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TWO SPECIAL KINDS OF LEAST SQUARES SOLUTIONS FOR THE QUATERNION MATRIX EQUATION $AXB + CXD = E$ *

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Abstract. By using the complex representation of quaternion matrices, the Moore–Penrose generalized inverse and the Kronecker product of matrices, the expressions of the least squares η-Hermitian solution with the least norm and the expressions of the least squares η-anti-Hermitian solution with the least norm are derived for the matrix equation $AXB+CXD = E$ over quaternions.

Key words. Matrix equation, Least squares solution, η-Hermitian matrix, η-Anti-Hermitian matrix, Moore–Penrose generalized inverse, Kronecker product, Quaternion matrices.

AMS subject classifications. 65F05, 65H10, 15A33.

1. Introduction.

Throughout this paper, let $\mathbb{Q}$, $\mathbb{R}^{m\times n}$, $\mathbb{SR}^{n\times n}$, $\mathbb{ASR}^{n\times n}$, $\mathbb{C}^{m\times n}$, and $\mathbb{Q}^{m\times n}$ be the skew field of quaternions, the set of all $m \times n$ real matrices, the set of all $n \times n$ real symmetric matrices, the set of all $n \times n$ real anti-symmetric matrices, the set of all $m \times n$ complex matrices, and the set of all $m \times n$ quaternion matrices, respectively. For $A \in \mathbb{C}^{m\times n}$, $\mathrm{Re}(A)$ and $\mathrm{Im}(A)$ denote the real part and the imaginary part of matrix A, respectively. For $A \in \mathbb{Q}^{m\times n}$, $\overline{A}$, A^T, A^H and A^+ denote the conjugate matrix, the transpose matrix, the conjugate transpose matrix, and the Moore–Penrose generalized inverse matrix of matrix A, respectively.

A quaternion a can be uniquely expressed as $a = a_0 + a_1 i + a_2 j + a_3 k$ with real coefficients a_0, a_1, a_2, a_3, and $i^2 = j^2 = k^2 = -1, ij = -ji = k$, and a can be uniquely expressed as $a = c_1 + c_2 j$, where c_1 and c_2 are complex numbers. The following quaternion involutions of a quaternion $a = a_0 + a_1 i + a_2 j + a_3 k$, defined as [4]

$$a^i = -iai = a_0 + a_1 i - a_2 j - a_3 k,$$

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$$a^j = -jaj = a_0 - a_1 i + a_2 j - a_3 k,$$
$$a^k = -kak = a_0 - a_1 i - a_2 j + a_3 k.$$

For any $A \in \mathbb{Q}^{m\times n}$, A can be uniquely expressed as $A = A_1 + A_2 j$, where $A_1, A_2 \in \mathbb{C}^{m\times n}$, and $A^H = \mathrm{Re}(A_1)^T - \mathrm{Im}(A_1)^T i - \mathrm{Re}(A_2)^T j - \mathrm{Im}(A_2)^T k$. The complex representation matrix of $A = A_1 + A_2 j \in \mathbb{Q}^{m\times n}$ is denoted by

$$f(A) = \begin{bmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{bmatrix} \in \mathbb{C}^{2m\times 2n}.$$

Notice that $f(A)$ is uniquely determined by A. For $A \in \mathbb{Q}^{m\times n}, B \in \mathbb{Q}^{n\times s}$, we have $f(AB) = f(A)f(B)$ (see [39]). We define the *inner product*: $\langle A, B\rangle = \mathrm{tr}(B^H A)$ for all $A, B \in \mathbb{Q}^{m\times n}$. Then $\mathbb{Q}^{m\times n}$ is a right Hilbert inner product space and the norm of a matrix generated by this inner product is the quaternion matrix Frobenius norm $\|\cdot\|$. For $A = (a_{ij}) \in \mathbb{Q}^{m\times n}, B = (b_{ij}) \in \mathbb{Q}^{p\times q}$, the symbol $A \otimes B = (a_{ij}B) \in \mathbb{Q}^{mp\times nq}$ stands for the Kronecker product of A and B.

Definition 1.1. [6, 18] A matrix $A \in \mathbb{Q}^{n\times n}$ is *η-Hermitian* if $A^{\eta H} = A$, and a matrix $A \in \mathbb{Q}^{n\times n}$ is *η-anti-Hermitian* if $A^{\eta H} = -A$, where $A^{\eta H} = -\eta A^H \eta$, $\eta \in \{i, j, k\}$. η-Hermitian matrices and η-anti -Hermitian matrices are denoted by $\eta\mathbb{HQ}^{n\times n}$ and $\eta\mathbb{AQ}^{n\times n}$, respectively.

Various aspects of the solutions for real, complex, or quaternion matrix equations such as $AXB = C$, $AX + XB = C$, $AXB + CX^T D = E$, $X - A\overline{X}B = C$, and $AXB + CYD = E$ have been widely investigated (see [2], [8]-[11], [12], [14], [15], [19]-[38], [40] and references cited therein). For the matrix equation

$$AXB + CXD = E, \tag{1.1}$$

if B and C are identity matrices, then the matrix equation (1.1) reduces to the well-known Sylvester equation [3]. If C and D are identity matrices, then the matrix equation (1.1) reduces to the well-known Stein equation. There are many important results about their solutions. For example, Mitra [14] and Tian [16] considered the solvability condition for the complex and real matrix equation (1.1), respectively. Hernández and Gassó [5] obtained the explicit solution of the matrix equation (1.1). Mansour [13] studied the solvability condition for the matrix equation (1.1) in the operator algebra. For the quaternion matrix equation (1.1), Huang [7] obtained necessary and sufficient conditions for the existence of a solution or a unique solution using the method of complex representation of quaternion matrices. Using the complex representation of quaternion matrices, the Moore–Penrose generalized inverse, and the Kronecker product of matrices, Yuan, Liao, and Lei [38] studied the least squares Hermitian problems for the quaternion matrix equation $(AXB, CXD) = (E, F)$. η-Hermitian and η-anti-Hermitian quaternion matrices, are important class of matrices

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applied in widely linear modelling due to the quaternion involution properties (see [4, 17, 18] for details). In this paper, we use the results of [38] to consider the least squares η-Hermitian and η-anti-Hermitian problems for quaternion matrix equation (1.1). The related problems are described as follows.

Problem I. Given $A \in \mathbb{Q}^{m\times n}$, $B \in \mathbb{Q}^{n\times s}$, $C \in \mathbb{Q}^{m\times n}$, $D \in \mathbb{Q}^{n\times s}$, and $E \in \mathbb{Q}^{m\times s}$, let

$$H_L = \{X | X \in \eta\mathbb{HQ}^{m\times n}, ||AXB + CXD - E|| = \min_{X_0\in\eta\mathbb{HQ}^{m\times n}} ||AX_0B + CX_0D - E||\}.$$

Find $X_H \in H_L$ such that

$$||X_H|| = \min_{X\in H_L} ||X||.$$

Problem II. Given $A \in \mathbb{Q}^{m\times n}$, $B \in \mathbb{Q}^{n\times s}$, $C \in \mathbb{Q}^{m\times n}$, $D \in \mathbb{Q}^{n\times s}$, and $E \in \mathbb{Q}^{m\times s}$, let

$$A_L = \{X | X \in \eta\mathbb{AQ}^{n\times n}, ||AXB + CXD - E|| = \min_{X_0\in\eta\mathbb{AQ}^{m\times n}} ||AX_0B + CX_0D - E||\}.$$

Find $X_A \in A_L$ such that

$$||X_A|| = \min_{X\in A_L} ||X||.$$

The solution X_H of Problem I is called the least squares η-Hermitian solution with the least norm, and the solution X_A of Problem II is called the least squares η-anti-Hermitian solution with the least norm for matrix equation (1.1) over quaternions.

Our approach to solving these problems is based on the way of studying vec(ABC) mentioned in [38], which can overcome the difficulty from the noncommutative multiplication of quaternions, and turns Problems I and II of quaternion matrix equation (1.1) into a system of real equations, respectively.

This paper is organized as follows. In Section 2, we analyze the structure of two special matrix sets: $\eta\mathbb{HQ}^{n\times n}$ and $\eta\mathbb{AQ}^{n\times n}$. In Section 3, we derive the explicit expression for the solution of Problem I. In Section 4, we derive the explicit expression for the solution of Problem II. Finally, in Section 5, we give numerical algorithms and numerical examples for Problems I and II, respectively.

2. The structure of $\eta\mathbb{HQ}^{n\times n}$ and $\eta\mathbb{AQ}^{n\times n}$. In this section, we analyze the structure of $\eta\mathbb{HQ}^{n\times n}$ and $\eta\mathbb{AQ}^{n\times n}$.

DEFINITION 2.1. For matrix $A \in \mathbb{Q}^{n\times n}$, let $a_1 = (a_{11}, \sqrt{2}a_{21}, \ldots, \sqrt{2}a_{n1})$, $a_2 = (a_{22}, \sqrt{2}a_{32}, \ldots, \sqrt{2}a_{n2})$, $\ldots$, $a_{n-1} = (a_{(n-1)(n-1)}, \sqrt{2}a_{n(n-1)})$, $a_n = a_{nn}$, and denote

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by $\mathrm{vec}_S(A)$ the following vector:

$$\mathrm{vec}_S(A) = (a_1, a_2, \ldots, a_{n-1}, a_n)^T \in \mathbb{Q}^{\frac{n(n+1)}{2}}. \tag{2.1}$$

Definition 2.2. For matrix $B \in \mathbb{Q}^{n\times n}$, let $b_1 = (b_{21}, b_{31}, \ldots, b_{n1})$, $b_2 = (b_{32}, b_{42}, \ldots, b_{n2})$, $\ldots$, $b_{n-2} = (b_{(n-1)(n-2)}, b_{n(n-2)})$, $b_{n-1} = b_{n(n-1)}$, and denote by $\mathrm{vec}_A(B)$ the following vector:

$$\mathrm{vec}_A(B) = \sqrt{2}(b_1, b_2, \ldots, b_{n-2}, b_{n-1})^T \in \mathbb{Q}^{\frac{n(n-1)}{2}}. \tag{2.2}$$

Lemma 2.3. [38] *Suppose $X \in \mathbb{R}^{n\times n}$. Then,*

(i) $X \in \mathbb{SR}^{n\times n} \iff \mathrm{vec}(X) = K_S\mathrm{vec}_S(X)$, *where* $\mathrm{vec}_S(X)$ *is represented as* (2.1), *and the matrix* $K_S \in \mathbb{R}^{n^2\times\frac{n(n+1)}{2}}$ *is of the following form:*

$$K_S = \frac{1}{\sqrt{2}}$$

$$\begin{bmatrix}
\sqrt{2}e_1 & e_2 & e_3 & \cdots & e_{n-1} & e_n & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0\\
0 & e_1 & 0 & \cdots & 0 & 0 & \sqrt{2}e_2 & e_3 & \cdots & e_{n-1} & e_n & \cdots & 0 & 0 & 0\\
0 & 0 & e_1 & \cdots & 0 & 0 & 0 & e_2 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots\\
0 & 0 & 0 & \cdots & e_1 & 0 & 0 & 0 & \cdots & e_2 & 0 & \cdots & \sqrt{2}e_{n-1} & e_n & 0\\
0 & 0 & 0 & \cdots & 0 & e_1 & 0 & 0 & \cdots & 0 & e_2 & \cdots & 0 & e_{n-1} & \sqrt{2}e_n
\end{bmatrix},$$

(ii) $X \in \mathbb{ASR}^{n\times n} \iff \mathrm{vec}(X) = K_A\mathrm{vec}_A(X)$, *where* $\mathrm{vec}_A(X)$ *is represented as* (2.2), *and the matrix* $K_A \in \mathbb{R}^{n^2\times\frac{n(n-1)}{2}}$ *is of the following form:*

$$K_A = \frac{1}{\sqrt{2}}\begin{bmatrix}
e_2 & e_3 & \cdots & e_{n-1} & e_n & 0 & \cdots & 0 & 0 & \cdots & 0\\
-e_1 & 0 & \cdots & 0 & 0 & e_3 & \cdots & e_{n-1} & e_n & \cdots & 0\\
0 & -e_1 & \cdots & 0 & 0 & -e_2 & \cdots & 0 & 0 & \cdots & 0\\
\vdots & \vdots & & \vdots & \vdots & \vdots & & 0 & 0 & & 0\\
0 & 0 & \cdots & -e_1 & 0 & 0 & \cdots & -e_2 & 0 & \cdots & e_n\\
0 & 0 & \cdots & 0 & -e_1 & 0 & \cdots & 0 & -e_2 & \cdots & -e_{n-1}
\end{bmatrix},$$

where e_i is the i-th column of I_n. Obviously, $K_S^TK_S = I_{\frac{n(n+1)}{2}}$ *and* $K_A^TK_A = I_{\frac{n(n-1)}{2}}$.

We identify $q \in \mathbb{Q}$ with a complex vector $\vec{q} \in \mathbb{C}^2$, and denote such an identification by the symbol $\cong$, that is,

$$c_1 + c_2j = q \cong \vec{q} = (c_1, c_2).$$

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For $A = A_1 + A_2 j \in \mathbb{Q}^{m\times n}$, we have $A \cong \Phi_A = (A_1, A_2)$,

$$\mathrm{vec}(A_1) + \mathrm{vec}(A_2)j = \mathrm{vec}(A) \cong \mathrm{vec}(\Phi_A) = \begin{bmatrix} \mathrm{vec}(A_1) \\ \mathrm{vec}(A_2) \end{bmatrix},$$

so that

$$\|\mathrm{vec}(A)\| = \|\mathrm{vec}(\Phi_A)\| = \left\| \begin{bmatrix} \mathrm{vec}(A_1) \\ \mathrm{vec}(A_2) \end{bmatrix} \right\|.$$

We denote $\overrightarrow{A} = (\mathrm{Re}(A_1), \mathrm{Im}(A_1), \mathrm{Re}(A_2), \mathrm{Im}(A_2))$,

$$\mathrm{vec}(\overrightarrow{A}) = \begin{bmatrix} \mathrm{vec}(\mathrm{Re}(A_1)) \\ \mathrm{vec}(\mathrm{Im}(A_1)) \\ \mathrm{vec}(\mathrm{Re}(A_2)) \\ \mathrm{vec}(\mathrm{Im}(A_2)) \end{bmatrix}.$$

Notice that $\|\mathrm{vec}(\Phi_A)\| = \|\mathrm{vec}(\overrightarrow{A})\|$. In particular, for $A = A_1 + A_2 i \in \mathbb{C}^{m\times n}$ with $A_1, A_2 \in \mathbb{R}^{m\times n}$, we have $A \cong \overrightarrow{A} = (A_1, A_2)$, and

$$\mathrm{vec}(A_1) + \mathrm{vec}(A_2)i = \mathrm{vec}(A) \cong \mathrm{vec}(\overrightarrow{A}) = \begin{bmatrix} \mathrm{vec}(A_1) \\ \mathrm{vec}(A_2) \end{bmatrix}.$$

Addition of two quaternion matrices $A = A_1 + A_2 j$ and $B = B_1 + B_2 j$ satisfies

$$(A_1 + B_1) + (A_2 + B_2)j = (A + B) \cong \Phi_{A+B} = (A_1 + B_1, A_2 + B_2),$$

whereas multiplication satisfies

$$AB = (A_1 + A_2 j)(B_1 + B_2 j) = (A_1 B_1 - A_2 \overline{B_2}) + (A_1 B_2 + A_2 \overline{B_1})j.$$

So $AB \cong \Phi_{AB}$, moreover, Φ_{AB} can be expressed as

$$\begin{aligned} \Phi_{AB} &= (A_1 B_1 - A_2 \overline{B_2}, A_1 B_2 + A_2 \overline{B_1}) \\ &= (A_1, A_2) \begin{bmatrix} B_1 & B_2 \\ -\overline{B_2} & \overline{B_1} \end{bmatrix} \\ &= \Phi_A f(B). \end{aligned}$$

For $X = X_1 + X_2 j \in \eta\mathbb{HQ}^{n\times n}$, by Definition 1.1, we have

$$\begin{aligned} & X^H \eta = \eta X \\ \Longleftrightarrow\ & (\mathrm{Re}(X_1)^T - \mathrm{Im}(X_1)^T i - \mathrm{Re}(X_2)^T j - \mathrm{Im}(X_2)^T k)\eta \end{aligned}$$

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$$= \eta(\mathrm{Re}(X_1) + \mathrm{Im}(X_1)i + \mathrm{Re}(X_2)j + \mathrm{Im}(X_2)k)$$

$$\Longleftrightarrow \mathrm{Re}(X_1)^T = \mathrm{Re}(X_1), \quad \mathrm{Im}(X_1)^T = \begin{cases} -\mathrm{Im}(X_1), & \eta = i \\ \mathrm{Im}(X_1), & \eta \neq i \end{cases},$$

$$\mathrm{Re}(X_2)^T = \begin{cases} -\mathrm{Re}(X_2), & \eta = j \\ \mathrm{Re}(X_2), & \eta \neq j \end{cases}, \quad \mathrm{Im}(X_2)^T = \begin{cases} -\mathrm{Im}(X_2), & \eta = k \\ \mathrm{Im}(X_2), & \eta \neq k \end{cases}.$$

THEOREM 2.4. *If $X \in \mathbb{Q}^{n\times n}$, then $X \in \eta\mathbb{H}\mathbb{Q}^{n\times n} \Longleftrightarrow \mathrm{vec}(\overrightarrow{X}) = K_{\eta H}\mathrm{vec}_{\eta H}(\overrightarrow{X})$, where*

$$K_{iH} = \begin{bmatrix} K_S & 0 & 0 & 0 \\ 0 & K_A & 0 & 0 \\ 0 & 0 & K_S & 0 \\ 0 & 0 & 0 & K_S \end{bmatrix}, \quad \mathrm{vec}_{iH}(\overrightarrow{A}) = \begin{bmatrix} \mathrm{vec}_S(\mathrm{Re}(A_1)) \\ \mathrm{vec}_A(\mathrm{Im}(A_1)) \\ \mathrm{vec}_S(\mathrm{Re}(A_2)) \\ \mathrm{vec}_S(\mathrm{Im}(A_2)) \end{bmatrix},$$

$$K_{jH} = \begin{bmatrix} K_S & 0 & 0 & 0 \\ 0 & K_S & 0 & 0 \\ 0 & 0 & K_A & 0 \\ 0 & 0 & 0 & K_S \end{bmatrix}, \quad \mathrm{vec}_{jH}(\overrightarrow{A}) = \begin{bmatrix} \mathrm{vec}_S(\mathrm{Re}(A_1)) \\ \mathrm{vec}_S(\mathrm{Im}(A_1)) \\ \mathrm{vec}_A(\mathrm{Re}(A_2)) \\ \mathrm{vec}_S(\mathrm{Im}(A_2)) \end{bmatrix},$$

$$K_{kH} = \begin{bmatrix} K_S & 0 & 0 & 0 \\ 0 & K_S & 0 & 0 \\ 0 & 0 & K_S & 0 \\ 0 & 0 & 0 & K_A \end{bmatrix}, \quad \mathrm{vec}_{kH}(\overrightarrow{A}) = \begin{bmatrix} \mathrm{vec}_S(\mathrm{Re}(A_1)) \\ \mathrm{vec}_S(\mathrm{Im}(A_1)) \\ \mathrm{vec}_S(\mathrm{Re}(A_2)) \\ \mathrm{vec}_A(\mathrm{Im}(A_2)) \end{bmatrix}.$$

Proof. We prove it only the case of $\eta = j$, similar arguments handle the cases that $\eta = i$ and $\eta = k$. For $X \in \mathbb{Q}^{n\times n}$, we have

$$\begin{aligned} & X \in j\mathbb{H}\mathbb{Q}^{n\times n} \\ \Longleftrightarrow\ & X^H j = jX \\ \Longleftrightarrow\ & \mathrm{Re}(X_1)^T = \mathrm{Re}(X_1),\ \mathrm{Im}(X_1)^T = \mathrm{Im}(X_1), \\ & \quad \mathrm{Re}(X_2)^T = -\mathrm{Re}(X_2),\ \mathrm{Im}(X_1)^T = \mathrm{Im}(X_1) \end{aligned}$$

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$$\Longleftrightarrow \operatorname{vec}(\overrightarrow{X}) = \begin{bmatrix} \operatorname{vec}(\operatorname{Re}(A_1)) \\ \operatorname{vec}(\operatorname{Im}(A_1)) \\ \operatorname{vec}(\operatorname{Re}(A_2)) \\ \operatorname{vec}(\operatorname{Im}(A_2)) \end{bmatrix} = \begin{bmatrix} K_S & 0 & 0 & 0 \\ 0 & K_S & 0 & 0 \\ 0 & 0 & K_A & 0 \\ 0 & 0 & 0 & K_S \end{bmatrix} \begin{bmatrix} \operatorname{vec}_S(\operatorname{Re}(A_1)) \\ \operatorname{vec}_S(\operatorname{Im}(A_1)) \\ \operatorname{vec}_A(\operatorname{Re}(A_2)) \\ \operatorname{vec}_S(\operatorname{Im}(A_2)) \end{bmatrix}$$

$$= K_{jH}\operatorname{vec}_{jH}(\overrightarrow{X}). \quad \square$$

Similar to the discussions mentioned above, we have

THEOREM 2.5. *If $X \in \mathbb{Q}^{n\times n}$, then $X \in \eta\mathbb{A}\mathbb{Q}^{n\times n} \Longleftrightarrow \operatorname{vec}(\overrightarrow{X}) = K_{\eta A}\operatorname{vec}_{\eta A}(\overrightarrow{X})$, where*

$$K_{iA} = \begin{bmatrix} K_A & 0 & 0 & 0 \\ 0 & K_S & 0 & 0 \\ 0 & 0 & K_A & 0 \\ 0 & 0 & 0 & K_A \end{bmatrix}, \quad \operatorname{vec}_{iA}(\overrightarrow{A}) = \begin{bmatrix} \operatorname{vec}_A(\operatorname{Re}(A_1)) \\ \operatorname{vec}_S(\operatorname{Im}(A_1)) \\ \operatorname{vec}_A(\operatorname{Re}(A_2)) \\ \operatorname{vec}_A(\operatorname{Im}(A_2)) \end{bmatrix},$$

$$K_{jA} = \begin{bmatrix} K_A & 0 & 0 & 0 \\ 0 & K_A & 0 & 0 \\ 0 & 0 & K_S & 0 \\ 0 & 0 & 0 & K_A \end{bmatrix}, \quad \operatorname{vec}_{jA}(\overrightarrow{A}) = \begin{bmatrix} \operatorname{vec}_A(\operatorname{Re}(A_1)) \\ \operatorname{vec}_A(\operatorname{Im}(A_1)) \\ \operatorname{vec}_S(\operatorname{Re}(A_2)) \\ \operatorname{vec}_A(\operatorname{Im}(A_2)) \end{bmatrix},$$

$$K_{kA} = \begin{bmatrix} K_A & 0 & 0 & 0 \\ 0 & K_A & 0 & 0 \\ 0 & 0 & K_A & 0 \\ 0 & 0 & 0 & K_S \end{bmatrix}, \quad \operatorname{vec}_{kA}(\overrightarrow{A}) = \begin{bmatrix} \operatorname{vec}_A(\operatorname{Re}(A_1)) \\ \operatorname{vec}_A(\operatorname{Im}(A_1)) \\ \operatorname{vec}_A(\operatorname{Re}(A_2)) \\ \operatorname{vec}_S(\operatorname{Im}(A_2)) \end{bmatrix}.$$

3. The solution of Problem I.

It is well known that for $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times s}$, and $C \in \mathbb{C}^{s\times t}$,

$$\operatorname{vec}(ABC) = (C^T \otimes A)\operatorname{vec}(B). \tag{3.1}$$

However, in the setting of quaternion matrices, if $A \in \mathbb{Q}^{m\times n}$, $B \in \mathbb{Q}^{n\times s}$, and $C \in \mathbb{Q}^{s\times t}$, (3.1) does not hold because $\mathbb{Q}$ is not commutative under multiplication.

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Nevertheless, by applying the method in [38] to study $\mathrm{vec}(ABC)$, we can turn Problem I for quaternion matrix equation (1.1) into the least squares problem with the least norm for a system of real equations, and solve Problem I.

LEMMA 3.1. [38] *Let $A = A_1 + A_2 j \in \mathbb{Q}^{m\times n}$, $B = B_1 + B_2 j \in \mathbb{Q}^{n\times s}$, and $C = C_1 + C_2 j \in \mathbb{Q}^{s\times t}$ be given. Then*

$$\mathrm{vec}(\Phi_{ABC}) = (f(C)^T \otimes A_1, f(Cj)^H \otimes A_2) \begin{bmatrix} \mathrm{vec}(\Phi_B) \\ \mathrm{vec}(-\Phi_{jBj}) \end{bmatrix}.$$

LEMMA 3.2. *For $X = X_1 + X_2 j \in \eta\mathbb{H}\mathbb{Q}^{n\times n}$, let*

$$W = \begin{bmatrix} I_{n^2} & iI_{n^2} & 0 & 0 \\ 0 & 0 & I_{n^2} & iI_{n^2} \\ I_{n^2} & -iI_{n^2} & 0 & 0 \\ 0 & 0 & I_{n^2} & -iI_{n^2} \end{bmatrix}.$$

Then

$$\begin{bmatrix} \mathrm{vec}(\Phi_X) \\ \mathrm{vec}(-\Phi_{jXj}) \end{bmatrix} = WK_{\eta H}\mathrm{vec}_{\eta H}(\overrightarrow{X}).$$

Proof. For $X = X_1 + X_2 j \in \eta HQ^{n\times n}$, by Theorem 2.4, we have

$$\begin{aligned} \begin{bmatrix} \mathrm{vec}(\Phi_X) \\ \mathrm{vec}(-\Phi_{jXj}) \end{bmatrix} &= \begin{bmatrix} \mathrm{vec}(X_1) \\ \mathrm{vec}(X_2) \\ \mathrm{vec}(\overline{X_1}) \\ \mathrm{vec}(\overline{X_2}) \end{bmatrix} \\ &= \begin{bmatrix} I_{n^2} & iI_{n^2} & 0 & 0 \\ 0 & 0 & I_{n^2} & iI_{n^2} \\ I_{n^2} & -iI_{n^2} & 0 & 0 \\ 0 & 0 & I_{n^2} & -iI_{n^2} \end{bmatrix} \begin{bmatrix} \mathrm{vec}(\mathrm{Re}(A_1)) \\ \mathrm{vec}(\mathrm{Im}(A_1)) \\ \mathrm{vec}(\mathrm{Re}(A_2)) \\ \mathrm{vec}(\mathrm{Im}(A_2)) \end{bmatrix} \\ &= WK_{\eta H}\mathrm{vec}_{\eta H}(\overrightarrow{X}). \quad \square \end{aligned}$$

By Theorem 2.4, and Lemmas 3.1 and 3.2, we have

LEMMA 3.3. *If $A = A_1 + A_2 j \in Q^{m\times n}$, $X = X_1 + X_2 j \in \eta HQ^{n\times n}$, and $B = B_1 + B_2 j \in Q^{n\times s}$, then*

$$\mathrm{vec}(\Phi_{AXB}) = (f(B)^T \otimes A_1, f(Bj)^H \otimes A_2)WK_{\eta H}\mathrm{vec}_{\eta H}(\overrightarrow{X}).$$

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LEMMA 3.4. [1] *The matrix equation $Ax = b$, with $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^n$, has a solution $x \in \mathbb{R}^n$ if and only if*

$$AA^+b = b,$$

in this case it has the general solution

$$x = A^+b + (I - A^+A)y,$$

where $y \in \mathbb{R}^n$ is an arbitrary vector.

LEMMA 3.5. [1] *The least squares solutions of the matrix equation $Ax = b$, with $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^n$, can be represented as*

$$x = A^+b + (I - A^+A)y,$$

where $y \in \mathbb{R}^n$ is an arbitrary vector, and the least squares solution of the matrix equation $Ax = b$ with the least norm is $x = A^+b$.

Based on our earlier discussions, we now turn our attention to Problem I. The following notation is necessary for deriving the solutions of Problem I. For $A = A_1 + A_2j \in \mathrm{Q}^{m\times n}$, $B \in \mathrm{Q}^{n\times s}$, $C = C_1 + C_2j \in \mathrm{Q}^{m\times n}$, $D \in \mathrm{Q}^{n\times s}$, $E \in \mathrm{Q}^{m\times s}$, set

$$P = [(f(B)^T \otimes A_1, f(Bj)^H \otimes A_2) + (f(D)^T \otimes C_1, f(Dj)^H \otimes C_2)]WK_{\eta H},$$

$$P_1 = \mathrm{Re}(P), \quad P_2 = \mathrm{Im}(P), \quad e = \begin{bmatrix} \mathrm{vec}(\mathrm{Re}(\Phi_E)) \\ \mathrm{vec}(\mathrm{Im}(\Phi_E)) \end{bmatrix}, \tag{3.2}$$

and

$$\begin{aligned}
R &= (I - P_1^+P_1)P_2^T,\\
Z &= (I + (I - R^+R)P_2P_1^+P_1^{+T}P_2^T(I - R^+R))^{-1},\\
H &= R^+ + (I - R^+R)ZP_2P_1^+P_1^{+T}(I - P_2^TR^+),\\
S_{11} &= I - P_1P_1^+ + P_1^{+T}P_2^TZ(I - R^+R)P_2P_1^+,\\
S_{12} &= -P_1^{+T}P_2^T(I - R^+R)Z,\\
S_{22} &= (I - R^+R)Z.
\end{aligned}$$

¿From the results in [12], we have

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix}^+ = (P_1^+ - H^TP_2P_1^+, H^T), \quad \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}^+ \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = P_1^+P_1 + RR^+,$$

$$I - \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}^+ \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix}.$$

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THEOREM 3.6. *Let $A \in \mathbb{Q}^{m\times n}$, $B \in \mathbb{Q}^{k\times s}$, $C \in \mathbb{Q}^{m\times n}$, $D \in \mathbb{Q}^{k\times s}$, and $E \in \mathbb{Q}^{m\times s}$. Let P_1, P_2, and e be as in* (3.2). *Then*

$$H_L = \left\{ X \,\middle|\, \mathrm{vec}(\overrightarrow{X}) = K_{\eta H}[(P_1^+ - H^T P_2 P_1^+, H^T)e + (I - P_1^+ P_1 - RR^+)y] \right\}, \tag{3.3}$$

where $y \in \mathbb{R}^{2n^2+n}$ is an arbitrary vector.

Proof. By Lemma 3.3,

$$\begin{aligned}
\|AXB + CXD - E\|^2 &= \|\Phi_{AXB} + \Phi_{CXD} - \Phi_E\|^2 \\
&= \|\mathrm{vec}(\Phi_{AXB}) + \mathrm{vec}(\Phi_{CXD}) - \mathrm{vec}(\Phi_E)\|^2 \\
&= \left\| Q\mathrm{vec}_{\eta H}(\overrightarrow{X}) - \begin{pmatrix} \mathrm{vec}(\mathrm{Re}(\Phi_E)) \\ \mathrm{vec}(\mathrm{Im}(\Phi_E)) \end{pmatrix} \right\|^2 \\
&= \left\| \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} \mathrm{vec}_{\eta H}(\overrightarrow{X}) - e \right\|^2.
\end{aligned}$$

By Lemma 3.5, it follows that

$$\mathrm{vec}_{\eta H}(\overrightarrow{X}) = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}^+ e + \left(I - \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}^+ (P_1, P_2) \right) y,$$

and thus,

$$\mathrm{vec}(\overrightarrow{X}) = K_{\eta H}(P_1^+ - H^T P_2 P_1^+, H^T)e + K_{\eta H}(I - P_1^+ P_1 - RR^+)y. \qquad \square$$

By Lemma 3.4 and Theorem 3.6, we get the following conclusion.

COROLLARY 3.7. *The quaternion matrix equation* (1.1) *has a solution $X \in \eta\mathbb{H}\mathbb{Q}^{m\times n}$ if and only if*

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix} e = 0. \tag{3.4}$$

In this case, denote by H_E the solution set of (1.1). *Then*

$$H_E = \left\{ X \,\middle|\, \mathrm{vec}(\overrightarrow{X}) = K_{\eta H}[(P_1^+ - H^T P_2 P_1^+, H^T)e + (I - P_1^+ P_1 - RR^+)y] \right\},$$

where $y \in \mathbb{R}^{2n^2+n}$ is an arbitrary vector.

Furthermore, if (3.4) *holds, then the quaternion matrix equation* (1.1) *has a unique solution $X \in H_E$ if and only if*

$$\mathrm{rank} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = 2n^2 + n. \tag{3.5}$$

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In this case,

$$H_E = \left\{ X \,\middle|\, \mathrm{vec}(\overrightarrow{X}) = K_{\eta H}(P_1^+ - H^T P_2 P_1^+, H^T)e \right\}. \tag{3.6}$$

THEOREM 3.8. *Problem I has a unique solution $X_H \in H_L$. This solution satisfies*

$$\mathrm{vec}(\overrightarrow{X_H}) = K_{\eta H}(P_1^+ - H^T P_2 P_1^+, H^T)e. \tag{3.7}$$

Proof. From (3.3), it is easy to verify that the solution set H_L is nonempty and is a closed convex set. Hence, Problem I has a unique solution $X_H \in H_L$.

We now prove that the solution X_H can be expressed as (3.7).

From (3.3), we have

$$\min_{X \in H_L} ||X|| = \min_{X \in H_L} \|\mathrm{vec}(\overrightarrow{X})\|,$$

by Lemma 3.5 and (3.3),

$$\mathrm{vec}(\overrightarrow{X_H}) = K_{\eta H} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}^+ e.$$

Thus,

$$\mathrm{vec}(\overrightarrow{X_H}) = K_{\eta H}(P_1^+ - H^T P_2 P_1^+, H^T)e. \qquad \square$$

COROLLARY 3.9. *The least norm problem*

$$||X_H|| = \min_{X \in H_E} ||X||$$

has a unique solution $X_H \in H_E$ and X_H can be expressed as (3.7).

4. The solution of Problem II. We now discuss the solution of Problem II. Similar to Lemmas 3.2 and 3.3, we have the following conclusions.

LEMMA 4.1. *For $X = X_1 + X_2 j \in \eta\mathbb{A}\mathbb{Q}^{n\times n}$, one gets*

$$\begin{bmatrix} \mathrm{vec}(\Phi_X) \\ \mathrm{vec}(-\Phi_{jXj}) \end{bmatrix} = W K_{\eta A} \mathrm{vec}_{\eta A}(\overrightarrow{X}).$$

LEMMA 4.2. *For $A = A_1 + A_2 j \in \mathbb{Q}^{m\times n}$, $X = X_1 + X_2 j \in \eta\mathbb{A}\mathbb{Q}^{n\times n}$, and $B = B_1 + B_2 j \in \mathbb{Q}^{n\times s}$, one gets*

$$\mathrm{vec}(\Phi_{AXB}) = (f(B)^T \otimes A_1, f(Bj)^H \otimes A_2) W K_{\eta A} \mathrm{vec}_{\eta A}(\overrightarrow{X}).$$

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For $A = A_1 + A_2 j \in \mathbb{Q}^{m\times n}$, $B \in \mathbb{Q}^{n\times s}$, $C = C_1 + C_2 j \in \mathbb{Q}^{m\times n}$, $D \in \mathbb{Q}^{n\times s}$, and $E \in \mathbb{Q}^{m\times s}$, set

$$Q = [(f(B)^T \otimes A_1, f(Bj)^H \otimes A_2) + (f(D)^T \otimes C_1, f(Dj)^H \otimes C_2)]WK_{\eta A},$$

$$Q_1 = \mathrm{Re}(Q), \quad Q_2 = \mathrm{Im}(Q), \quad e = \begin{bmatrix} \mathrm{vec}(\mathrm{Re}(\Phi_E)) \\ \mathrm{vec}(\mathrm{Im}(\Phi_E)) \end{bmatrix}, \tag{4.1}$$

and

$$\begin{aligned}
R_1 &= (I - Q_1^+ Q_1)Q_2^T, \\
Z_1 &= (I + (I - R_1^+ R_1)Q_2 Q_1^+ Q_1^{+T} Q_2^T (I - R_1^+ R_1))^{-1}, \\
H_1 &= R_1^+ + (I - R_1^+ R_1)Z_1 Q_2 Q_1^+ Q_1^{+T}(I - Q_2^T R_1^+), \\
\Delta_{11} &= I - Q_1 Q_1^+ + Q_1^{+T} Q_2^T Z_1 (I - R_1^+ R_1) Q_2 Q_1^+, \\
\Delta_{12} &= -Q_1^{+T} Q_2^T (I - R_1^+ R_1) Z_1, \\
\Delta_{22} &= (I - R_1^+ R_1) Z_1.
\end{aligned}$$

¿From the results in [12], we have

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}^+ = (Q_1^+ - H_1^T Q_2 Q_1^+, H_1^T), \quad \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}^+ \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = Q_1^+ Q_1 + R_1 R_1^+,$$

$$I - \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}^+ \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{12}^T & \Delta_{22} \end{bmatrix}.$$

THEOREM 4.3. *Let $A \in \mathbb{Q}^{m\times n}$, $B \in \mathbb{Q}^{k\times s}$, $C \in \mathbb{Q}^{m\times n}$, $D \in \mathbb{Q}^{k\times s}$, and $E \in \mathbb{Q}^{m\times s}$. Let Q_1, Q_2, and e be as in* (4.1). *Then*

$$A_L = \left\{ X \,\middle|\, \mathrm{vec}(\overrightarrow{X}) = K_{\eta A}[(Q_1^+ - H_1^T Q_2 Q_1^+, H_1^T)e + (I - Q_1^+ Q_1 - R_1 R_1^+)y] \right\},$$

where $y \in \mathbb{R}^{2n^2-n}$ is an arbitrary vector.

By Lemma 3.4 and Theorem 4.3, we get the following conclusion.

COROLLARY 4.4. *The quaternion matrix equation* (1.1) *has a solution $X \in \eta\mathbb{H}\mathbb{Q}^{m\times n}$ if and only if*

$$\begin{bmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{12}^T & \Delta_{22} \end{bmatrix} e = 0. \tag{4.2}$$

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In this case, denote by A_E *the solution set of* (1.1). *Then*

$$A_E = \left\{ X \,\middle|\, \mathrm{vec}(\overrightarrow{X}) = K_{\eta A}[(Q_1^+ - H_1^T Q_2 Q_1^+, H_1^T)e + (I - Q_1^+ Q_1 - R_1 R_1^+)y] \right\},$$

where $y \in \mathbb{R}^{2n^2-n}$ *is an arbitrary vector.*

Furthermore, if (4.2) *holds, then the quaternion matrix equation* (1.1) *has a unique solution* $X \in A_E$ *if and only if*

$$\mathrm{rank} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = 2n^2 - n. \tag{4.3}$$

In this case,

$$A_E = \left\{ X \,\middle|\, \mathrm{vec}(\overrightarrow{X}) = K_{\eta A}(Q_1^+ - H_1^T Q_2 Q_1^+, H_1^T)e \right\}. \tag{4.4}$$

Theorem 4.5. *Problem II has a unique solution* $X_A \in A_L$. *This solution satisfies*

$$\mathrm{vec}(\overrightarrow{X_A}) = K_{\eta A}(Q_1^+ - H_1^T Q_2 Q_1^+, H_1^T)e. \tag{4.5}$$

Corollary 4.6. *The least norm problem*

$$||X_A|| = \min_{X \in A_E} ||X||$$

has a unique solution $X_A \in A_E$ *and* X_A *can be expressed as* (4.5).

5. Numerical verification. Based on the discussions in Sections 2, 3 and 4, we report two numerical algorithms and three numerical examples to find the solutions of Problems I, II in this section.

Algorithms 5.1 and 5.2 provide the methods to find the solutions of Problems I and II. When the consistent conditions for matrix equation (1.1) hold, Examples 5.3 and 5.4 consider the numerical solutions of Problems I and II for $X \in j\mathbb{HQ}^{n\times n}$ and $X \in k\mathbb{AQ}^{n\times n}$, respectively. In Example 5.5, we use a matrix to perturb the matrix E of Example 5.4, obtain the inconsistent matrix equation (1.1), thus we can analyze the least squares solution with the least norm for matrix equation (1.1) in Problem II. For demonstration purpose and avoiding the matrices with large norm to interrupt the solutions of Problems I and II, we only consider the cases of small $n = 5$ and take the coefficient matrices in Examples 5.3 and 5.4.

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ALGORITHM 5.1. (for Problem I)

(1) Input A, B, C, D, and E ($A \in \mathbb{Q}^{m\times n}$, $B \in \mathbb{Q}^{n\times s}$, $C \in \mathbb{Q}^{m\times n}$, $D \in \mathbb{Q}^{n\times s}$, and $E \in \mathbb{Q}^{m\times s}$).
(2) Compute P_1, P_2, R, H, S_{11}, S_{12}, S_{22}, and e.
(3) If (3.4) and (3.5) hold, then calculate $X_H(X_H \in H_E)$ according to (3.6).
(4) If (3.4) holds, then calculate $X_H(X_H \in H_E)$ according to (3.7), otherwise go to next step.
(5) Calculate $X_H(X_H \in H_L)$ according to (3.7).

ALGORITHM 5.2. (for Problem II)

(1) Input A, B, C, D, and E ($A \in \mathbb{Q}^{m\times n}$, $B \in \mathbb{Q}^{n\times s}$, $C \in \mathbb{Q}^{m\times n}$, $D \in \mathbb{Q}^{n\times s}$, and $E \in \mathbb{Q}^{m\times s}$).
(2) Compute Q_1, Q_2, R_1, H_1, Δ_{11}, Δ_{12}, Δ_{22}, and e.
(3) If (4.2) and (4.3) hold, then calculate $X_A(X_A \in A_E)$ according to (4.4).
(4) If (4.2) holds, then calculate $X_A(X_A \in A_E)$ according to (4.4), otherwise go to next step.
(5) Calculate $X_A(X_A \in A_L)$ according to (4.5).

EXAMPLE 5.3. Taking

$$A = A_1 + A_2 j, \quad B = B_1 + B_2 j, \quad C = C_1 + C_2 j, \quad D = D_1 + D_2 j, \quad X = X_1 + X_2 j,$$

$E = AXB + CXD$, where

$$A_1 = \begin{bmatrix} I_5 \\ \text{ones}(3,5) \end{bmatrix} i, \; A_2 = \begin{bmatrix} -I_5 \\ 0_{3\times 5} \end{bmatrix}, \; C_1 = \begin{bmatrix} I_5 \\ 0_{3\times 5} \end{bmatrix}, \; C_2 = \begin{bmatrix} I_5 \\ 0_{3\times 5} \end{bmatrix},$$

$$B_1 = (I_5, 0_{5\times 1}), \quad B_2 = (-I_5, 0_{5\times 1})i, \quad D_1 = \text{ones}(5,6)i, \quad D_2 = \text{ones}(5,6),$$

$X_1 =$

$$\begin{bmatrix} 1.0000 + 0.4000i & 0.5000 + 1.0000i & -2.0000 + 0.2500i & -1.0000 - 1.0000i & 0.2500 + 0.5000i \\ 0.5000 + 1.0000i & 2.0000 + 2.0000i & 1.0000 + 1.0000i & 2.0000 - 0.5000i & -0.5000 + 1.0000i \\ -2.0000 + 0.2500i & 1.0000 + 1.0000i & -1.0000 + 4.0000i & 0.5000 - 2.0000i & 1.0000 + 0.2500i \\ -1.0000 - 1.0000i & 2.0000 - 0.5000i & 0.5000 - 2.0000i & 1.0000 + 3.0000i & 2.0000 - 1.0000i \\ 0.2500 + 0.5000i & -0.5000 + 1.0000i & 1.0000 + 0.2500i & 2.0000 - 1.0000i & -2.0000 + 2.0000i \end{bmatrix},$$

$X_2 =$

$$\begin{bmatrix} 0 + 2.0000i & 0.5000 + 0.5000i & -1.0000 + 1.0000i & 0.2500 - 0.2000i & 2.0000 + 1.0000i \\ -0.5000 + 0.5000i & 0 + 0.5000i & 2.0000 + 2.0000i & -1.0000 - 0.5000i & 0.2500 + 1.0000i \\ 1.0000 + 1.0000i & -2.0000 + 2.0000i & 0 + 1.0000i & 1.0000 + 1.0000i & -2.0000 + 0.5000i \\ -0.2500 - 0.2000i & 1.0000 - 0.5000i & -1.0000 + 1.0000i & 0 - 1.0000i & 1.0000 - 1.0000i \\ -2.0000 + 1.0000i & -0.2500 + 1.0000i & 2.0000 + 0.5000i & -1.0000 - 1.0000i & 0 - 4.0000i \end{bmatrix},$$

Obviously, $X \in j\mathbb{HQ}^{5\times 5}$. Let

$$\Phi_A = (A_1, A_2), \quad \Phi_B = (B_1, B_2), \quad \Phi_C = (C_1, C_2), \quad \Phi_D = (D_1, D_2),$$

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$$\Phi_X = (X_1, X_2), \quad \Phi_E = \Phi_A f(X) f(B) + \Phi_C f(X) f(D).$$

By using Matlab 7.7 and Algorithm 5.1, we obtain

$$\operatorname{rank} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = 55, \quad \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix} e = 1.1249 \times 10^{-13}.$$

According to Algorithm 5.1 (3), we can see the matrix equation $AXB + CXD = E$ has a unique j-Hermitian solution and a unique Hermitian solution with the least norm $X_H \in H_E$, and we can get $\|X_H - X\| = 1.5131 \times 10^{-14}$.

EXAMPLE 5.4. Suppose A, B, C, D are the same as in Example 5.3, $X = X_1 + X_2 j \in k\mathbb{AQ}^{n\times n}$ and $E = AXB + CXD$, where

$X_1 =$

$$\begin{bmatrix} 0 & 0.5189 + 0.5000i & -2.0000 + 1.0000i & -1.0000 - 0.2564i & 0.2500 + 1.0000i \\ -0.5189 - 0.5000i & 0 & 1.0000 + 2.0000i & 2.0000 - 0.5000i & -0.5000 + 1.0000i \\ 2.0000 - 1.0000i & -1.0000 - 2.0000i & 0 & 0.5000 + 1.0000i & 1.0000 + 0.5000i \\ 1.0000 + 0.2564i & -2.0000 + 0.5000i & -0.5000 - 1.0000i & 0 & 2.0000 - 1.0000i \\ -0.2500 - 1.0000i & 0.5000 - 1.0000i & -1.0000 - 0.5000i & -2.0000 + 1.0000i & 0 \end{bmatrix},$$

$X_2 =$

$$\begin{bmatrix} 0 + 0.4000i & 0.5000 + 1.0000i & -1.0000 + 0.2500i & 0.2500 - 1.0000i & 2.0000 + 0.5000i \\ -0.5000 + 1.0000i & 0 + 2.0000i & 2.0000 + 1.0000i & -1.0000 - 0.5000i & 0.2500 + 1.0000i \\ 1.0000 + 0.2500i & -2.0000 + 1.0000i & 0 + 4.0000i & 1.0000 - 2.0000i & -2.0000 + 0.2500i \\ -0.2500 - 1.0000i & 1.0000 - 0.5000i & -1.0000 - 2.0000i & 0 + 3.0000i & 1.0000 - 1.0000i \\ -2.0000 + 0.5000i & -0.2500 + 1.0000i & 2.0000 + 0.2500i & -1.0000 - 1.0000i & 0 + 2.0000i \end{bmatrix}.$$

Let

$$\Phi_A = (A_1, A_2), \quad \Phi_B = (B_1, B_2), \quad \Phi_C = (C_1, C_2), \quad \Phi_D = (D_1, D_2),$$

$$\Phi_X = (X_1, X_2), \quad \Phi_E = \Phi_A f(X) f(B) + \Phi_C f(X) f(D).$$

By using Matlab 7.7 and Algorithm 5.2, we obtain

$$\operatorname{rank} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = 45, \quad \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix} e = 8.2499 \times 10^{-15}.$$

According to Algorithm 5.2 (4), we can see the matrix equation $AXB + CXD = E$ has infinite k-anti-Hermitian solutions and a unique k-anti-Hermitian solution with the least norm $X_A \in A_E$ for Problem II and we can get $\|X_A - X\| = 1.4058 \times 10^{-14}$.

EXAMPLE 5.5. Suppose

$$A, \quad B, \quad C, \quad D, \quad X, \quad \Phi_A, \quad \Phi_B, \quad \Phi_C, \quad \Phi_D, \quad \Phi_X$$

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are the same as in Example 5.4, $\Phi_G = \text{ones}(8,12)$ and let $\Phi_E = \Phi_A f(X) f(B) + \Phi_C f(X) f(D) + \Phi_G$. By using Matlab 7.7 and Algorithm 5.2, we obtain

$$\text{rank}\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = 45, \quad \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix} e = 9.5570.$$

According to Algorithm 5.2 (5), we can see the matrix equation $AXB + CXD = E$ has infinite least squares k-anti-Hermitian solutions and a unique least squares k-anti-Hermitian solution with the least norm $X_A \in A_L$ for Problem II and we can get $\|X_A - X\| = 0.1866$. and $X_A = X_{A1} + X_{A2}j$, where

$X_{A1} =$

$$\begin{bmatrix} 0 & 0.5189+0.5000i & -2.0000+1.0000i & -1.0000-0.2564i & 0.2500+1.0000i \\ -0.5189-0.5000i & 0 & 1.0000+2.0000i & 2.0000-0.5000i & -0.5000+1.0000i \\ 2.0000-1.0000i & -1.0000-2.0000i & 0 & 0.5000+1.0000i & 1.0000+0.5000i \\ 1.0000+0.2564i & -2.0000+0.5000i & -0.5000-1.0000i & 0 & 2.0000-1.0000i \\ -0.2500-1.0000i & 0.5000-1.0000i & -1.0000-0.5000i & -2.0000+1.0000i & 0 \end{bmatrix},$$

$X_{A2} =$

$$\begin{bmatrix} 0+0.3627i & 0.5000+0.9627i & -1.0000+0.2127i & 0.2500-1.0373i & 2.0000+0.4627i \\ -0.5000+0.9627i & 0+1.9627i & 2.0000+0.9627i & -1.0000-0.5373i & 0.2500+0.9627i \\ 1.0000+0.2127i & -2.0000+0.9627i & 0+3.9627i & 1.0000-2.0373i & -2.0000+0.2127i \\ -0.2500-1.0373i & 1.0000-0.5373i & -1.0000-2.0373i & 0+2.9627i & 1.0000-1.0373i \\ -2.0000+0.4627i & -0.2500+0.9627i & 2.0000+0.2127i & -1.0000-1.0373i & 0+1.9627i \end{bmatrix}.$$

In addition, the related numerical results are also verified and listed in Table 1 and Table 2, where

$$N(P_1,P_2) = \left\| I - \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}^+ - \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix} \right\| \quad \text{and}$$

$$N(Q_1,Q_2) = \left\| I - \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}^+ - \begin{bmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{12}^T & \Delta_{22} \end{bmatrix} \right\|.$$

Table 1. Numerical results for Example 5.3.

	$\|\|S_{11}\|\|$	$\|\|S_{12}\|\|$	$\|\|S_{22}\|\|$	$\|H\|$	$\|Z\|$	$N(P_1,P_2)$
$E5.3$	8.2938	1.4272	174.4133	2.2601	9.1727	$3.2778e-014$

Table 2. Numerical results for Example 5.4 and Example 5.5.

	$\|\|\Delta_{11}\|\|$	$\|\|\Delta_{12}\|\|$	$\|\|\Delta_{22}\|\|$	$\|H_1\|$	$\|Z_1\|$	$N(Q_1,Q_2)$
$E5.4$, $E5.5$	8.3695	1.3086	8.5747	1.7614	9.2480	$3.6131e-014$

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Examples 5.3, 5.4, and 5.5 are used to show the feasibility of Algorithms 5.1 and 5.2.

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