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THE MOORE–PENROSE INVERSE IS AN ADJOINT*

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Abstract. This article provides a construction of the Moore–Penrose inverse of a linear map f between vector spaces with suitable inner products as the adjoint of f with respect to some modified inner product on the target space. Furthermore, it proves, in a precise sense, that this modified metric is the one that is closer to turning f into an isometry.

Key words. Generalized inverse, Moore–Penrose inverse, Adjoints, Isometries.

AMS subject classifications. 15A09, 15A04, 15A63, 47A05.

1. Introduction. The Moore–Penrose inverse of a matrix A or, in general, of a linear map $f\colon V \to W$ between vector spaces with suitable inner products, was formally introduced in [2] and has proven to be one of the most important tools of modern linear algebra, since it provides the "least squares solutions" (minimal norm) to all linear systems of the form

$$f(x) = w,$$

with $w \in W$. The Moore–Penrose inverse was characterized axiomatically by Penrose in [6] as the unique linear map $f^\dagger\colon W \to V$ satisfying the following conditions:

$$\begin{aligned} &f \circ f^\dagger \circ f = f \text{ (it is a 1-inverse);}\\ &f^\dagger \circ f \circ f^\dagger = f^\dagger \text{ (it is a 2-inverse; a reflexive inverse together with the previous fact);}\\ &f^\dagger \circ f \text{ and } f \circ f^\dagger \text{ are self-adjoint idempotents, i.e., orthogonal projectors.} \end{aligned} \tag{1.1}$$

Based on this definition and derived properties, multiple techniques for its calculation have been developed over the decades. Perhaps most notably, one simply has to notice that f, in the presence of suitable inner products, always induces an isomorphism

$$f_{|\ker(f)^\perp}\colon \ker(f)^\perp \to \operatorname{Im}(f);$$

and $f^\dagger$ is the inverse of this map extended by zero over the orthogonal complement $\operatorname{Im}(f)^\perp$. Another known closed formula is

$$f^\dagger = f^{\mathrm{ad}} \circ (f \circ f^{\mathrm{ad}})^D; \tag{1.2}$$

where ^{D} denotes the Drazin inverse and $^{\mathrm{ad}}$ the adjoint map, see, for example [3], where it is proved for regular elements of C^*-algebras.

The Moore–Penrose inverse has some properties that are reminiscent of those of an adjoint, most notably the fact that $f^{\dagger\dagger} = f$, and also the value of its kernel and image. While much has been written relating $f^\dagger$

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and the adjoint f^{ad}, we have failed to find a clear connection in the literature that expresses $f^\dagger$ as *some* adjoint map, giving it a clarifying and meaningful interpretation in that context.

This paper consists of two main sections, after some quick preliminaries in Section 2:

- In Section 3, we realize $f^\dagger$ as an adjoint with respect to a modified product in the target space. For this purpose, we give a notion of *orthogonal push-forward* $f_\perp\phi_V$ of an inner product ϕ_V on V and show that $f^\dagger$ is indeed just the adjoint with respect to it. Such an inner product can be written in terms of the Drazin inverse of the operator $f \circ f^{\mathrm{ad}}$, which relates it to the classically known formula of equation (1.2). We would like to highlight that the construction is pedagogic in the sense that it leads one to the MP-inverse without knowing about its existence or properties beforehand.
- To complete the paper, in Section 4, we show that $f_\perp\phi_V$ admits an interpretation as the inner product on the target space that is closer to turning f into an isometry: if $T_2(f)\colon T_2(W) \to T_2(V)$ is the pullback map induced by f between rank 2 tensors, then $f_\perp\phi_V$ is the "least squares solution" to the problem
$$T_2(f)(X) = \phi_V,$$
or in other words and using the Moore–Penrose inverse at the level of tensors: $(T_2(f))^\dagger(\phi_V) = f_\perp\phi_V$; where the † here is taken with respect to the natural products induced on $T_2(W)$ and $T_2(V)$ by those of W and V.

At the end, we provide examples of computation. All results are valid over arbitrary base fields and vector spaces, provided the required hypotheses on the inner products and related decompositions are satisfied.

2. Notations and other considerations. Let k be an arbitrary field, which will remain fixed. For the rest of the paper, $\mathrm{Hom}_k(V, W)$ will denote the space of linear maps from a vector space V to a vector space W. Let $V^* = \mathrm{Hom}_k(V, k)$ denote the algebraic dual of V. A linear inner product or bilinear form on V is a bilinear map
$$\langle -, - \rangle_V \colon V \times V \to k.$$
By fixing the second component, this is equivalent to giving a linear map
$$\phi_V \colon V \to V^* \quad \text{such that} \quad \phi_V(v)(v') = \langle v', v \rangle_V,$$
that is sometimes called the (left) *polarity* of the inner product due to reasons related to projective geometry. It is well known that $\langle -, - \rangle_V$ is non-degenerate if and only if ϕ_V is injective and, in this case, it is an isomorphism when V is finite-dimensional. Furthermore, in this situation, the bilinear form is symmetric (called, *a metric*) if and only if ϕ_V is self-dual, i.e., if it is isomorphic to its transpose map via reflexivity $V \simeq V^{**}$.

> *In this paper, we will regard inner products as linear maps* $\phi_V \colon V \to V^*$, hence a vector space with such a product will be a pair (V, ϕ_V).

If V is infinite dimensional, the issue with ϕ_V not being an isomorphism—it never is due to cardinalities of bases reasons—can be fixed by regarding V as a topological vector space with the topology induced by the inner product and considering the topological dual V' instead of the algebraic dual. If $(V, \langle -, - \rangle_V)$ is a Hilbert space, then the content of the Riesz representation theorem (see [7, Theorem 4.12]) states that $\phi_V \colon V \to V'$ is indeed an (isometric) isomorphism. Furthermore, in this case, V is reflexive for the topological dual.

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Note that, when working with $k = \mathbb{C}$, it is typical to consider sesquilinear inner products. In that case, the map ϕ_V is sesquilinear, but otherwise, the previous discussion holds with minor modifications. More generally, the theory makes sense for *semilinear* maps with respect to any fixed automorphism $\sigma\colon k \to k$ of the base field with $\sigma^2 = \mathrm{id}$ (in the case of $\mathbb{C}$, it is complex conjugation). For the rest of the paper and for simplicity, we will assume that all our products are given by bilinear forms, but the complete discussion holds for sesquilinear (or semilinear) ones.

On a related note, recall that given a subspace $Z \subseteq V$, we can define its *incident subspace* as the space of linear forms that vanish on Z, i.e.,

$$Z^0 = \{\omega \in V^* : \omega_{|Z} = 0\} \subseteq V^*,$$

which can be interpreted as the space of implicit equations of Z. In finite dimensions, reflexivity $V \simeq V^{**}$ restricts to an isomorphism $Z \simeq Z^{00}$, which can be interpreted as "any subspace is the solution set of its implicit equations." Furthermore, if $\phi_V\colon V \to V^*$ is (the polarity of) a product on V, the orthogonal of Z with respect to ϕ_V is exactly

$$Z^\perp = \phi_V^{-1}(Z^0);$$

thus, if ϕ_V is an isomorphism, Z^0 and $Z^\perp$ formally share the same properties.

Definition 2.1. *If (V, ϕ_V) and (W, ϕ_W) are vector spaces equipped with inner products and ϕ_V is an isomorphism, given a linear map $f\colon V \to W$, its adjoint*[1] *(with respect to the given inner products)*

$$f^{\mathrm{ad}} = \phi_V^{-1} \circ f^* \circ \phi_W\colon W \to V,$$

where $f^*\colon W^* \to V^*$ is the transpose map: $f^*(\omega) = \omega \circ f$.

In other words, the adjoint map is the composition of the three maps at the bottom of the diagram

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ {\scriptstyle \phi_V}\downarrow & & \downarrow{\scriptstyle \phi_W} \\ V^* & \xleftarrow[f^*]{} & W^*. \end{array}$$

On the other hand, and also motivated by the diagram above, given such a map f and just a product ϕ_W on the target space, one defines the *pullback* of ϕ_W through f as the composition

$$f^*\phi_W = f^* \circ \phi_W \circ f\colon V \to V^*.$$

Notice that the linear map f is an *isometry* exactly when $\phi_V = f^*\phi_W$. Indeed, under this hypothesis, for each $v, v' \in V$ one has, unveiling the definitions:

$$\langle v, v'\rangle_V = \phi_V(v')(v) = (f^*\phi_W)(v')(v) = (f^* \circ \phi_W)(f(v'))(v) = \phi_W(f(v'))(f(v)) = \langle f(v), f(v')\rangle_W.$$

[1] If ϕ_V is degenerate, one can define *an adjoint* g of f to be any linear map such that $\phi_V \circ g = f^* \circ \phi_W$. However, this g is not necessarily unique in this case. On the other hand, if ϕ_V is degenerate, but it is non-degenerate over $\phi_V^{-1}(\mathrm{Im}(f^*)) = \ker(f)^\perp$, it makes sense to define the adjoint of f using only the restricted inner product.

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If ϕ_V is injective (non-degenerate), then all isometries are injective and thus, in finite dimensions, isomorphisms (called isometric isomorphisms). Finally, an isomorphism f is an isometry if and only if $f^{\mathrm{ad}} = f^{-1}$ because in the isometry condition

$$\phi_V = f^* \circ \phi_W \circ f,$$

one can compose with ϕ_V^{-1} on the left and with f^{-1} on the right to obtain

$$f^{-1} = f^{\mathrm{ad}};$$

and similarly in the other direction.

3. The Moore–Penrose inverse is an adjoint. The fact that $f^{\dagger\dagger} = f$ together with its norm-minimizing properties—related to inner products—and the fact that $\mathrm{Im}(f^\dagger) = \mathrm{Im}(f^{\mathrm{ad}})$ and $\ker(f^\dagger) = \ker(f^{\mathrm{ad}})$ should make the novice reader think that the MP-inverse has properties similar to those of an adjoint map, but, in general, it is not the adjoint of f (far from it). However, it is possible to realize the Moore–Penrose inverse as an adjoint in a way that is conceptually meaningful.

First, note that if f is an isomorphism, its MP-inverse is always its inverse $f^\dagger = f^{-1}$, while its adjoint f^{ad} is not: it is if and only if f is an isometric isomorphism. However, turning an isomorphism into an isometric isomorphism is easy: given

$$f\colon (V, \phi_V) \xrightarrow{\sim} (W, \phi_W),$$

we may simply consider the push-forward of ϕ_V onto W, which is given by

$$f_*\phi_V := (f \circ \phi_V^{-1} \circ f^*)^{-1}\colon W \to W^*.$$

(A priori, this makes sense because both f and f^* are also isomorphisms. Equivalently, this is the pullback through f^{-1}.) Now, f becomes an isometry

$$f\colon (V, \phi_V) \to (W, f_*\phi_V),$$

because $f^* f_*\phi_V := f^* \circ (f_*\phi_V) \circ f = \phi_V$, hence

$$f^{\mathrm{ad}} = f^{-1} = f^\dagger,$$

with respect to this new "push-forward" product.

For an arbitrary linear map, push-forward of inner products in the sense described above is not well defined, but we can define a good replacement by using a classical trick that can be used to deduce the classification of inner products from that of endomorphisms. Let us denote $\mathrm{End}_k(W) = \mathrm{Hom}_k(W, W)$.

LEMMA 3.1. *Let (W, ϕ_W) be a vector space equipped with an isomorphism $\phi_W\colon W \to W^*$ corresponding to a non-degenerate bilinear form. Then, there is an isomorphism of k-vector spaces*

$$\mathrm{End}_k(W) \simeq \mathrm{Hom}_k(W^*, W),$$

given by $\alpha \mapsto \alpha \circ \phi_W^{-1}$ with inverse $\psi \mapsto \psi \circ \phi_W$. If ϕ_W is symmetric as a bilinear form (self-dual as a linear map) and under this correspondence, a map α is self-adjoint with respect to ϕ_W if and only if its image $\alpha \circ \phi_W^{-1}$ is self-dual.

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Proof. Both correspondences are clearly linear and mutually inverse. For the second part: $\alpha\colon W \to W$ is self adjoint if and only if $\alpha = \phi_W^{-1} \circ \alpha^* \circ \phi_W$, i.e., when $\phi_W \circ \alpha = \alpha^* \circ \phi_W$. On the other hand, if $\mathrm{ev}\colon W \xrightarrow{\sim} W^{**}$ is the isomorphism of the reflexivity theorem (evaluation of forms at vectors), then $\alpha \circ \phi_W^{-1}$ is self-dual if and only if

$$\alpha \circ \phi_W^{-1} = \mathrm{ev}^{-1} \circ (\phi_W^{-1})^* \circ \alpha^* = (\phi_W^* \circ \mathrm{ev})^{-1} \circ \alpha^*;$$

but $\phi_W^* \circ \mathrm{ev} = \phi_W$ because ϕ_W is symmetric, hence both conditions are exactly the same. ∎

Remark 3.2. *This Lemma can be slightly sharpened: the map*

$$\Psi_W\colon \mathrm{Hom}_k(W^*, W) \to \mathrm{End}_k(W),$$

is always defined and its image are precisely those endomorphisms α such that ϕ_W is non-degenerate over $\phi_W^{-1}(\mathrm{Im}(\alpha))$. This is actually the form we will use.

Recall that the index of an endomorphism $g\colon W \to W$ is the smallest non-negative integer $m \geq 0$, if it exists, such that

$$\mathrm{Im}(g^{m+1}) = \mathrm{Im}(g^m) \text{ and } \ker(g^{m+1}) = \ker(g^m).$$

In infinite dimensions, an endomorphism may not have an index. Endomorphisms admitting a notion of index are called *core-nilpotent*, see [4] and [5]. If g has index m, there is a g-invariant decomposition

$$W = \mathrm{Im}(g^m) \oplus \ker(g^m),$$

such that $g_{|\,\mathrm{Im}(g^m)}$ is an isomorphism and $g_{|\,\ker(g^m)}$ is nilpotent: this decomposition of W is the so-called AST decomposition induced by g ([5, Thm. 3.6]). The *Drazin* inverse of an endomorphism g of index m is, in this case, the unique endomorphism $g^D\colon W \to W$ such that

$$(g^D)_{|\,\mathrm{Im}(g^m)} = (g_{|\,\mathrm{Im}(g^m)})^{-1} \text{ and } (g^D)_{|\,\ker(g^m)} = 0. \tag{3.3}$$

Equivalently, it is the unique endomorphism such that

$$g^{m+1}g^D = g^m, \quad g^D g g^D = g^D \text{ (it is a 2-inverse) and } g^D g = g g^D \text{ (commutes with } g). \tag{3.4}$$

Lemma 3.3. *Let (V, ϕ_V) and (W, ϕ_W) be spaces equipped with maps into their duals and let ϕ_W be such that its restriction to $\mathrm{Im}(f)$ is non-degenerate and so that $\mathrm{Im}(f) \oplus \mathrm{Im}(f)^\perp = W$.[2] Furthermore, assume that the restriction of ϕ_V to $\ker(f)^\perp = \phi_V^{-1}(\ker(f)^0)$ is non-degenerate, i.e.,*

$$(\phi_V)_{|\,\ker(f)^\perp}\colon \ker(f)^\perp \to \ker(f)^0$$

is an isomorphism and that $\ker(f) \oplus \ker(f)^\perp = V$. Then, the endomorphism corresponding to

$$f \circ ((\phi_V)_{|\,\ker(f)^\perp})^{-1} \circ f^*\colon W^* \to W,$$

via the isomorphism in Lemma 3.1 is

$$f \circ f^{\mathrm{ad}}\colon W \to W,$$

and, furthermore, $f \circ f^{\mathrm{ad}}$ has index ≤ 1 with $\mathrm{Im}(f \circ f^{\mathrm{ad}}) = \mathrm{Im}(f)$ and $\ker(f \circ f^{\mathrm{ad}}) = \mathrm{Im}(f)^\perp$.[3]

[2] This is actually automatic in the finite-dimensional case once non-degeneracy over $\mathrm{Im}(f)$ is assumed, but in the usual, topologized infinite-dimensional frameworks, one has to assume here that f has closed image. The same thing happens for $\ker(f)$ below, but in that case, the common assumption of f being continuous (bounded) automatically makes $\ker(f)$ closed.

[3] Where the adjoint f^{ad} with our restricted map $(\phi_V)_{|\,\ker(f)^\perp}$, even if ϕ_V is not an isomorphism, because $\mathrm{Im}(f^*) = \ker(f)^0$. Note that the non-degeneracy condition over $\ker(f)^\perp$ does not follow from global non-degeneracy in general but, if ϕ_V is non-degenerate and our hypotheses hold, the restriction and inversion operations commute. Also note that we are omitting the inclusion $\ker(f)^\perp \hookrightarrow V$ from the notation of the composition.

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Proof. For the first part, note that, under the correspondence of Lemma 3.1, we are looking at

$$f \circ ((\phi_V)_{|\ker(f)^\perp})^{-1} \circ f^* \circ \phi_W \colon W \to W.$$

The three maps on the right-hand side of the expression compose exactly to f^{ad}, so our claim follows. As for the index 1 claim: note that $\ker(f^{\mathrm{ad}}) = \mathrm{Im}(f)^\perp$ and $\mathrm{Im}(f^{\mathrm{ad}}) = \ker(f)^\perp$. Thanks to the hypothesis, we have that both $(f^{\mathrm{ad}})_{|\mathrm{Im}(f)^\perp} \colon \mathrm{Im}(f) \to \ker(f)^\perp$ and $f_{|\ker(f)^\perp} \colon \ker(f)^\perp \to \mathrm{Im}(f)$ are isomorphisms. Since $\ker(f) \oplus \ker(f)^\perp = V$ and $\mathrm{Im}(f) \oplus \mathrm{Im}(f)^\perp = W$, we conclude that

$$f \circ f^{\mathrm{ad}},$$

is an isomorphism over $\mathrm{Im}(f)$ and zero over $\mathrm{Im}(f)^\perp$, so it has index ≤ 1. ∎

Since $f \circ f^{\mathrm{ad}}$ is of index ≤ 1, its Drazin inverse $(f \circ f^{\mathrm{ad}})^D$ (also known as *group inverse* in this case) is defined: it is the inverse of $(f \circ f^{\mathrm{ad}})_{|\mathrm{Im}(f)}$ over $\mathrm{Im}(f)$ extended by zero over $\mathrm{Im}(f)^\perp$ (with respect to ϕ_W). Now, the idea is to pull this Drazin inverse back to a morphism $W \to W^*$ composing with ϕ_W, as is justified by Lemma 3.1.

Definition 3.4. *Given a linear map $f \colon (V, \phi_V) \to (W, \phi_W)$ between vector spaces so that:*

- *ϕ_V corresponds to a product that is non-degenerate over $\ker(f)^\perp$ and such that $\ker(f) \oplus \ker(f)^\perp = V$;*
- *ϕ_W corresponds to a product that is non-degenerate over $\mathrm{Im}(f)$ so that $\mathrm{Im}(f) \oplus \mathrm{Im}(f)^\perp = W$;*

then, we define the orthogonal push-forward of ϕ_V to be

$$f_\perp \phi_V := \phi_W \circ (f \circ f^{\mathrm{ad}})^D.$$

Definition 3.5 (Equivalent Definition of the MP-inverse). *Let $f \colon (V, \phi_V) \to (W, \phi_W)$ be a linear map in the hypotheses of Definition 3.4. Then, we define the Moore–Penrose inverse $f^\dagger$ of f as the adjoint of the map*

$$f \colon (V, \phi_V) \to (W, f_\perp \phi_V).$$

Explicitly, unveiling our constructions, Definition 3.5 yields

$$f^\dagger = \phi_V^{-1} \circ f^* \circ f_\perp \phi_V = \phi_V^{-1} \circ f^* \circ \phi_W \circ (f \circ f^{\mathrm{ad}})^D = f^{\mathrm{ad}} \circ (f \circ f^{\mathrm{ad}})^D,$$

so it does indeed recover the classical Moore–Penrose inverse of f, as Equation (1.2) is the same formula we have calculated.

Remark 3.6. *Note that, for a bounded operator between Hilbert spaces, the first condition of Definition 3.4 holds by default and the second condition requires them to have closed image.*

> From this point on and for the sake of simplifying notations, we are assuming that ϕ_V and ϕ_W are not only in the hypotheses of Definition 3.4 but also globally non-degenerate. All the discussion that follows holds true with minor modifications.

Finally, we can use this description to compute $f^* f_\perp \phi_V$. More precisely,

$$f^* f_\perp \phi_V = f^* \circ \phi_W \circ (f \circ f^{\mathrm{ad}})^D \circ f.$$

This says that f is an isometry with respect to the orthogonal push-forward bilinear form if and only if

$$f^{\mathrm{ad}} \circ (f \circ f^{\mathrm{ad}})^D \circ f = \mathrm{id}_V.$$

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We can also see that $f_\perp \phi_V = \phi_W$ if and only if $f \circ f^{\mathrm{ad}} = \mathrm{id}_W$ (because an endomorphism $g\colon W \to W$ satisfies that $g^D = \mathrm{id}_W$ if and only if $g = \mathrm{id}_W$), so we have that $f^\dagger = f^{\mathrm{ad}}$ if and only if that is the case.

REMARK 3.7. *Since the construction of adjoints is well-behaved, this point of view reduces the study of properties of the MP-inverse to the study of properties of the orthogonal push-forward construction $f_\perp$ (which ultimately depends on properties of the Drazin inverse). For instance, one can compute images and inverse images of arbitrary subspaces through the MP-inverse using orthogonality with respect to the orthogonal push-forward product.*

4. The push-forward metric as the closest to make f an isometry. Consider a linear map $f\colon (V, \phi_V) \to (W, \phi_W)$ as before. The orthogonal push-forward $f_\perp \phi_V$ that we have constructed can also be constructed purely geometrically within the space of inner products. Let us denote

$$T_2(V) := \mathrm{Hom}_k(V, V^*),$$

which is a k-vector space naturally isomorphic (in finite-dimensions) to the space of $(2,0)$-tensors[4].

REMARK 4.1. *By considering the vector space $T_2(V) := \mathrm{Hom}_k(V, V^*)$ as our "ambient" space we are, a priori, admitting non-symmetric bilinear forms in our theory, which may make sense in certain contexts. Our main result, however—Theorem 4.8—critically requires symmetry. All statements that follow hold true, with identical proofs, when the ambient space is restricted to that of symmetric bilinear forms, which is identified with*

$$S_2(V) = \{\phi \in T_2(V) : \phi^* \circ \mathrm{ev}_V = \phi\} \subseteq T_2(V),$$

where $\mathrm{ev}_V\colon V \to V^{**}$ is the canonical evaluation map.

Each linear map $g\colon V \to W$ induces a linear map $T_2(g)\colon T_2(W) \to T_2(V)$ given by $\alpha \mapsto g^* \circ \alpha \circ g$. Additionally, for a fixed non-degenerate product ϕ_V on V, we have a natural product on $T_2(V)$ (same for W), whose construction we provide below. For simplicity, assume that ϕ_V is globally non-degenerate.

REMARK 4.2 (On finite-dimensionality assumptions). *In the discussion that follows, we will endow $T_2(V)$ with a product that is related with the trace operator. In infinite-dimensions (for example, in infinite-dimensional Hilbert spaces), this turns the discussion more delicate, as not every operator has a well-defined trace, and requires one to talk about Schatten class operators—in this case, asking our elements in $T_2(V)$ to live in the class S_2, see [1, S.19]. In order to keep our discussion relatively elementary and purely algebraic in nature we are going to tacitly assume that all our spaces from this point on are finite-dimensional.*

CONSTRUCTION 4.3 (Inner product on $T_2(V)$). *Equip V^* with the product given by $(\phi_V^*)^{-1}\colon V^* \to V^{**}$, which allows us to describe the adjoint of $\phi \in T_2(V)$ with respect to ϕ_V and $(\phi_V^*)^{-1}$ as*

$$\phi^{\mathrm{ad}} = \phi_V^{-1} \circ \phi^* \circ (\phi_V^*)^{-1}.$$

Now, we have the Frobenius bilinear form (symmetric if and only if ϕ_V is symmetric) T_F on $T_2(V)$ induced by ϕ_V: given $\phi_1, \phi_2 \in T_2(V)$:

$$T_F(\phi_1, \phi_2) := \mathrm{tr}(\phi_2^{\mathrm{ad}} \circ \phi_1).$$

If we see it as a linear map $\phi_F\colon T_2(V) \to T_2(V)^*$, it is

$$\phi_F(\phi_2) = \mathrm{tr}(\phi_2^{\mathrm{ad}} \circ -).$$

[4] One has an injective linear map $\psi\colon V^* \otimes_k V^* \to \mathrm{Hom}_k(V, V^*)$ given by $\omega_1 \otimes \omega_2 \mapsto \psi_{\omega_1 \otimes \omega_2}$ with $\psi_{\omega_1 \otimes \omega_2}(v) = \omega_1(v) \cdot \omega_2$. It is an isomorphism in the finite-dimensional case by dimensionality reasons.

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REMARK 4.4 (Naturality of the Frobenius bilinear form). *We will now show that the induced Frobenius product on $T_2(V)$ is actually $T_2(\phi_V^{-1})$ up to composition with the injective map $\eta\colon T_2(V^*) \hookrightarrow T_2(V)^*$ so that*

$$\begin{aligned}\eta\colon \operatorname{Hom}_k(V^*, V^{**}) &\hookrightarrow (\operatorname{Hom}_k(V, V^*))^* \\ \psi &\mapsto \eta(\psi)\colon \operatorname{Hom}_k(V, V^*) \to k,\end{aligned}$$

with $\eta(\psi)(\phi) = \operatorname{tr}(\phi^* \circ \psi)$ for all $\phi \in T_2(V)$. Indeed, since ϕ_V is an isomorphism by our hypothesis in this section and the operation $T_2(-)$ satisfies, for any two linear maps g_1 and g_2, that

$$T_2(g_1 \circ g_2) = T_2(g_2) \circ T_2(g_1) \text{ and also that } T_2(\mathrm{id}_V) = \mathrm{id}_{T_2(V)},$$

(i.e. T_2 is a functor); it follows that $T_2(\phi_V)\colon T_2(V^*) \to T_2(V)$ is an isomorphism and $T_2(\phi_V)^{-1} = T_2(\phi_V^{-1})$. Now we may consider the composition

$$T_2(V) \overset{T_2(\phi_V)^{-1}}{\longrightarrow} T_2(V^*) \xrightarrow{\eta} T_2(V)^*,$$

and see that it yields the Frobenius bilinear form ϕ_F, i.e., abusing notation: $T_2(\phi_V^{-1}) = \phi_F$. Indeed, for any $\phi_2 \in T_2(V)$, we have $T_2(\phi_V^{-1})(\phi_2) = (\phi_V^{-1})^* \circ \phi_2 \circ \phi_V^{-1}$, and thus, for any $\phi_1 \in T_2(V)$, we have

$$\eta(T_2(\phi_V)(\phi_2))(\phi_1) = \operatorname{tr}(\phi_1^* \circ (\phi_V^{-1})^* \circ \phi_2 \circ \phi_V^{-1}) = \operatorname{tr}(\phi_V^{-1} \circ \phi_1^* \circ (\phi_V^{-1})^* \circ \phi_2) = \operatorname{tr}(\phi_1^{\mathrm{ad}} \circ \phi_2);$$

where in the second equality we have used that taking trace on a composition is commutative.

Now, in $(T_2(V), \phi_F)$, regarding f as a map $\overline{f}$ with codomain $\operatorname{Im}(f)$, denoted $\overline{f}\colon V \to \operatorname{Im}(f)$, we may consider the induced pullback

$$T_2(\overline{f})\colon T_2(\operatorname{Im}(f)) \to T_2(V) \text{ so that } \phi \mapsto (\overline{f})^* \circ \phi \circ \overline{f},$$

which is injective because $\overline{f}$ is surjective. For the image of this map, we have an orthogonal decomposition

$$T_2(V) = \operatorname{Im}(T_2(\overline{f})) \oplus \operatorname{Im}(T_2(\overline{f}))^{\perp}$$

and a canonical orthogonal projection $\pi\colon T_2(V) \to \operatorname{Im}(T_2(\overline{f}))$. The image of ϕ_V itself via this projection is, precisely, the bilinear form on the image of $T_2(\overline{f})$ that best approximates ϕ_V, and, by injectivity of $T_2(\overline{f})$, it must be the image of some unique $\overline{\phi} \in T_2(\operatorname{Im}(f))$. Finally, we extend this $\overline{\phi}$ to $T_2(W)$ by zero over the ortogonal complement $\operatorname{Im}(f)^{\perp}$ (with respect to ϕ_W). This construction yields, almost by definition, the least squares solution to the problem

$$T_2(f)(X) = \phi_V,$$

which can be formulated in terms of the Moore–Penrose inverse of the pull-back, i.e.,

$$X = T_2(f)^{\dagger}(\phi_V).$$

We will show that the result is indeed the orthogonal push-forward bilinear form $f_{\perp}\phi_V$, but first, let us first see a constructive example, step by step.

EXAMPLE 4.5. *Consider $f \in \operatorname{End}_k(k^{\oplus 3})$ given by multiplication by*

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

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and assume that $V = k^{\oplus 3}$ comes equipped with the usual inner product, denoted ϕ_V for consistency with the previous discussion, whose associated matrix with respect to the canonical basis is the identity matrix.

We have that $\mathrm{Im}(f) = \langle (1,0,0),(0,0,1)\rangle_k$ *and that the induced map*

$$\overline{f}\colon k^{\oplus 3} \to \mathrm{Im}(f),$$

is given by multiplication by the matrix

$$M = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

with respect to the chosen basis. The pullback $T_2(\overline{f})\colon T_2(\mathrm{Im}(f)) \to T_2(k^{\oplus 3})$ *is given by mapping an inner product with matrix K to $M^t \cdot K \cdot M$ with respect to the given basis on* $\mathrm{Im}(f)$ *and the canonical basis on $k^{\oplus 3}$, respectively. A basis of* $\mathrm{Im}(T_2(\overline{f}))$ *is*

$$\mathrm{Im}(T_2(\overline{f})) = \left\langle \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \right\rangle_k \subseteq T_2(k^{\oplus 3}).$$

The standard inner product on $T_2(k^{\oplus 3})$ *restricted to* $T_2(\mathrm{Im}(f))$ *is given, with respect to the basis chosen above, by the matrix*

$$C = \begin{pmatrix} 4 & 2 & 2 & 1 \\ 2 & 4 & 1 & 2 \\ 2 & 1 & 4 & 2 \\ 1 & 2 & 2 & 4 \end{pmatrix}.$$

The orthogonal projection $\pi\colon T_2(k^{\oplus 3}) \to \mathrm{Im}(T_2(\overline{f}))$ *is the adjoint to the inclusion* $\mathrm{Im}(T_2(\overline{f})) \hookrightarrow T_2(k^{\oplus 3})$*, where the Frobenius product on* $T_2(k^{\oplus 3})$ *coincides with the standard metric for matrices and where* $\mathrm{Im}(T_2(\overline{f}))$ *is endowed with the one given by C. It is given by multiplication by*

$$\Pi = C^{-1} \cdot \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^t \cdot \mathrm{Id} = \begin{pmatrix} \frac{1}{9} & \frac{2}{9} & \frac{-1}{9} & \frac{2}{9} & \frac{4}{9} & \frac{-2}{9} & \frac{-1}{9} & \frac{-2}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{-1}{9} & \frac{2}{9} & \frac{2}{9} & \frac{-2}{9} & \frac{4}{9} & \frac{-1}{9} & \frac{1}{9} & \frac{-2}{9} \\ \frac{1}{9} & \frac{2}{9} & \frac{-1}{9} & \frac{-1}{9} & \frac{-2}{9} & \frac{1}{9} & \frac{2}{9} & \frac{4}{9} & \frac{-2}{9} \\ \frac{1}{9} & \frac{-1}{9} & \frac{2}{9} & \frac{-1}{9} & \frac{1}{9} & \frac{-2}{9} & \frac{2}{9} & \frac{-2}{9} & \frac{4}{9} \end{pmatrix},$$

Note that the matrix in the middle is the one whose columns are exactly the coordinates of the four basis vectors of $\mathrm{Im}(T_2(\overline{f}))$ *with respect to the standard basis; it coincides with $M \otimes M$, where $\otimes$ denotes the Kronecker product.*

Now we compute that, in coordinates with respect to the basis above, the orthogonal projection of the canonical metric—whose coordinates are $(1,0,0,0,1,0,0,0,1)$*—onto* $\mathrm{Im}(T_2(\overline{f}))$ *is*

$$\pi(\phi_V) \equiv (2/3, -1/3, -1/3, 2/3);$$

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in other words, $\pi(\phi_V) \in \mathrm{Im}(T_2(\overline{f})) \subseteq T_2(k^{\oplus 3})$ *has matrix, with respect to the standard basis of* $k^{\oplus 3}$*:*

$$\pi(\phi_V) \equiv \begin{pmatrix} 2/3 & 1/3 & 1/3 \\ 1/3 & 2/3 & -1/3 \\ 1/3 & -1/3 & 2/3 \end{pmatrix} \in \mathrm{Im}(T_2(\overline{f})).$$

Now we may regard $T_2(\overline{f})$ *as an isomorphism onto its image*

$$\overline{T_2(\overline{f})} \colon T_2(\mathrm{Im}(f)) \xrightarrow{\sim} \mathrm{Im}(T_2(\overline{f})).$$

Computing the inverse image of $\pi(\phi_V)$ *via this isomorphism may be done in coordinates. The matrix of* $\overline{T_2(\overline{f})}$ *with respect to our already chosen bases is exactly the identity matrix, hence,* $(\overline{T_2(\overline{f})})^{-1}(\pi(\phi_V))$ *also has coordinates*

$$(\overline{T_2(\overline{f})})^{-1} \equiv (2/3, -1/3, -1/3, 2/3),$$

but this time with respect to the standard basis on the space of 2×2 *matrices that we have identified* $T_2(\mathrm{Im}(f))$ *with, i.e.,*

$$(\overline{T_2(\overline{f})})^{-1}(\pi(\phi_V)) = \begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix} \in T_2(\mathrm{Im}(f)),$$

with respect to our chosen basis for $\mathrm{Im}(f)$*.*

Finally, we extend this metric by zero over $\mathrm{Im}(f)^\perp = \langle 0,1,0\rangle_k$ *to get our desired product*

$$f_\perp\phi_V = \begin{pmatrix} 2/3 & 0 & -1/3 \\ 0 & 0 & 0 \\ -1/3 & 0 & 2/3 \end{pmatrix}.$$

The Moore–Penrose inverse is the adjoint of f *with respect to this product on the codomain and the standard product on the domain:*

$$f^\dagger \equiv A^\dagger = \mathrm{Id}^{-1} \cdot A^t \cdot f_\perp\phi_V = \begin{pmatrix} 1/3 & 0 & 1/3 \\ 2/3 & 0 & -1/3 \\ -1/3 & 0 & 2/3 \end{pmatrix}.$$

We now prove two small lemmas before the main result of this section.

LEMMA 4.6. *For any linear map* $f\colon V \to W$ *between vector spaces equipped with non-degenerate inner products* ϕ_V *and* ϕ_W[5]*, then*

$$(T_2(f))^{\mathrm{ad}} = T_2(f^{\mathrm{ad}});$$

where the first adjoint is taken with respect to the Frobenius products induced on $T_2(V)$ *and* $T_2(W)$ *by* ϕ_V *and* ϕ_W*.*

[5]In fact, ϕ_V and ϕ_W only need to be in the hypothesis of Definition 3.4. Indeed, non-degeneracy of ϕ_V over $\ker(f)^\perp$ guarantees the existence of a unique adjoint f^{ad} (Footnote 1) and the fact that $\ker(f)^\perp$ is a complementary of $\ker(f)$ gives one the natural way define the Frobenius metric on $T_2(V)$ as the extension by zero of the Frobenius metric associated to $(\phi_V)_{|\ker(f)^\perp}$ over the complementary of $T_2(\ker(f)^\perp)$ in $T_2(V) = T_2(\ker(f)^\perp) \oplus T_2(\ker(f)) \oplus \mathrm{Hom}_k(\ker(f), (\ker(f)^\perp)^*) \oplus \mathrm{Hom}_k(\ker(f)^\perp, \ker(f)^*)$. One proceeds similarly for W to guarantee the existence of a unique adjoint $T_2(f)^{\mathrm{ad}}$. The reader may check that the resulting Frobenius metrics are also in the hypothesis of Definition 3.4 for $T_2(f)\colon T_2(W) \to T_2(V)$. Now, the general proof of the Lemma remains the same modulo domain and codomain restrictions.

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Proof. As seen in Remark 4.4, the construction of $T_2(-)$ satisfies, for any linear maps $g_1\colon V_1 \to V_2$ and $g_2\colon V_2 \to V_3$,

$$T_2(g_2 \circ g_1) = T_2(g_1) \circ T_2(g_2),$$

$T_2(\mathrm{id}_V) = \mathrm{id}_{T_2(V)}$ (i.e. T_2 is a functor), $T_2(g_1^*) = T_2(g_1)^*$ (same for g_2) and $T_2(\phi_V) = \phi_{F,V}$ is the corresponding Frobenius inner product on $T_2(V)$ (similar for W) via the identification $T_2(V^*) \simeq T_2(V)^*$. Due to these reasons and bearing in mind Remark 4.4, it is immediate that

$$T_2(f^{\mathrm{ad}}) = T_2(\phi_V^{-1} \circ f^* \circ \phi_W) = T_2(\phi_W) \circ T_2(f^*) \circ T_2(\phi_V^{-1}) = \phi_{F,W}^{-1} \circ T_2(f)^* \circ \phi_{F,V} = T_2(f)^{\mathrm{ad}},$$

which concludes the proof. The last part of the statement follows by functoriality of T_2. ∎

LEMMA 4.7. *For any linear map $f\colon V \to W$ between vector spaces equipped with respective suitable*[6] *symmetric inner products ϕ_V and ϕ_W, then*

$$T_2(f)^\dagger = T_2(f^\dagger).$$

That is, taking $T_2(-)$ commutes with taking the Moore–Penrose inverse.

Proof. This is, once again, a consequence of the functoriality of $T_2(-)$ and of the uniqueness of the Moore–Penrose inverse as the only linear map satisfying the Equations (1.1). Indeed, $T_2(f^\dagger)$ satisfies the first two equations of $T_2(f)^\dagger$ (it is a reflexive inverse),

$$\begin{aligned} T_2(f) \circ T_2(f^\dagger) \circ T_2(f) &= T_2(f \circ f^\dagger \circ f) = T_2(f); \\ T_2(f^\dagger) \circ T_2(f) \circ T_2(f^\dagger) &= T_2(f^\dagger \circ f \circ f^\dagger) = T_2(f^\dagger). \end{aligned}$$

Similarly, $T_2(f) \circ T_2(f^\dagger) = T_2(f^\dagger \circ f)$ is idempotent by functoriality and self-adjoint because $f^\dagger \circ f$ is self-adjoint and one can apply Lemma 4.6. In the same way, $T_2(f^\dagger) \circ T_2(f)$ is also a self-adjoint idempotent. This concludes the proof. ∎

We now arrive to the main result of this section.

THEOREM 4.8. *For any linear map $f\colon V \to W$ between vector spaces equipped with respective suitable*[6] *symmetric inner products ϕ_V and ϕ_W, then*

$$(T_2(f))^\dagger(\phi_V) = f_\perp \phi_V.$$

In other words, $f_\perp \phi_V$ is the "least squares solution" to the problem

$$T_2(f)(X) = \phi_V,$$

with respect to the induced product on $T_2(W)$ by ϕ_W, i.e., $f_\perp \phi_V$ is the inner product on W that is the closest to make f into an isometry.

Proof. Thanks to Lemma 4.7 and by definition of pullback, one has

$$\begin{aligned} T_2(f)^\dagger(\phi_V) = T_2(f^\dagger)(\phi_V) &= (f \circ f^{\mathrm{ad}})^{D*} f^{\mathrm{ad}*} \phi_V f^{\mathrm{ad}} (f \circ f^{\mathrm{ad}})^D \\ &= (f \circ f^{\mathrm{ad}})^{D*} \phi_W^* f^{**} (\phi_V^{-1})^* \phi_V f^{\mathrm{ad}} (f \circ f^{\mathrm{ad}})^D \\ &\overset{(\star)}{=} (f \circ f^{\mathrm{ad}})^{D*} \phi_W (f \circ f^{\mathrm{ad}}) (f \circ f^{\mathrm{ad}})^D; \end{aligned}$$

[6] In the hypotheses of Definition 3.4 for $T_2(f)$. See also 5.

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where the third equality is the definition of $f^{\mathrm{ad}*}$ (see Definition 2.1) and in $(\star)$ we have used that $f^{**} \circ \mathrm{ev}_V = \mathrm{ev}_W \circ f$ (with $\mathrm{ev}_V, \mathrm{ev}_W$ the respective evaluation maps) and that $\phi_V^* \circ \mathrm{ev}_V = \phi_V$ (and same for ϕ_W) because we have assumed our bilinear forms to be symmetric.

Now, composing on the left by $\phi_W \circ \phi_W^{-1}$, we get

$$T_2(f)^\dagger = \phi_W \circ ((f \circ f^{\mathrm{ad}})^D)^{\mathrm{ad}} (f \circ f^{\mathrm{ad}})(f \circ f^{\mathrm{ad}})^D,$$

where the new adjoint is taken with respect to ϕ_W. Now, note that taking Drazin inverse commutes with taking adjoint because the latter operation is compatible with composition (it is "functorial") and the Drazin inverse is the unique endomorphism satisfying the defining equations of (3.4). Hence,

$$((f \circ f^{\mathrm{ad}})^D)^{\mathrm{ad}} = ((f \circ f^{\mathrm{ad}})^{\mathrm{ad}})^D = (f \circ f^{\mathrm{ad}})^D,$$

because $f \circ f^{\mathrm{ad}}$ is self-adjoint[7] with respect to ϕ_W. We have arrived, since the Drazin inverse is a 2-inverse, to

$$T_2(f)^\dagger = \phi_W \circ (f \circ f^{\mathrm{ad}})^D = f_\perp \phi_V,$$

which completes the proof. ∎

Example 4.9. *Once again, with the same data as in the previous examples, the pull-back of metrics associated to f given by*

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

is described, with respect to the usual basis for spaces of matrices, with

$$T_2(A) \equiv A^t \otimes A^t = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Its Moore–Penrose inverse with respect to the Frobenius metric on $T_2(k^{\oplus 3})$ (on both source and target), that

[7] This critically requires the bilinear forms ϕ_V and ϕ_W to be symmetric; otherwise, one has to alternate between left and right adjoints for the double adjoint duality to hold, which will never lead to self-adjointness in this expression because it mixes the original map with its adjoint.

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is, the ordinary Moore–Penrose inverse matrix is

$$T_2(A)^\dagger = \begin{pmatrix} \frac{1}{9} & \frac{2}{9} & \frac{-1}{9} & \frac{2}{9} & \frac{4}{9} & \frac{-2}{9} & \frac{-1}{9} & \frac{-2}{9} & \frac{1}{9} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{9} & \frac{-1}{9} & \frac{2}{9} & \frac{2}{9} & \frac{-2}{9} & \frac{4}{9} & \frac{-1}{9} & \frac{1}{9} & \frac{-2}{9} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{9} & \frac{2}{9} & \frac{-1}{9} & \frac{-1}{9} & \frac{-2}{9} & \frac{1}{9} & \frac{2}{9} & \frac{4}{9} & \frac{-2}{9} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{9} & \frac{-1}{9} & \frac{2}{9} & \frac{-1}{9} & \frac{1}{9} & \frac{-2}{9} & \frac{2}{9} & \frac{-2}{9} & \frac{4}{9} \end{pmatrix};$$

and thus the image by this map of the canonical metric of $k^{\oplus 3}$, which in coordinates may be written as $\phi_V = (1,0,0,0,1,0,0,0,1)$ is

$$T_2(A)^\dagger(\phi_V) = \left(\frac{2}{3}, 0, -\frac{1}{3}, 0, 0, 0, -\frac{1}{3}, 0, \frac{2}{3}\right);$$

which, as a matrix, is exactly

$$f_\perp \phi_V = \begin{pmatrix} 2/3 & 0 & -1/3 \\ 0 & 0 & 0 \\ -1/3 & 0 & 2/3 \end{pmatrix},$$

coinciding with our previous calculations.

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