its laplacian. Then,* $\det(\mathcal{L}_{q,t}) = 1 - qt$.

Proof. We induct on n, the number of vertices of T. The base case when $n = 2$ is clear. Let T be a tree with $n+1$ vertices and let vertex $n+1$ be a leaf vertex

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A publication of the International Linear Algebra Society
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connected to vertex n. Let $T' = T - \{n+1\}$. Let $\mathcal{L}'_{q,t}$ be the q,t-analogue of the laplacian of T'. Let $\mathbf{e_n}$ be the $n \times 1$ column vector with a 1 in position n and zeroes elsewhere. Let the arc (e_n, e_{n+1}) have the label q and the arc (e_{n+1}, e_n) have the label t (the other case is identically proved as will be clear). It is clear that $\mathcal{L}_{q,t} = \left[\begin{array}{c|c} \mathcal{L}'_{q,t} + qt\mathbf{e_n} \times \mathbf{e_n}^T & -q\mathbf{e_n} \\ \hline -t\mathbf{e_n}^T & 1 \end{array}\right]$. By the multiplicative property of block determinants (see [7, p. 475]) we get $\det(\mathcal{L}_{q,t}) = \det(1) \cdot \det(\mathcal{L}'_{q,t} + qt\mathbf{e_n} \times \mathbf{e_n}^T - qt\mathbf{e_n} \times \mathbf{e_n}^T) = \det(\mathcal{L}'_{q,t}) = 1 - qt$, completing the proof. $\square$

4. A bivariate Ihara-Selberg zeta function. We extend the definition of the q,t-laplacian given in Section 3 to connected graphs using (3.1); i.e., we duplicate edges, assign directions to them, assign weights q,t to each arc arbitrarily and consider the matrix $\mathcal{L}_{q,t} = I - W + K$, where W records the arc variable as q or t, and K is a diagonal matrix with (v,v) entry being $qt(\deg(v) - 1)$.

Thus, $\mathcal{L}_{q,t}$ is a bivariate generalization of the laplacian matrix L of a graph (i.e., on setting $q = t = 1$, we get $\mathcal{L}_{q,t} = L$) and a generalization of the q-analogue of the laplacian matrix $\mathcal{L}_q$ of G (when $q = t$, we get $\mathcal{L}_{q,t} = \mathcal{L}_q$).

We define a bivariate Ihara-Selberg zeta function for a connected graph G motivated by the q,t-laplacian $\mathcal{L}_{q,t}$ of G. This is towards proving a bivariate version of a result of Bass [3]. Our proof is a reasonably straightforward generalisation of a combinatorial proof of the univariate result of Bass given by Foata and Zeilberger [5].

Foata and Zeilberger [5, Theorem 1.1] give a slightly more general result on Lyndon words from which a more general edge-weighted version of (1.2) follows. If instead of assigning each arc a weight q, we assign arc (i,j) a weight $w_{i,j}$ and for a prime and reduced cycle C, consider its weight $w(C) = \prod_{a \in C} w(a)$, then there is a $2m \times 2m$ size matrix S_w depending on w such that $\prod_{C \in \mathcal{C}}(1 - w(C)) = \det(I - S_w)$. Thus, an analogue of (1.2) exists for a modified version of $\eta(q)$ for arbitrary edge weights. Analogues of (1.3) do not seem to exist for arbitrary weights. Below, we prove a bivariate analogue of (1.3).

Let G be a connected graph and let G_d be as above. In G_d, assign each arc $e = (u,v)$, a weight q and its reverse arc, a weight t. Let C be a directed prime and reduced cycle. Let C have $a(C)$ arcs with weight q and $b(C)$ arcs with weight t. It is clear that $a(C) + b(C) = |C|$ and that the numbers $a(C)$ and $b(C)$ are independent of the starting vertex of C. For graph G_d, arising from a connected G, consider

$$\eta(q,t) = \prod_{C \in \mathcal{C}} (1 - q^{a(C)} t^{b(C)}). \tag{4.1}$$

where $\mathcal{C}$ is the set of prime and reduced cycles of G_d. It is easy to see that if $q = t$, then $\eta(q,t) = \eta(q)$ and $\mathcal{L}_{q,t} = \mathcal{L}_q$. We show the following generalisation of

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A publication of the International Linear Algebra Society
Volume 23, pp. 275-286, March 2012

Theorem 1.5.

THEOREM 4.1. *Let G_d be obtained as above from a connected graph G with m edges and n vertices. Then, $\eta(q,t) = (1-qt)^{m-n}\det(\mathcal{L}_{q,t})$.*

Our approach is very similar to that of Foata and Zeilberger [5] and we briefly go over a few preliminaries before proving Theorem 4.1. Given a bidirected connected graph with arcs assigned weights, label the arcs as $e_1, e_2, \ldots, e_{2m}$ in an arbitrary manner and consider the $2m \times 2m$ matrices $\mathrm{Succ}_{q,t}, T_{q,t}$ and $J_{q,t}$ whose entries are defined below. Each row and column of all the above matrices are labelled by $e_1, e_2, \ldots, e_{2m}$. Since each arc is directed, we use standard directed graph terms like *initial vertex* and *terminal vertex* of arcs. $J_{q,t}$ is the arc reversal map; i.e., the row corresponding to arc e_i has only one non-zero entry. Recall the for an arc e, e_{rev} is its unique "reverse arc". This entry is in the column corresponding to e_{rev} and if e_i is labelled q (or t respectively), this entry is $-q$ ($-t$ respectively). In the matrix $\mathrm{Succ}_{q,t}$, the row corresponding to arc e_i has non-zero entries only in those columns e_j which "succeed" e_i, i.e., in those arcs e_j whose initial vertex coincide with the terminal vertex of e_i. In these columns, the entry is $-q$ (respectively $-t$) if e_i is labelled q (respectively t).

Define the $2m \times 2m$ "common origin map" matrix $\mathrm{Com}_{q,t}$ as follows. The row corresponding to arc e_i has non-zero entries only in columns e_j which are different from e_i and yet have the same initial vertex as e_i. In such columns, the entry is qt. As an example, for the graph in Figure 4.1, the relevant matrices are given below.

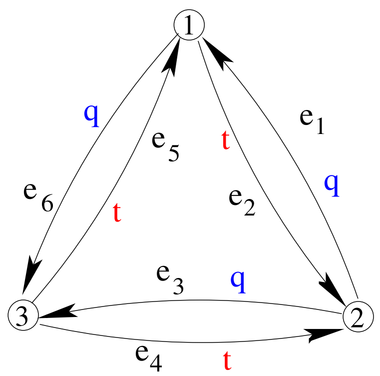

FIG. 4.1. *A directed graph with labels.*

$$J_{q,t} = \begin{bmatrix} 0 & -q & 0 & 0 & 0 & 0 \\ -t & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -q & 0 & 0 \\ 0 & 0 & -t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -t \\ 0 & 0 & 0 & 0 & -q & 0 \end{bmatrix}, \quad \mathrm{Com}_{q,t} = \begin{bmatrix} 0 & 0 & qt & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & qt \\ qt & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & qt & 0 \\ 0 & 0 & 0 & qt & 0 & 0 \\ 0 & qt & 0 & 0 & 0 & 0 \end{bmatrix},$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 275-286, March 2012

$$\mathrm{Succ}_{q,t} = \begin{bmatrix} 0 & -q & 0 & 0 & 0 & -q \\ -t & 0 & -t & 0 & 0 & 0 \\ 0 & 0 & 0 & -q & -q & 0 \\ -t & 0 & -t & 0 & 0 & 0 \\ 0 & -t & 0 & 0 & 0 & -t \\ 0 & 0 & 0 & -q & -q & 0 \end{bmatrix}, \quad \mathcal{L}_{q,t} = \begin{bmatrix} 1+qt & -q & -t \\ -t & 1+qt & -t \\ -q & -q & 1+qt \end{bmatrix}.$$

Define $A_{q,t} = I + \mathrm{Succ}_{q,t} + \mathrm{Com}_{q,t}$ and $S_{q,t} = \mathrm{Succ}_{q,t} - J_{q,t}$. With these definitions, it follows from [5, Theorem 1.1], that

$$\eta(q,t) = \det(I - S_{q,t}). \tag{4.2}$$

The next proposition follows immediately from [5, Proposition 8.1] which we state for easy reference. For a graph with n vertices and m edges, let there be a set of commuting edge variables $s_{u,v}$ and another set of commuting vertex variables a_v. Bidirect the edges of G to get two arcs and consider the $2m \times 2m$ dimensional "common origin" matrix Com defined as follows: the row corresponding to arc e of Com has non-zero entries only in the columns corresponding to arcs f where f is an arc different from e, but with the same start vertex v as e, in which case $\mathrm{Com}_{e,f} = a_v$. Similarly, define the $2m \times 2m$ matrix Succ whose row corresponding to arc e is as follows. This row has non-zero entries only in columns f such that the terminal vertex of e and the initial vertex of f coincide. In such a case, we set $\mathrm{Succ}_{e,f} = s_{u,v}$ where $e = (u,v)$. Define the $2m \times 2m$ matrix $A = I + \mathrm{Succ} + \mathrm{Com}$. Recall that $\deg(u)$ is the degree of vertex u in G (before bidirection) and let $\mathrm{Adj} = (b_{u,v})_{u,v \in V}$ be the 0/1 adjacency matrix of G. Define the $n \times n$ matrix Δ as follows. $\Delta_{u,u} = 1 + b_{u,u}s_{u,u} + (\deg(u)-1)a_v$ and $\Delta_{u,v} = b_{u,v}s_{u,v}$. The following proposition is due to Foata and Zeilberger.

Proposition 4.2. [5, Proposition 8.1] *With the definitions as above,*

$$\det(A) = \det(\Delta) \prod_{v \in V(G)} (1 - a_v)^{\deg(v)-1}.$$

Specializing to matrices in our context, we get the following.

Lemma 4.3. *With the above definitions,*

$$\det(A_{q,t}) = \det(\mathcal{L}_{q,t}) \prod_{v \in V} (1 - qt)^{\deg(v)-1}.$$

Proof. By setting $a_v = qt$ for all v and $q_{u,v} = q$ (or t respectively) if the label on arc (u,v) is q (or t respectively), we get $A_{q,t} = A$ and $\mathcal{L}_{q,t} = \Delta$. The proof follows. ∎

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 275-286, March 2012

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Proof of Theorem 4.1. We recall $S_{q,t} = \text{Succ}_{q,t} - J_{q,t}$. It is easy to see that $\text{Com}_{q,t} = J_{q,t} \times S_{q,t}$. Thus, $A_{q,t} = I - (S_{q,t} + J_{q,t}) + J_{q,t} \times S_{q,t} = (I - J_{q,t}) \times (I - S_{q,t})$. Thus, $\det(A_{q,t}) = \det(I - S_{q,t}) \det(I - J_{q,t})$. It is simple to note that $\det(I - J_{q,t}) = (1-qt)^m$. Hence, by Lemma 4.3 and (4.2), $(1-qt)^{m-n} \det(\mathcal{L}_{q,t}) = \eta(q,t)$, completing the proof. ◻

In the remainder of this section, we give a partial derivative based analogue of Theorem 1.4. There have been analogues of Theorem 1.4 using partial derivatives (see Kim, Kwon and Lee [6]), but these results are motivated by connections to the Bartholdi zeta function as opposed to any variant of the Ihara-Selberg zeta function of a graph. We denote the vertices of G as $1, 2, \ldots, n$.

THEOREM 4.4. *Let G be a connected graph with m edges, n vertices, κ spanning trees and let $\mathcal{L}_{q,t}$ be the q,t-analogue of its laplacian. If $D(q,t) = \det(\mathcal{L}_{q,t})$, let $f(q,t) = \dfrac{\partial D(q,t)}{\partial q}$ and let $g(q,t) = \dfrac{\partial D(q,t)}{\partial t}$. Then $f(1,1) = g(1,1) = (m-n)\kappa$.*

Proof. It follows from the multilinearity of the determinant that the derivative (or partial derivative) of the determinant of $\mathcal{L}_{q,t}$ can be computed in the following manner. For $1 \le i \le n$, let $\mathcal{L}^i_{q,t}$ be the matrix $\mathcal{L}_{q,t}$ with the following change: all elements of the ith column are replaced by their partial derivative with respect to q. Then $f(q,t) = \sum_{i=1}^n \det(\mathcal{L}^i_{q,t})$.

Thus, $f(1,1) = \sum_{i=1}^n \det(\mathcal{L}^i_{1,1})$. Since we are considering the partial derivative with respect to q, it is easy to see that $\mathcal{L}^i_{1,1} = L^i$ where L^i is the laplacian matrix of G with the ith column having all entries $-q$ replaced by -1, entries $-t$ replaced by 0 and the diagonal entry being $(\deg(i)-1)t$ where $\deg(i)$ is the degree of vertex i in G. We note that setting $q = t = 1$ gives $\mathcal{L}_{q,t} = L$, the laplacian matrix of G and by the Matrix Tree Theorem (see [4]), that the minor of L obtained by deleting any row and column of L is κ, the number of spanning trees of G. Thus, if we compute $\det(L^i)$ by expanding along the ith column, we get $\det(\mathcal{L}^i_{1,1}) = (\deg(i)-1)\kappa - d^o_i.\kappa$, where d^o_i is the number of arcs coming into vertex i labelled q. Since each edge of G is bidirected and one of each of the arcs is labelled q, it is easy to note that $\sum_{i=1}^n d^o_i = m$. Hence,

$$f(1,1) = \sum_{i=1}^n [(\deg(i)-1) - d^o_i]\kappa = (2m-n)\kappa - \sum_{i=1}^n d^o_i \kappa = (m-n)\kappa.$$

The argument for $g(1,1)$ is identical and is omitted. The proof is complete. ◻

Acknowledgment. Some theorems in this work were in conjecture form, tested using the computer package “Sage”. We thank the authors of “Sage” for generously releasing their software as an open-source package.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 275-286, March 2012

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