Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 874-895, November 2026.

COMPUTATION OF A GCRD OF SEVERAL POLYNOMIAL MATRICES USING TRIMMED SYLVESTER MATRICES*

ANJALI BENIWAL† AND SWANAND R. KHARE†

Abstract. This paper proposes a method for computing an exact Greatest Common Right Divisor (GCRD) of several univariate polynomial matrices using Trimmed Sylvester matrices. It establishes a novel relationship: for a full rank polynomial matrix with specified column degrees, constructed by stacking the given set of polynomial matrices one below the other, the degree of the determinant of any GCRD is related to the rank deficiency of a particular Trimmed Sylvester matrix. It characterizes the nullity of Trimmed Sylvester matrices contrasting it with the nullity of the generalized Sylvester matrices, proving that both of them are governed by different forms of the Index Sum Theorem. The coefficients of a GCRD are extracted using an Effectively Eliminating QR ($EEQR$) decomposition of the Trimmed Sylvester matrix. The effectiveness of this approach is demonstrated through numerical examples, which reveal that the Trimmed Sylvester matrix approach offers a computationally advantageous alternative to the standard generalized Sylvester matrices, particularly when column degrees vary a lot.

Key words. Polynomial matrix, Index Sum Theorem, Trimmed Sylvester matrix, Greatest Common Right Divisor, QR.

AMS subject classifications. 15A54, 15B05, 65F25, 65F50.

1. Introduction.

The computation of a Greatest Common Right Divisor (GCRD) is a cornerstone problem in control theory, with applications including the identification of controllable and uncontrollable subsystems, the construction of minimal kernel representations, the computation of the intersection of several behaviors, and the determination of the radius of uncontrollability [4, 12]. Given the central role of the GCRD in these applications, we need efficient and computationally cost-effective methods for its computation.

Several algorithms exploit matrix operations on coefficient matrices [1, 11, 17, 18, 23]. Among the most common methods are state-space realizations [14, 19] and Sylvester resultant matrices [6, 15, 16, 22]. Recently, the authors in [13] proposed an extension of the polynomial GCD algorithm based on Sylvester matrices to generalized Sylvester matrices for computation of a GCRD of two polynomial matrices. This idea was later extended to generalized Sylvester matrices for the case of several polynomial matrices in [5]. These methods address the computation of a square GCRD for a given set of polynomial matrices. The authors in [19] study the case of nonsquare GCRDs and their computation via state-space techniques and the staircase algorithm. However, when the column degrees vary a lot, these methods become computationally inefficient because the associated matrices become excessively large and sparse. Such matrices are dominated by zero blocks that do not contribute, thereby inflating the computational cost.

In this work, we address this limitation for the case of square GCRDs by utilizing the **Trimmed Sylvester matrix** (see [10]), a structure that systematically removes redundant zero blocks found in the generalized Sylvester matrix. This approach allows for the computation of a GCRD at a much earlier stage than the generalized Sylvester matrix approach. Particularly when column degrees vary a lot, constructing

*Received by the editors on February 27, 2026. Accepted for publication on September 21, 2026. Handling Editor: Vanni Noferini. Corresponding Author: Swanand Khare.

†Department of Mathematics, Indian Institute of Technology Kharagpur (anjalibeniwal1811@gmail.com, srkhare@maths.iitkgp.ac.in).

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 874-895, November 2026.

a matrix of substantially smaller dimensions and removing the redundant zero blocks results in a reduced computational complexity compared to the algorithms described in [5, 19] for computation of square GCRDs. Finally, the coefficients of the GCRD are extracted from a unique QR decomposition of a particular Trimmed Sylvester matrix.

The paper is organized as follows: In Section 2, we fix notation and recall the basics of polynomial matrices and the Index Sum Theorem, formulate the problem, and introduce the generalized Sylvester matrices and the Trimmed Sylvester matrices. Section 3 contains the core results: we relate the left nullities of the sequence of Trimmed Sylvester matrices to the left minimal indices and show that the rank deficiency (equivalently the dimension of the right kernel) of a particular Trimmed Sylvester matrix is related to the degree of determinant of a GCRD of the given set of polynomial matrices. We further discuss some special cases of this theorem. In Section 4, we extract the coefficients of a GCRD via an Effectively Eliminating QR ($EEQR$) decomposition of a particular Trimmed Sylvester matrix and then analyze the computational complexity of this approach, comparing it with both the generalized Sylvester approach and the staircase algorithm on a state-space realization for the case of square GCRDs. Section 5 provides numerical examples.

2. Preliminaries. In this section, we establish the necessary framework of polynomial matrices and formally state the problem.

Let $\mathbb{R}^{g\times w}[s]$ denote the set of polynomial matrices and $\mathbb{R}^{g\times w}(s)$ the set of rational matrices with real coefficients of size $g \times w$. Consider a given set of polynomial matrices $B_i(s) \in \mathbb{R}^{g_i\times w}[s]$, for $i = 1, 2, \ldots, t$. Define $P(s)$ by stacking these matrices one below the other. We assume that $P(s)$ satisfies the following conditions:

$$\begin{aligned} P(s) &= \begin{bmatrix} B_1(s) \\ \vdots \\ B_t(s) \end{bmatrix} \in \mathbb{R}^{g\times w}[s], \quad g = \sum_{i=1}^{t} g_i,\ g > w,\ \operatorname{rank} P(s) = w, \\ &= P_d s^d + \cdots + P_1 s + P_0, \quad P_d \neq 0, \\ &= \begin{bmatrix} P_1(s) & \cdots & P_w(s) \end{bmatrix}, \quad P_j(s) = \sum_{\ell=0}^{c_j} P_{\ell,j} s^\ell, \quad P_{hc} = \begin{bmatrix} P_{c_1,1} & \cdots & P_{c_w,w} \end{bmatrix}. \end{aligned} \tag{2.1}$$

Here, $P_{\ell,j} \in \mathbb{R}^{g\times 1}$ is the coefficient of s^ℓ in the jth column $P_j(s) \in \mathbb{R}^{g\times 1}[s]$, c_j is the degree of that column, and $P_{hc} \in \mathbb{R}^{g\times w}$ is the highest column-degree coefficient matrix. Define $d = \max_{1\le j\le w} c_j$, the degree of $P(s)$, and $\mathbf{c} = (c_1, \ldots, c_w)$ is the column-degree vector, arranged in nondecreasing order. We say $P(s)$ has uniform degree if all columns have the same degree, i.e., $c_1 = c_2 = \cdots = c_w = d$. Throughout the paper, $P(s)$ is assumed to be tall ($g > w$) and full column rank ($\operatorname{rank} P(s) = w$) with given column-degree vector $\mathbf{c}$ unless stated otherwise.

2.1. Structural invariants of a polynomial matrix. In this subsection, we define the invariants of $P(s)$ defined in (2.1) that will be used in the subsequent sections.

Right primeness. The polynomial matrix $P(s) \in \mathbb{R}^{g\times w}[s]$ ($g > w$) is right prime if, as a complex matrix, it has full column rank for every $s \in \mathbb{C}$ [16].

Maximal-minor degree $\mathfrak{M}_{P(s)}$. The maximal-minor degree of $P(s)$, denoted by $\mathfrak{M}_{P(s)}$, is the maximum degree among its maximal minors. For $P(s) \in \mathbb{R}^{g\times w}[s]$ ($g > w$), the maximal minors are the minors of size $w \times w$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 874-895, November 2026.

Nullspace and minimal indices. The left nullspace of $P(s)$, defined in (2.1), is

$$\mathcal{N}^{\ell}(P) = \left\{ y(s) \in \mathbb{R}^{1\times g}[s] \ : \ y(s)P(s) \equiv 0^{\top} \right\}.$$

A basis $\{y_1(s), \ldots, y_{g-w}(s)\}$ of $\mathcal{N}^{\ell}(P)$ with degrees $\beta_1 \leq \cdots \leq \beta_{g-w}$ is a minimal basis if, for any other basis with degrees $\gamma_1 \leq \cdots \leq \gamma_{g-w}$, one has $\beta_i \leq \gamma_i$ for every $i = 1, \ldots, g-w$. The ordered integers $\beta_1 \leq \cdots \leq \beta_{g-w}$ are the left minimal indices of $P(s)$, and their sum is denoted by $\mathfrak{L}_{P(s)} = \sum_{i=1}^{g-w} \beta_i$. Because $P(s)$ defined in (2.1) is tall and has full column rank, it has no nonzero vector in the right nullspace and by extension has no right minimal indices; thus, $\mathfrak{R}_{P(s)} = 0$.

Unimodular equivalence. A square polynomial matrix $U(s) \in \mathbb{R}^{m\times m}[s]$ is *unimodular* when $\det(U(s))$ is a nonzero constant. The polynomial matrix $P(s)$ is *left unimodularly equivalent* to the polynomial matrix $T(s)$ if there exist unimodular matrix $L(s)$ such that $P(s) = L(s)T(s)$ and *unimodularly equivalent* if there exist unimodular matrices $U(s)$ and $V(s)$ such that $P(s) = U(s)T(s)V(s)$ [16].

The unimodular polynomial matrices form a transformation group under which a unique canonical form of an arbitrary polynomial or rational matrix can be obtained [2, 8]. We first use unimodular equivalence to define the Smith form of a polynomial matrix and the Smith–McMillan form of a rational matrix. We then use these forms to define the invariants of the matrix $P(s)$ in (2.1), together with a structural invariant associated with its column degrees.

We begin with the Smith form of a polynomial matrix $\mathcal{P}(s)$ of rank r.

THEOREM 2.1 (Smith Form of a Polynomial Matrix [7, 8]). *Let $\mathcal{P}(s) \in \mathbb{R}^{g\times w}[s]$ be any polynomial matrix. Then, there exists unimodular matrices $\mathcal{U}(s) \in \mathbb{R}^{g\times g}[s]$ and $\mathcal{V}(s) \in \mathbb{R}^{w\times w}[s]$ such that*

$$\mathcal{U}(s)\mathcal{P}(s)\mathcal{V}(s) = \mathcal{D}(s) = \left[\begin{array}{c|c} \mathit{diag}(p_1(s), \ldots, p_r(s)) & 0_{r\times(w-r)} \\ \hline 0_{(g-r)\times r} & 0_{(g-r)\times(w-r)} \end{array}\right], \tag{2.2}$$

where $p_1(s), \ldots, p_r(s)$ are monic polynomials and $p_j(s)$ is a divisor of $p_{j+1}(s)$, for $j = 1, \ldots, r-1$. The matrix $\mathcal{D}(s)$ is unique and called the Smith form. The integer r is the rank of $\mathcal{P}(s)$ considered as a matrix over the field $\mathbb{R}^{g\times w}(s)$, denoted by rank$(\mathcal{P}(s))$. $\mathcal{P}(s)$ is said to have full rank if $r = \min\{g, w\}$. A finite eigenvalue of $\mathcal{P}(s)$ is a number $\alpha \in \mathbb{C}$ such that $p_j(\alpha) = 0$ for some $j = 1, \ldots, r$. The partial multiplicity sequence of $\mathcal{P}(s)$ at the finite eigenvalue α is the sequence

$$0 \leq \xi_1(\alpha) \leq \xi_2(\alpha) \leq \cdots \leq \xi_r(\alpha), \tag{2.3}$$

such that $p_j(s) = (s-\alpha)^{\xi_j(\alpha)} q_j(s)$ with $q_j(\alpha) \neq 0$, for $j = 1, \ldots, r$. Let $\sigma_f(\mathcal{P})$ denote the total number of finite eigenvalues (counting multiplicity). It is given by

$$\sigma_f(\mathcal{P}) := \sum_{\alpha} \sum_{j=1}^{r} \xi_j(\alpha) = \sum_{j=1}^{r} \deg p_j(s). \tag{2.4}$$

Polynomial matrices may also have infinity as an eigenvalue. Its definition is based on the so-called reversal polynomial matrix.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 874-895, November 2026.

DEFINITION 2.2 (Reversal of a Polynomial matrix [7, 8]). *Let $\mathcal{P}(s) = \sum_{j=0}^{d} \mathcal{P}_j s^j$ be a polynomial matrix of degree d; the reversal polynomial matrix* $\operatorname{rev}\mathcal{P}(s)$ *of* $\mathcal{P}(s)$ *is* $\operatorname{rev}\mathcal{P}(s) := s^d\mathcal{P}(1/s) = \mathcal{P}_d + \mathcal{P}_{d-1}s + \cdots + \mathcal{P}_0 s^d$.

DEFINITION 2.3 (Eigenvalues at Infinity [7, 8]). *Let $\mathcal{P}(s) \in \mathbb{R}^{g\times w}[s]$ be a polynomial matrix of degree d. Then, ∞ is an eigenvalue of $\mathcal{P}(s)$ if and only if the reversal polynomial matrix* $\operatorname{rev}\mathcal{P}(s) = s^d\mathcal{P}(1/s)$ *has* 0 *as an eigenvalue. The partial multiplicities of $\mathcal{P}(s)$ at the eigenvalue ∞ are given by the partial multiplicities of* $\operatorname{rev}\mathcal{P}(s)$ *at the eigenvalue* 0. *Denote $\sigma_\infty(\mathcal{P})$ as the total number of infinite eigenvalues (counting multiplicity).*

Polynomial matrices in $\mathbb{R}^{g\times w}[s]$ may also be considered as a special case of rational matrices $\mathbb{R}^{g\times w}(s)$, for which the canonical form is given by the Smith–McMillan form. The Smith–McMillan form is a generalization of the Smith form, and it captures both finite and infinite zeros and poles of a rational matrix. The next definition describes the finite zero and finite pole structure of a general rational matrix. This provides the framework that will later be specialized to a polynomial matrix.

THEOREM 2.4 (Smith–McMillan form). *[2][16, pp. 443–450] Given any $g\times w$ rational matrix $\mathcal{R}(s)$ there exist unimodular polynomial matrices $U(s)$ and $V(s)$ such that*

$$U(s)\mathcal{R}(s)V(s) = D(s) = \operatorname{diag}\left(\frac{\theta_1(s)}{\psi_1(s)}, \ldots, \frac{\theta_r(s)}{\psi_r(s)}, 0, \ldots, 0\right),$$

where each pair $(\theta_i(s), \psi_i(s))$ is formed by coprime monic polynomials for $i = 1, \ldots, r$, and $\theta_j(s)$ divides $\theta_{j+1}(s)$ and $\psi_{j+1}(s)$ divides $\psi_j(s)$, for $j = 1, \ldots, r-1$. The matrix $D(s)$ is unique and is called the Smith–McMillan form of $\mathcal{R}(s)$. The roots of $\theta_1(s), \ldots, \theta_r(s)$ are called the finite zeros of $\mathcal{R}(s)$, while the roots of $\psi_1(s), \ldots, \psi_r(s)$ are called the finite poles of $\mathcal{R}(s)$. If $\alpha \in \mathbb{C}$ is a finite zero or a finite pole (or simultaneously both) of $\mathcal{R}(s)$, then one can write

$$\frac{\theta_j(s)}{\psi_j(s)} = (s-\alpha)^{\chi_j(\alpha)}\left(\frac{\tilde{\theta}_j(s)}{\tilde{\psi}_j(s)}\right) \quad \text{with} \quad \tilde{\theta}_j(\alpha) \neq 0,\ \tilde{\psi}_j(\alpha) \neq 0,$$

for $j = 1, \ldots, r$, and the sequence $\chi_1(\alpha) \leq \cdots \leq \chi_r(\alpha)$ is the sequence of structural indices of $\mathcal{R}(s)$ at α. If $\chi_j(\alpha) < 0$, then α is a finite pole, and if $\chi_j(\alpha) > 0$, then α is a finite zero. The total finite zero and finite pole degrees of $\mathcal{R}(s)$ are defined by

$$\begin{aligned} \delta_z^{\text{fin}}(\mathcal{R}) &:= \sum_{i=1}^{r} \deg\theta_i(s) = \sum_{\alpha} \sum_{j:\,\chi_j(\alpha)>0} \chi_j(\alpha), \\ \delta_p^{\text{fin}}(\mathcal{R}) &:= \sum_{i=1}^{r} \deg\psi_i(s) = \sum_{\alpha} \sum_{j:\,\chi_j(\alpha)<0} (-\chi_j(\alpha)). \end{aligned} \tag{2.5}$$

Rational matrices may also have poles and zeros at infinity. The following definition introduces structural indices at infinity.

DEFINITION 2.5 (Poles and Zeros at Infinity). *[2, Definition 2.8][16, p. 450] The sequence of structural indices of $\mathcal{R}(s)$ at ∞ is defined to be the sequence of structural indices of $\mathcal{R}(1/s)$ at 0. If the Smith–McMillan form of $\mathcal{R}(1/s)$ is unimodularly equivalent to*

$$\left[\begin{array}{c|c} \operatorname{diag}(s^{\chi_1(\infty)}, \ldots, s^{\chi_r(\infty)}) & 0_{r\times(w-r)} \\ \hline 0_{(g-r)\times r} & 0_{(g-r)\times(w-r)} \end{array}\right],$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 874-895, November 2026.

then every structural index $\chi_j(\infty)$ partitions as

$$\chi_j(\infty) = \begin{cases} -\chi_j(\infty), & \text{if } \chi_j(\infty) < 0 \quad (\textit{pole at infinity}) \\ \chi_j(\infty), & \text{if } \chi_j(\infty) > 0 \quad (\textit{zero at infinity}) \end{cases}$$

The total degrees of the poles and zeros at infinity are defined by

$$\delta_p^{\infty}(\mathcal{R}) := \sum_{\chi_j(\infty)<0} (-\chi_j(\infty)), \quad \delta_z^{\infty}(\mathcal{R}) := \sum_{\chi_j(\infty)>0} \chi_j(\infty). \tag{2.6}$$

Therefore, the total degree of zeros and poles of $\mathcal{R}(s)$ are defined by

$$\delta_z(\mathcal{R}) := \delta_z^{\text{fin}}(\mathcal{R}) + \delta_z^{\infty}(\mathcal{R}), \qquad \delta_p(\mathcal{R}) := \delta_p^{\text{fin}}(\mathcal{R}) + \delta_p^{\infty}(\mathcal{R}). \tag{2.7}$$

We use the above definitions to analyze the invariants of the polynomial matrix $P(s)$ defined in (2.1). We know that $P(s) \in \mathbb{R}^{g\times w}[s]$ is a tall ($g > w$) full rank ($\operatorname{rank}(P(s)) = w$) polynomial matrix. Therefore, its zeros and poles as a rational matrix can be defined using Theorem 2.4. Since the Smith–McMillan form of a polynomial matrix is its Smith form [8], $P(s)$ has no finite poles and hence $\delta_p^{\text{fin}}(P) = 0$. Its finite zeros are equal to its finite eigenvalues, and the structural indices of $P(s)$ at a finite zero α coincide with the partial multiplicities of $P(s)$ at the corresponding finite eigenvalue. Consequently, we have

$$\delta_z^{\text{fin}}(P) = \sigma_f(P). \tag{2.8}$$

As for the partial multiplicity sequence at ∞ of a polynomial matrix $P(s)$ as defined in Definition 2.3, it is different from the structural indices of $P(s)$ at ∞ as defined in Definition 2.5. Observe that the Smith form of the reversal polynomial

$$\operatorname{rev} P(s) = s^d P(1/s)$$

is just s^d times the Smith–McMillan form of the rational matrix $P(1/s)$. This implies that if $\chi_1(\infty) \le \cdots \le \chi_r(\infty)$ is the sequence of structural indices of $P(s)$ at ∞, then

$$0 \le d + \chi_1(\infty) \le \cdots \le d + \chi_r(\infty),$$

is the partial multiplicity sequence of $P(s)$ at the eigenvalue ∞. Therefore, we have

$$\sigma_\infty(P) = dw - \delta_p^{\infty}(P) + \delta_z^{\infty}(P). \tag{2.9}$$

We know that $P(s)$ has specified column degrees. We now define the reversal of a polynomial matrix $P(s)$ with respect to the given column-degree vector $\mathbf{c} = (c_1, \ldots, c_w)$.

Definition 2.6 (Column-Degree Reversal of a Polynomial Matrix). *Let $P(s) \in \mathbb{R}^{g\times w}[s]$ be a polynomial matrix with column-degree vector $\mathbf{c} = (c_1, \ldots, c_w)$, arranged in nondecreasing order. The column-degree reversal of $P(s)$ is defined as*

$$\operatorname{rev}_{\mathbf{c}} P(s) := P(1/s)\operatorname{diag}(s^{c_1}, \ldots, s^{c_w}). \tag{2.10}$$

Observe that the reversal of $P(s)$ in Definition 2.2 is a special case of the column-degree reversal when all columns have the same degree d, i.e., $\mathbf{c} = (d, \ldots, d)$. We redefine the eigenvalues of $P(s)$ at ∞ with respect to the column-degree vector.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 874-895, November 2026.

DEFINITION 2.7 (Eigenvalues at Infinity w.r.t. Column-Degree Vector $\mathbf{c}$). *Let $P(s) \in \mathbb{R}^{g\times w}[s]$ be as defined in (2.1) with column-degree vector $\mathbf{c}$. Then, ∞ is an eigenvalue of $P(s)$ w.r.t. $\mathbf{c}$ if and only if the reversal polynomial $\operatorname{rev}_{\mathbf{c}} P(s) := P(1/s)\operatorname{diag}(s^{c_1},\dots,s^{c_w})$ has 0 as an eigenvalue. The partial multiplicities of $P(s)$ at the eigenvalue ∞ are given by the partial multiplicities of $\operatorname{rev}_{\mathbf{c}} P(s)$ at the eigenvalue 0. Denote $\sigma_\infty^{(\mathbf{c})}(P)$ as the total number of infinite eigenvalues (counting multiplicity).*

If $\chi_1(\infty) \le \cdots \le \chi_r(\infty)$ is the sequence of structural indices of $P(s)$ at ∞, then

$$0 \le c_w + \chi_1(\infty) \le \cdots \le c_1 + \chi_r(\infty),$$

are the multiplicities of 0 as a root of the invariant polynomials of $\operatorname{rev}_{\mathbf{c}} P(s)$. Therefore, we have

$$\sigma_\infty^{(\mathbf{c})}(P) = \sum_{j=1}^{w} c_j - \delta_p^\infty(P) + \delta_z^\infty(P). \tag{2.11}$$

The relationships among the reversals are given by

$$\sigma_\infty(P) = \sigma_\infty^{(\mathbf{c})}(P) + \sum_{j=1}^{w}(d - c_j). \tag{2.12}$$

2.2. Index Sum Theorems. In this subsection, we state the Index Sum Theorems for rational and polynomial matrices. The Index Sum Theorem for polynomial matrices is a special case of the Index Sum Theorem for rational matrices.

THEOREM 2.8 (Rational Index Sum Theorem). *[2] Let $\mathcal{R}(s) \in \mathbb{R}^{g\times w}(s)$ be a rational matrix. Let $\delta_p(\mathcal{R})$ and $\delta_z(\mathcal{R})$ be the total number of poles and zeros as defined in (2.7), of $\mathcal{R}(s)$, and let $\tau(\mathcal{R})$ be the sum of the left and right minimal indices of $\mathcal{R}(s)$. Then,*

$$\delta_p(\mathcal{R}) = \delta_z(\mathcal{R}) + \tau(\mathcal{R}).$$

For a full-column-rank polynomial matrix, the rational index-sum theorem can be expressed using finite and infinite eigenvalues and the left minimal indices. Let $P(s) \in \mathbb{R}^{g\times w}[s]$ be as defined in (2.1). The two forms of the Index Sum Theorem defined below are based on a uniform degree d and on the column-degree vector $\mathbf{c}$, respectively.

THEOREM 2.9 (Polynomial Matrix Index Sum Theorem). *Let $P(s) \in \mathbb{R}^{g\times w}[s]$ be the polynomial matrix defined in (2.1). Let $\sigma_f(P)$ be the total number of finite eigenvalues, $\sigma_\infty(P)$ be the total number of infinite eigenvalues, and $\mathfrak{L}_{P(s)}$ be the sum of the left minimal indices. Then,*

$$dw = \sigma_f(P) + \sigma_\infty(P) + \mathfrak{L}_{P(s)}. \tag{2.13}$$

THEOREM 2.10 (Column-Degree Index Sum Theorem). *Let $P(s) \in \mathbb{R}^{g\times w}[s]$ be the polynomial matrix defined in (2.1). Let $\sigma_f(P)$ be the total number of finite eigenvalues, $\sigma_\infty^{(\mathbf{c})}(P)$ be the total number of infinite eigenvalues w.r.t. the column-degree vector $\mathbf{c}$, and $\mathfrak{L}_{P(s)}$ be the sum of the left minimal indices. Then,*

$$\sum_{j=1}^{w} c_j = \sigma_f(P) + \sigma_\infty^{(\mathbf{c})}(P) + \mathfrak{L}_{P(s)}. \tag{2.14}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 874-895, November 2026.

2.3. GCRD. In this subsection, we define the GCRD of a set of polynomial matrices and state the problem.

DEFINITION 2.11 (GCRD). *[19] Let $B_i(s) \in \mathbb{R}^{g_i \times w}[s]$, for $i = 1, \ldots, t$. A GCRD of the polynomial matrices $B_1(s), \ldots, B_t(s)$ is a polynomial matrix $G(s) \in \mathbb{R}^{w \times w}[s]$, which satisfies the following conditions:*

1. *$B_i(s) = \widehat{B}_i(s)G(s)$, for some $\widehat{B}_i(s) \in \mathbb{R}^{g_i \times w}[s]$, for $i = 1, \ldots, t$, and*
2. *If $B_i(s) = \widetilde{B}_i(s)F(s)$, for $i = 1, \ldots, t$, then $G(s) = \widehat{G}(s)F(s)$ for some $\widehat{G}(s)$, $F(s) \in \mathbb{R}^{w \times w}[s]$.*

*If a GCRD of the given polynomial matrices is unimodular, they are said to be **coprime**.*

PROBLEM 2.12. *Given a set of polynomial matrices $B_i(s) \in \mathbb{R}^{g_i \times w}[s]$, for $i = 1, \ldots, t$, and let $P(s)$ be as defined in (2.1) with given column degrees $(c_1, c_2, \ldots, c_w)$, compute a GCRD of $B_1(s), \ldots, B_t(s)$.*

In [16], the authors define a relationship between the general stacked matrix $P(s)$ and a GCRD of the given set of polynomial matrices.

LEMMA 2.13. *[16, Section 6.3-3] Let $B_i(s) \in \mathbb{R}^{g_i \times w}[s]$, for $i = 1, \ldots, t$. Let $P(s) \in \mathbb{R}^{g \times w}[s]$ be as defined in (2.1). Then, there exists a unimodular matrix $U(s) \in \mathbb{R}^{g \times g}[s]$ such that $U(s)P(s)$ has at least $g - w$ rows identically zero. That is, $U(s)P(s) = \begin{bmatrix} G(s) \\ 0 \end{bmatrix}$, then the matrix $G(s)$ is a GCRD of $B_1(s), \ldots, B_t(s)$.*

Remark 2.14. The preceding lemma shows that a GCRD is unique up to left multiplication by a unimodular matrix. In the general setting, a GCRD need not be square; nonsquare GCRDs are treated by the generalization of Definition 2.11 in [19]. In this paper, we restrict our attention to square compact[1] GCRDs in $\mathbb{R}^{w \times w}[s]$ because $P(s)$ has full column rank w.

The matrix $P(s)$ and a GCRD $G(s)$ have the same Smith form because they are related by a left unimodular transformation. In particular, their finite eigenvalues (counting multiplicity) coincide. For a square polynomial matrix $G(s)$ with full rank, $\sigma_f(G) = \deg(\det G(s))$. Therefore, we have

$$\sigma_f(P) = \sigma_f(G) = \deg(\det G(s)). \tag{2.15}$$

Next, we state the relationship between the coprimeness of the set $B_1(s), \ldots, B_t(s)$ to the right primeness of the matrix $P(s)$.

LEMMA 2.15. *[5] Let $B_i(s) \in \mathbb{R}^{g_i \times w}[s]$, for $i = 1, \ldots, t$. Let $P(s)$ be as defined in (2.1). The following are equivalent.*

1. *The polynomial matrices $B_1(s), \ldots, B_t(s)$ are coprime.*
2. *$P(s)$ is a right prime matrix.*
3. *$\mathfrak{M}_{P(s)} = \mathfrak{L}_{P(s)}$.*

Remark 2.16. In general, $\mathfrak{M}_{P(s)} \geq \mathfrak{L}_{P(s)}$. If $\mathfrak{M}_{P(s)} \neq \mathfrak{L}_{P(s)}$, then $P(s)$ has a finite eigenvalue. If the Smith form of $P(s)$ is $D(s)$, then the existence of a finite eigenvalue implies that there is a degree 1 or greater polynomial on the leading diagonal of $D(s)$. Furthermore, $D(s)$ loses its rank on that eigenvalue. The rank of $P(s)$ is equal to the rank of $D(s)$, and hence $P(s)$ also loses rank on that eigenvalue. Therefore, $P(s)$ is not right prime [19].

[1] A compact GCRD is a GCRD that has the same number of rows as the rank of the stacked matrix $P(s)$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 874-895, November 2026.

2.4. Trimmed Sylvester Matrices. In practice, we need efficient numerical methods to compute a GCRD. The key challenge is that the size of standard Sylvester matrices becomes very large when column degrees vary a lot, as they contain many zero columns that contribute no information for the computation of the GCRD. To address this, we employ Trimmed Sylvester matrices, which reduce the overall size of the matrix by removing redundant columns while preserving the structural information necessary for the computation of a GCRD at a much earlier stage than generalized Sylvester matrices.

For $P(s)$ defined in (2.1), we construct a sequence of generalized Sylvester matrices (see [21]). Let A be the block-Toeplitz matrix of coefficients:

$$A := \begin{bmatrix} P_d & P_{d-1} & \cdots & P_0 & 0 & 0 & \cdots \\ 0 & P_d & P_{d-1} & \cdots & P_0 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}. \tag{2.16}$$

We define the truncated submatrix A_i (using MATLAB notation for indexing):

$$A_i = A(1:(i+1)g,\ 1:(d+i+1)w), \quad \text{for } i = 0,\ 1,\ 2, \ldots. \tag{2.17}$$

We observe that when column degrees vary a lot, A_i contains significant sparsity. Specifically, consider the submatrix $A(P_j)$ corresponding to the coefficients of the j-th column $P_j(s)$:

$$A(P_j) = \begin{bmatrix} 0_{g\times(d-c_j)} & P_{c_j,j} & P_{c_j-1,j} & \cdots & P_{0,j} & 0 & \cdots \\ 0 & 0_{g\times 1} & P_{c_j,j} & P_{c_j-1,j} & \cdots & P_{0,j} & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \end{bmatrix}. \tag{2.18}$$

We define the truncated submatrix $A_i(P_j)$ (using MATLAB notation for indexing):

$$A_i(P_j) = A(P_j)(1:(i+1)g,\ 1:d+i+1), \quad \text{for } i = 0,\ 1,\ 2, \ldots. \tag{2.19}$$

For any i, the submatrix $A_i(P_j)$ identifies exactly $(d - c_j)$ zero columns at the beginning of the block corresponding to the j-th column. Across all w columns, this results in a total of $dw - \sum_{j=1}^{w} c_j$ zero columns in A_i.

Since these zero columns contribute no information, we eliminate them to define the corresponding truncated Trimmed Sylvester matrix:

$$T_i(P_j) = A_i(P_j)(\ :\ ,\ d - c_j + 1 : d + i + 1), \quad \text{for } i = 0,\ 1,\ 2, \ldots. \tag{2.20}$$

Definition 2.17 (Trimmed Sylvester Matrix). *[10] The $(i-1)$-th Trimmed Sylvester matrix of $P(s)$, denoted T_{i-1}, is the submatrix of A_{i-1} obtained by removing the $(d - c_j)$ zero columns at the beginning of each block $A_{i-1}(P_j)$, for $j = 1, \ldots, w$. The dimension of the resulting matrix, for $i = 1,\ 2, \ldots$, is given by*

$$T_{i-1} \in \mathbb{R}^{ig \times \left(iw + \sum_{j=1}^{w} c_j\right)}. \tag{2.21}$$

For more information about Trimmed Sylvester matrices, refer to [10].

3. Main Results. In this section, we establish the fundamental relationship between the rank properties of the Trimmed Sylvester matrix and a GCRD. To achieve this, we first relate the left nullities of

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 874-895, November 2026.

Trimmed Sylvester matrices to the left minimal indices of the stacked matrix $P(s)$ defined in (2.1) (see Theorem 3.1), then connect the rank deficiency of Trimmed Sylvester matrices to the left minimal indices (see Theorem 3.5), and finally prove that the dimension of the right kernel of an appropriately chosen Trimmed Sylvester matrix is related to the determinant degree of a GCRD (see Theorem 3.8).

We first relate the left nullities of the sequence of Trimmed Sylvester matrices to the left minimal indices of $P(s)$. An analogous result for right minimal indices can be found in [10, Theorem 4.1].

THEOREM 3.1. *Let $P(s) \in \mathbb{R}^{g\times w}[s]$ be defined in (2.1). Construct the sequence of Trimmed Sylvester matrices $\{T_i\}_{i=0,1,2,\dots}$ as in (2.21). Let $\mathcal{K}_i^\ell$ be the left kernel of T_i and $\eta_i = \dim(\mathcal{K}_i^\ell)$ be the associated left nullity. Let the sequence of first differences be defined as*

$$\eta_0^{(1)} = \eta_0; \quad \eta_i^{(1)} = \eta_i - \eta_{i-1}, \quad for \quad i = 1, 2, \dots,$$

The sequence of second differences is defined as

$$\eta_0^{(2)} = \eta_0^{(1)}; \quad \eta_i^{(2)} = \eta_i^{(1)} - \eta_{i-1}^{(1)}, \quad for \quad i = 1, 2, \dots.$$

Then, the number of polynomially independent vectors of degree i in the left minimal nullspace basis $\mathcal{N}^\ell(P)$ of $P(s)$ is $\eta_i^{(2)}$.

Remark 3.2. If $P(s) \in \mathbb{R}^{g\times w}[s]$ is of normal rank w, $\eta_{i+1} - \eta_i = g - w$ for $i \geq \nu$, where $\nu + 1$ is the largest left minimal index [10]. All terms in $\left\{\eta_i^{(2)}\right\}$ are zero after stage $\nu + 1$. Moreover, their sum is $\mathfrak{L}_{P(s)} = \sum_{i=0}^{j} i\eta_i^{(2)} = j(g-w) - \eta_{j-1}$ for all $j \geq \nu + 1$.

We now relate the left rank deficiencies of the sequence of Trimmed Sylvester matrices and left minimal indices of $P(s)$. We begin by defining left rank deficiency of any real matrix.

DEFINITION 3.3. *The left rank deficiency κ of any matrix $T \in \mathbb{R}^{m\times n}$ is defined as $\kappa = \min\{m, n\} - \operatorname{rank}(T)$.*

Remark 3.4. If η is the left nullity of T, it can be observed that if T is a tall matrix, i.e., $m \geq n$, then the left rank deficiency of T is $\kappa = \eta - (m - n)$. If T is a wide matrix, i.e., $m \leq n$, then the left rank deficiency of T is $\kappa = \eta$. Also, $\kappa = 0$ if and only if T is a full rank matrix.

In [20], the authors relate the rank deficiency of generalized Sylvester matrices to the sum of right minimal indices. We extend this result to relate the rank deficiency of Trimmed Sylvester matrices to the sum of left minimal indices $\mathfrak{L}_{P(s)}$.

THEOREM 3.5. *Let $P(s) \in \mathbb{R}^{g\times w}[s]$ be defined in (2.1) with column degrees $(c_1, \dots, c_w)$. Construct the sequence of Trimmed Sylvester matrices $\{T_i\}_{i=0,1,2,\dots}$ as in (2.21). Let $\mathcal{K}_i^\ell$ be the left kernel of T_i and $\eta_i = \dim(\mathcal{K}_i^\ell)$ be the corresponding left nullity. Let ν be the first integer such that $\eta_{\nu+1} - \eta_\nu = g - w$, $\mathfrak{L}_{P(s)}$ be the sum of left minimal indices of $P(s)$, and $\kappa_{(i)}$ = left rank deficiency of T_i, then*

$$\kappa_{(i-1)} = \begin{cases} i(g-w) - \mathfrak{L}_{P(s)}, & \text{if } i \geq \nu + 1 \text{ and } i \leq \dfrac{\sum_{j=1}^{w} c_j}{g-w}, \\ \sum_{j=1}^{w} c_j - \mathfrak{L}_{P(s)}, & \text{if } i \geq \max\left\{\nu + 1, \dfrac{\sum_{j=1}^{w} c_j}{g-w}\right\}. \end{cases} \tag{3.1}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 874-895, November 2026.

Additionally, $P(s)$ is right prime if and only if

$$\kappa_{(i-1)} = \begin{cases} i(g-w) - \mathfrak{M}_{P(s)}, & \text{if } i \geq \nu+1 \text{ and } i \leq \dfrac{\sum_{j=1}^{w} c_j}{g-w}, \\ \sum_{j=1}^{w} c_j - \mathfrak{M}_{P(s)}, & \text{if } i \geq \max\left\{\nu+1, \dfrac{\sum_{j=1}^{w} c_j}{g-w}\right\}. \end{cases} \tag{3.2}$$

Proof. From Remark 3.2, we have $\mathfrak{L}_{P(s)} = i(g-w) - \eta_{i-1}$, for all $i \geq \nu+1$.

Case I: Assume $i \geq \nu+1$ and $T_{i-1} \in \mathbb{R}^{ig \times (\sum_{j=1}^{w} c_j + iw)}$ is a wide matrix, then $i \leq \frac{\sum_{j=1}^{w} c_j}{g-w}$. The left rank deficiency of T_{i-1} is

$$\begin{aligned} \kappa_{(i-1)} &= \eta_{i-1}, \\ \implies \mathfrak{L}_{P(s)} &= i(g-w) - \kappa_{(i-1)}, \\ \implies \kappa_{(i-1)} &= i(g-w) - \mathfrak{L}_{P(s)}. \end{aligned}$$

Case II: Assume $i \geq \nu+1$ and $T_{i-1} \in \mathbb{R}^{ig \times (\sum_{j=1}^{w} c_j + iw)}$ is a tall matrix, then $i \geq \frac{\sum_{j=1}^{w} c_j}{g-w}$. The left rank deficiency of T_{i-1} is

$$\begin{aligned} \kappa_{(i-1)} &= \eta_{i-1} - \left[ig - \left(iw + \sum_{j=1}^{w} c_j \right) \right] = \eta_{i-1} - i(g-w) + \sum_{j=1}^{w} c_j, \\ \implies \mathfrak{L}_{P(s)} &= i(g-w) - \left[\kappa_{(i-1)} + i(g-w) - \sum_{j=1}^{w} c_j \right], \\ \implies \kappa_{(i-1)} &= \sum_{j=1}^{w} c_j - \mathfrak{L}_{P(s)}. \end{aligned}$$

From Lemma 2.15, $P(s)$ is right prime if and only if $\mathfrak{M}_{P(s)} = \mathfrak{L}_{P(s)}$. Hence, the equations (3.1) and (3.2) hold true. ∎

Corollary 3.6. *Let $P(s) \in \mathbb{R}^{g \times w}[s]$ be defined in (2.1) with column degrees $(c_1, \ldots, c_w)$ and $\kappa_{(i)}$ = left rank deficiency of T_i. Then,*

$$\kappa_{(\sum_{j=1}^{w} c_j - 1)} = \sum_{j=1}^{w} c_j - \mathfrak{L}_{P(s)}. \tag{3.3}$$

Furthermore, $P(s)$ is right prime if and only if $\kappa_{(\sum_{j=1}^{w} c_j - 1)} = \sum_{j=1}^{w} c_j - \mathfrak{M}_{P(s)}$.

Proof. By Remark 3.2, the largest left minimal index is $\nu+1$ and $\mathfrak{L}_{P(s)}$ is the sum of all left minimal indices, so $\nu+1 \leq \mathfrak{L}_{P(s)}$. Every maximal minor for the polynomial matrix defined in (2.1) has degree at most $\sum_{j=1}^{w} c_j$ and Remark 2.16 implies $\sum_{j=1}^{w} c_j \geq \mathfrak{M}_{P(s)} \geq \mathfrak{L}_{P(s)}$. Hence,

$$\sum_{j=1}^{w} c_j \geq \mathfrak{M}_{P(s)} \geq \mathfrak{L}_{P(s)} \geq \nu+1.$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 874-895, November 2026.

Since $P(s)$ is tall, we have $g - w \geq 1$ and consequently $\sum_{j=1}^{w} c_j \geq \sum_{j=1}^{w} c_j/(g-w)$. Theorem 3.5, applied at the stage $i = \sum_j c_j$, gives

$$\kappa_{(\sum_{j=1}^{w} c_j - 1)} = \sum_{j=1}^{w} c_j - \mathfrak{L}_{P(s)}.$$

From Lemma 2.15 and (3.3), the second assertion holds true. ∎

Remark 3.7. From (2.21), we have $T_{\sum_{j=1}^{w} c_j - 1} \in \mathbb{R}^{\left(g \sum_{j=1}^{w} c_j\right) \times \left((w+1) \sum_{j=1}^{w} c_j\right)}$.

Given that $P(s)$ defined in (2.1) is a tall matrix such that

$$g > w \implies g \geq (w+1) \implies g \sum_{j=1}^{w} c_j \geq (w+1) \sum_{j=1}^{w} c_j.$$

This implies $T_{\sum_{j=1}^{w} c_j - 1}$ is either tall or square. The left rank deficiency of $T_{\sum_{j=1}^{w} c_j - 1}$ is $\kappa_{(\sum_{j=1}^{w} c_j - 1)} = (w+1)\sum_{j=1}^{w} c_j - \operatorname{rank}(T_{\sum_{j=1}^{w} c_j - 1})$.

Let $\dim(\ker(T_{\sum_{j=1}^{w} c_j - 1}))$ denote the dimension of the right kernel of $T_{\sum_{j=1}^{w} c_j - 1}$, then $\dim(\ker(T_{\sum_{j=1}^{w} c_j - 1})) = (w+1)\sum_{j=1}^{w} c_j - \operatorname{rank}(T_{\sum_{j=1}^{w} c_j - 1})$. This implies that $\kappa_{(\sum_{j=1}^{w} c_j - 1)} = \dim(\ker(T_{\sum_{j=1}^{w} c_j - 1}))$. Henceforth, we will use them interchangeably.

We now relate the existence of a GCRD to the right kernel of a particular Trimmed Sylvester matrix.

THEOREM 3.8. *Let $B_i(s) \in \mathbb{R}^{g_i \times w}[s]$, for $i = 1, \ldots, t$. Let $P(s) \in \mathbb{R}^{g \times w}[s]$ be defined in (2.1) with column degrees $(c_1, \ldots, c_w)$. Construct the sequence of Trimmed Sylvester matrices $\{T_i\}_{i=0,1,2,\ldots}$ as in (2.21). Then,*

1. *the polynomial matrices $B_1(s), \ldots, B_t(s)$ are coprime if and only if $\dim(\ker(T_{\sum_{j=1}^{w} c_j - 1})) = \sigma_\infty^{(\mathbf{c})}(P)$.*
2. *the determinant degree of every GCRD is equal to $\dim(\ker(T_{\sum_{j=1}^{w} c_j - 1})) - \sigma_\infty^{(\mathbf{c})}(P)$.*

Proof. By Corollary 3.6,

$$\kappa_{(\sum_{j=1}^{w} c_j - 1)} = \sum_{j=1}^{w} c_j - \mathfrak{L}_{P(s)}. \tag{3.4}$$

Since $g > w$, $T_{\sum_{j=1}^{w} c_j - 1}$ is tall or square, so Remark 3.7 gives $\dim(\ker(T_{\sum_{j=1}^{w} c_j - 1})) = \kappa_{\sum_{j=1}^{w} c_j - 1}$. From Theorem 2.10, the Column-Degree Index Sum Theorem gives

$$\dim(\ker(T_{\sum_{j=1}^{w} c_j - 1})) = \sum_{j=1}^{w} c_j - \mathfrak{L}_{P(s)} = \sigma_f(P) + \sigma_\infty^{(\mathbf{c})}(P). \tag{3.5}$$

1. By Lemma 2.15, the matrices $B_1(s), \ldots, B_t(s)$ are coprime if and only if $P(s)$ is right prime. This is equivalent to $P(s)$ having no finite eigenvalues, or equivalently to $\sigma_f(P) = 0$. Hence, (3.5) gives

$$\dim(\ker(T_{\sum_{j=1}^{w} c_j - 1})) = \sigma_\infty^{(\mathbf{c})}(P) \iff B_1(s), \ldots, B_t(s) \text{ are coprime.}$$

2. Let $G(s)$ be any GCRD of $B_1(s), \ldots, B_t(s)$. By (2.15), $\deg(\det G(s)) = \sigma_f(P)$. Subtracting $\sigma_\infty^{(\mathbf{c})}(P)$ from (3.5) therefore yields

$$\deg(\det G(s)) = \dim(\ker(T_{\sum_{j=1}^{w} c_j - 1})) - \sigma_\infty^{(\mathbf{c})}(P),$$

which proves the second assertion. ∎

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 874-895, November 2026.

3.1. Special Cases. Theorem 3.8 gets simplified in two settings. First, for a column-reduced matrix, the $\sigma_\infty^{(\mathbf{c})}(P)$ term vanishes and the rank of the Trimmed Sylvester matrix implies whether the matrices are coprime or not. Second, for a matrix with uniform column degrees $\mathbf{c} = (d, \ldots, d)$, the Trimmed Sylvester matrix becomes exactly the generalized Sylvester matrix. We discuss these two settings in turn.

3.1.1. Column-reduced matrices.

Definition 3.9 (Column-Reduced). *[3, Section 7.2] Let $P(s)$ be defined in (2.1) with column degrees $(c_1, \ldots, c_w)$. The highest column-degree coefficient matrix $P_{hc} \in \mathbb{R}^{g\times w}$ has as its jth column the coefficient of s^{c_j} in $P_j(s)$. The matrix $P(s)$ is column-reduced if P_{hc} has full column rank.*

For a column-reduced polynomial matrix $P(s)$, the structural indices of $P(s)$ at infinity are uniquely determined by the column degrees $-c_w \leq \ldots \leq -c_1$ [3, 16]. Because $c_j \geq 0$, all structural indices at infinity are nonpositive. Therefore, $P(s)$ has no infinite zeros; it has only infinite poles. By the definitions in Definition 2.5, their total degree is

$$\delta_z^\infty(P) = 0, \qquad \delta_p^\infty(P) = \sum_{j=1}^{w} c_j. \tag{3.6}$$

From (2.9), we have

$$\sigma_\infty(P) = dw - \delta_p^\infty(P) + \delta_z^\infty(P) = dw - \sum_{j=1}^{w} c_j = \sum_{j=1}^{w}(d - c_j). \tag{3.7}$$

From (2.11), we have

$$\sigma_\infty^{(\mathbf{c})}(P) = \sum_{j=1}^{w} c_j - \delta_p^\infty(P) + \delta_z^\infty(P) = \sum_{j=1}^{w} c_j - \sum_{j=1}^{w} c_j = 0. \tag{3.8}$$

Corollary 3.10 (Column-Reduced Index Sum Theorem). *Let $P(s) \in \mathbb{R}^{g\times w}[s]$ be the polynomial matrix defined in (2.1). Let $\sigma_f(P)$ be the total number of finite eigenvalues of $P(s)$ and $\mathfrak{L}_{P(s)}$ be the sum of left minimal indices of $P(s)$. If $P(s)$ is column-reduced, then*

$$\sum_{j=1}^{w} c_j = \sigma_f(P) + \mathfrak{L}_{P(s)}. \tag{3.9}$$

Proof. If $P(s)$ is column-reduced, all forms of the Index Sum Theorem reduce to a single identity. Since $P(s)$ is a polynomial matrix, it has no finite poles, so $\delta_p^{\text{fin}}(P) = 0$. The column-reduced property implies $\delta_z^\infty(P) = 0$ and $\delta_p^\infty(P) = \sum_{j=1}^{w} c_j$. Moreover, the tall, full rank polynomial matrix $P(s)$ has no right minimal indices, so the sum of its minimal indices is $\tau(P) = \mathfrak{L}_{P(s)}$. From (3.7) and (3.8), we have

$$\sigma_\infty(P) = dw - \sum_{j=1}^{w} c_j, \qquad \sigma_\infty^{(\mathbf{c})}(P) = 0. \tag{3.10}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 874-895, November 2026.

Rational Index Sum Theorem. Substitution into Theorem 2.8 gives

$$\begin{aligned}\delta_p(P) &= \delta_z(P) + \tau(P),\\ \delta_p^{\text{fin}}(P) + \delta_p^{\infty}(P) &= \delta_z^{\text{fin}}(P) + \delta_z^{\infty}(P) + \tau(P),\\ 0 + \sum_{j=1}^{w} c_j &= \sigma_f(P) + 0 + \mathfrak{L}_{P(s)},\\ \sum_{j=1}^{w} c_j &= \sigma_f(P) + \mathfrak{L}_{P(s)}.\end{aligned}$$

Polynomial Matrix Index Sum Theorem. Substitution into Theorem 2.9 gives

$$\begin{aligned}dw &= \sigma_f(P) + \sigma_{\infty}(P) + \mathfrak{L}_{P(s)},\\ dw &= \sigma_f(P) + dw - \sum_{j=1}^{w} c_j + \mathfrak{L}_{P(s)},\\ \sum_{j=1}^{w} c_j &= \sigma_f(P) + \mathfrak{L}_{P(s)}.\end{aligned}$$

Column-Degree Index Sum Theorem. Substitution into Theorem 2.10 gives

$$\begin{aligned}\sum_{j=1}^{w} c_j &= \sigma_f(P) + \sigma_{\infty}^{(\mathbf{c})}(P) + \mathfrak{L}_{P(s)},\\ \sum_{j=1}^{w} c_j &= \sigma_f(P) + 0 + \mathfrak{L}_{P(s)},\\ \sum_{j=1}^{w} c_j &= \sigma_f(P) + \mathfrak{L}_{P(s)}.\end{aligned}$$

∎

COROLLARY 3.11. *Let $P(s) \in \mathbb{R}^{g\times w}[s]$ be defined in (2.1) with column degrees $(c_1, \ldots, c_w)$ and $\kappa_{(i)}$ = left rank deficiency of T_i. If $P(s)$ is column-reduced, we have*

$$\kappa_{\left(\sum_{j=1}^{w} c_j - 1\right)} = \sum_{j=1}^{w} c_j - \mathfrak{L}_{P(s)}. \tag{3.11}$$

Furthermore, $P(s)$ is right prime if and only if $\kappa_{\left(\sum_{j=1}^{w} c_j - 1\right)} = \sum_{j=1}^{w} c_j - \mathfrak{M}_{P(s)}$.

Proof. Follows from Corollary 3.6 with $\mathfrak{M}_{P(s)} = \sum_{j=1}^{w} c_j$. ∎

The following corollary is a direct consequence of Theorem 3.8 specialized to column-reduced matrices using Corollary 3.10 and Corollary 3.11.

COROLLARY 3.12. *Let $P(s) \in \mathbb{R}^{g\times w}[s]$ be defined in (2.1) with column degrees $(c_1, \ldots, c_w)$ and $\kappa_{(i)}$ = left rank deficiency of T_i. If $P(s)$ is column-reduced, we have*

1. *the matrices $B_1(s), \ldots, B_t(s)$ are coprime if and only if $T_{\left(\sum_{j=1}^{w} c_j - 1\right)}$ is full rank;*
2. *for any GCRD $G(s)$, $\deg(\det G(s)) = \dim \ker T_{\left(\sum_{j=1}^{w} c_j - 1\right)}$.*

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 874-895, November 2026.

Proof. If $P(s)$ is column-reduced, then $\sigma_\infty^{(\mathbf{c})}(P) = 0$. The two assertions follow immediately from Theorem 3.8. ∎

3.1.2. Uniform-degree matrices. Suppose that $c_1 = \cdots = c_w = d$. In the uniform-degree case, the stage is $\sum_j c_j - 1 = dw - 1$, and no columns are removed, so the Trimmed Sylvester matrix is exactly the generalized Sylvester matrix of [5]. Therefore, Theorem 3.8 becomes identical to [5, Theorem 2.10].

COROLLARY 3.13. *[5, Theorem 2.10] Let $B_i(s) \in \mathbb{R}^{g_i \times w}[s]$, for $i = 1, \ldots, t$. Define $P(s)$ as in (2.1). Let $\{A_i\}_{i=0,1,2,\ldots}$ be the sequence of generalized Sylvester matrices, constructed as in (2.17). Then,*

1. *the polynomial matrices $B_1(s), \ldots, B_t(s)$ are coprime if and only if $\dim(\ker(A_{dw-1})) = \sigma_\infty(P)$.*
2. *the determinant degree of every GCRD of $B_1(s), \ldots, B_t(s)$, say $G(s)$, is equal to $\dim(\ker(A_{dw-1})) - \sigma_\infty(P)$.*

Full Rank Leading Coefficient. If, in addition, P_d (defined in (2.1)) has full column rank, then uniform degree polynomial matrix $P(s)$ has no eigenvalues at infinity, i.e., $\sigma_\infty(P) = 0$. Corollary 3.13 simplifies to the following corollary.

COROLLARY 3.14. *[5, Corollary 2.11] Let $B_i(s) \in \mathbb{R}^{g_i \times w}[s]$, for $i = 1, \ldots, t$. Define $P(s)$ as in (2.1). Assume that the leading coefficient matrix P_d is full column rank. Let $\{A_i\}$ be the sequence of generalized Sylvester matrices, constructed as in (2.17). Then*

1. *the polynomial matrices $B_1(s), \ldots, B_t(s)$ are coprime if and only if A_{dw-1} is a full rank matrix.*
2. *the determinant degree of every GCRD of $B_1(s), \ldots, B_t(s)$, say $G(s)$, is equal to $\dim(\ker(A_{dw-1}))$.*

Remark 3.15. Using (2.12), the relation between the left rank deficiency of the Trimmed Sylvester matrix and the generalized Sylvester matrix can be established. Observe that

$$\dim(\ker(A_{dw-1})) - \dim(\ker(T_{\sum_{j=1}^{w} c_j - 1})) = \sum_{j=1}^{w} (d - c_j), \tag{3.12}$$

which is exactly equal to the number of columns removed from the generalized Sylvester matrix to form the Trimmed Sylvester matrix at each stage.

4. Algorithm and Complexity.

4.1. Extraction via $EEQR$ Decomposition. Based on Theorem 3.8, the kernel of $T_{\sum_{j=1}^{w} c_j - 1}$ contains the information required to construct a GCRD of a given set of polynomial matrices. Having established this relationship between the Trimmed Sylvester matrix $T_{\sum_{j=1}^{w} c_j - 1}$ and a GCRD, we now propose an algorithm to extract coefficients of a GCRD using the "Effectively Eliminating' 'QR ($EEQR$) decomposition. For a rank-deficient matrix $T_{\sum_{j=1}^{w} c_j - 1}$ when a Givens rotation acts on $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$-entries, the choice of the orthogonal matrix becomes arbitrary, permitting nonunique factorizations. To resolve this, we employ the $EEQR$ decomposition, which imposes a strict constraint: a Givens rotation is only allowed to be non-trivial (i.e., not the identity matrix) if it is used to eliminate a nonzero entry in a sparsely structured matrix. Otherwise, the Givens rotation is simply chosen to be identity. This approach ensures the decomposition is unique (up to multiplication by an orthogonal diagonal matrix) [9, Theorem 6 (Lazy variant)]. To compute a GCRD,

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 874-895, November 2026.

we follow the algorithm similar to the one in [5] that proposes a method based on computing the $EEQR$ decomposition of a particular generalized Sylvester matrix.

Our argument follows that of [5], with a subtle difference in the stage at which a GCRD is extracted. We adopt the definition of equivalence in terms of the Smith Form for two matrices with different sizes from [5].

Remark 4.1 (**Equivalence of Smith Forms**). Two tall polynomial matrices $X_{n_1\times m}(s)$ and $Y_{n_2\times m}(s)$ with full normal ranks but different numbers of rows have equivalent Smith Forms, denoted by '$X(s) \sim Y(s)$', if

$$U_1(s)X(s)V_1(s) = \begin{bmatrix} \mathfrak{D}(s) \\ 0_{(n_1-m)\times m} \end{bmatrix}, \text{ and}$$
$$U_2(s)Y(s)V_2(s) = \begin{bmatrix} \mathfrak{D}(s) \\ 0_{(n_2-m)\times m} \end{bmatrix},$$

where $U_1(s)$, $V_1(s)$, $U_2(s)$, $V_2(s)$ are unimodular polynomial matrices of appropriate sizes and $\mathfrak{D}(s) \in \mathbb{R}^{m\times m}[s]$ is the Smith Form, as defined in Theorem 2.1.

From [5, Remark 3.4], the Smith Form of $P(s)$ is equivalent to the polynomial matrix formed by generalized Sylvester matrices A_i defined in (2.17):

$$\mathfrak{A}_i(s) = A_i \begin{bmatrix} s^{d+i}I_w \\ \vdots \\ sI_w \\ I_w \end{bmatrix}_{(d+i+1)w\times w} = \begin{bmatrix} s^iP(s) \\ \vdots \\ sP(s) \\ P(s) \end{bmatrix},$$
$$\implies \mathfrak{A}_i(s) \sim P(s).$$

Specifically, consider the submatrix $A_i(P_j)$ (see (2.19)) corresponding to the coefficients of the j-th column $P_j(s)$ in the generalized Sylvester matrix A_i:

$$\mathfrak{A}_{ij}(s) = A_i(P_j) \begin{bmatrix} s^{d+i} \\ \vdots \\ s \\ 1 \end{bmatrix}_{(d+i+1)\times 1} = \begin{bmatrix} s^iP_j(s) \\ \vdots \\ sP_j(s) \\ P_j(s) \end{bmatrix},$$
$$\implies \mathfrak{A}_{ij}(s) \sim P_j(s).$$

Consider submatrices $T_i(P_j)$ of $A_i(P_j)$ defined in (2.20). The submatrix $T_i(P_j)$ corresponding to the coefficients of the j-th column $P_j(s)$ in the Trimmed Sylvester matrix is given by

$$\mathfrak{T}_{ij}(s) = T_i(P_j) \begin{bmatrix} s^{c_j+i} \\ \vdots \\ s \\ 1 \end{bmatrix}_{(c_j+i+1)\times 1} = \begin{bmatrix} s^iP_j(s) \\ \vdots \\ sP_j(s) \\ P_j(s) \end{bmatrix},$$
$$\implies \mathfrak{T}_{ij}(s) \sim P_j(s) \sim \mathfrak{A}_{ij}(s),$$
$$\implies \mathfrak{T}_i(s) \sim P(s) \sim \mathfrak{A}_i(s).$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 874-895, November 2026.

Therefore, the Smith Form of $\mathfrak{T}_i(s)$, for $i = 0,\ 1,\ 2, \ldots$, is equivalent to the Smith Form of $P(s)$.

We present modifications of several lemmas from [5] to assist the proof that a GCRD of a set of several coprime polynomial matrices has a very specific structure in $EEQR$ decomposition of $T_{\sum_{j=1}^{w} c_j - 1}$.

LEMMA 4.2. *[5, Lemma 3.2] Let $P(s) \in \mathbb{R}^{g\times w}[s]$ be defined in (2.1) with column degrees $(c_1, \ldots, c_w)$. If $\operatorname{rank}(P(s)) = w$ and T_0 is defined as in (2.21), then $\operatorname{rank}(T_0) \geq w$.*

LEMMA 4.3. *[10, Theorem 4.1][5, Lemma 3.3] Let $P(s) \in \mathbb{R}^{g\times w}[s]$ be defined in (2.1) with column degrees $(c_1, \ldots, c_w)$. Construct the sequence of Trimmed Sylvester matrices $\{T_i\}_{i=0,1,2,\ldots}$ as in (2.21) and $r_i = rank(T_i)$. If $\operatorname{rank}(P(s)) = w$, then*

$$r_i - r_{i-1} = w, \qquad for\ i \geq \sum_{j=1}^{w} c_j - 1. \tag{4.1}$$

LEMMA 4.4. *Let $B_i(s) \in \mathbb{R}^{g_i\times w}[s]$, for $i = 1, \ldots, t$ be a set of several polynomial matrices. Let $P(s) \in \mathbb{R}^{g\times w}[s]$ be defined in (2.1) with column degrees $(c_1, \ldots, c_w)$. Suppose the $EEQR$ decomposition of $T_{\sum_{j=1}^{w} c_j - 1}$ is $T_{\sum_{j=1}^{w} c_j - 1} = QR$, where Q is orthogonal and $R \in \mathbb{R}^{(g\sum_{j=1}^{w} c_j)\times((w+1)\sum_{j=1}^{w} c_j)}$ is upper triangular. Let $G(s)$ be a polynomial matrix constructed from the last w nonzero rows of R. If $B_1(s), \ldots, B_t(s)$ are coprime, then $G(s)$ is a constant, full rank, square upper triangular matrix of size $w \times w$.*

Proof. First, by applying Lemma 4.2, Lemma 4.3 and the arguments of [5, Theorem 3.6, $(b) \Leftrightarrow (c)$] to the matrix $T_{\sum_{j=1}^{w} c_j - 1}$, it can be shown that the sequence of Givens rotations preserves the Smith Form. Specifically, $G(s)$, a polynomial matrix constructed from the last w nonzero rows, has the same Smith Form as the polynomial matrix $P(s)$ defined in (2.1), implying the equivalence $P(s) \sim G(s)$.

Second, by invoking Lemma 4.3 and [5, Theorem 3.6, $(c) \Leftrightarrow (d)$] on $T_{\sum_{j=1}^{w} c_j - 1}$, it follows that if $B_1(s), \ldots, B_t(s)$ are coprime, then the last w nonzero rows of R form a constant, full rank, square upper triangular matrix of size $w \times w$. ∎

THEOREM 4.5. *Let $B_i(s) \in \mathbb{R}^{g_i\times w}[s]$, for $i = 1, \ldots, t$ be a set of several polynomial matrices. Let $P(s) \in \mathbb{R}^{g\times w}[s]$ be defined in (2.1) with column degrees $(c_1, \ldots, c_w)$. Suppose the $EEQR$ decomposition of $T_{\sum_{j=1}^{w} c_j - 1}$ is $T_{\sum_{j=1}^{w} c_j - 1} = QR$, where Q is orthogonal, and R is upper triangular. Then, the last w nonzero rows of R define the coefficients of a GCRD of $B_1(s), \ldots, B_t(s)$*[2].

Proof. Let R be the upper triangular matrix obtained from the $EEQR$ decomposition of the Trimmed Sylvester matrix $T_{\sum_{j=1}^{w} c_j - 1}$. Let $G(s)$ be the polynomial matrix formed by the last w nonzero rows of R. Using Lemma 4.4 and [5, Theorem 3.9] on $T_{\sum_{j=1}^{w} c_j - 1}$, the last w nonzero rows of R define the coefficients of a GCRD. ∎

Remark 4.6 (Computation of determinant degree of a GCRD and infinite eigenvalues). Let $G(s)$ be a GCRD of size $w \times w$ and determinant degree $\deg(\det G(s))$. The structure of the $EEQR$ factorization shows that, once the block row corresponding to the GCRD has been selected, the number of zero rows below that block in the square part is exactly equal to the degree of determinant of $G(s)$. The remaining rank deficiency appears as zeros above the GCRD block row in R and represents the total number of infinite eigenvalues. This can be verified for the generalized Sylvester matrix using Corollary 3.13 as

$$\dim\ker A_{dw-1} = \deg(\det G(s)) + \sigma_\infty(P), \tag{4.2}$$

[2]Assume that if $P(s) = \widehat{P}(s)G(s)$, $\deg P = d$, $\deg G = N$, $\deg \widehat{P}(s) = d - N > 0$, we have $P_d = \widehat{P}_{d-N}G_N \neq 0$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 874-895, November 2026.

where the zeros on the diagonal and zero columns above the GCRD block row account for $\sigma_\infty(P)$. The same interpretation applies to the Trimmed Sylvester matrix

$$\dim\ker T_{\sum_{j=1}^{w} c_j - 1} = \deg(\det G(s)) + \sigma_\infty^{(\mathbf{c})}(P), \tag{4.3}$$

where the zeros on the diagonal above the GCRD block row account for $\sigma_\infty^{(\mathbf{c})}(P)$. In the column-reduced case, $\sigma_\infty^{(\mathbf{c})}(P) = 0$, so the Trimmed Sylvester matrix has no additional diagonal zeros from infinity. This is analogous to the generalized Sylvester case with a full rank leading coefficient matrix P_d, where $\sigma_\infty(P) = 0$. Consequently, the sum of left minimal indices of $P(s)$ can also be computed directly using the relationship:

$$\mathfrak{L}_{P(s)} = \sum_{j=1}^{w} c_j - \dim(\ker(T_{\sum_{j=1}^{w} c_j - 1})), \qquad \text{(Trimmed Sylvester matrix)}$$

$$\mathfrak{L}_{P(s)} = dw - \dim(\ker(A_{dw-1})). \qquad \text{(Generalized Sylvester Matrix).}$$

Thus, for the polynomial matrix $P(s)$, the determinant degree of a GCRD can be computed as the number of zero rows in the square part of R after selecting the block row corresponding to the GCRD. The remaining rank deficiency corresponds to the total number of infinite eigenvalues. Additionally, the difference between the rank deficiency of the Trimmed Sylvester matrix and the generalized Sylvester matrix is equal to the number of columns removed from the generalized Sylvester matrix to form the Trimmed Sylvester matrix at each stage.

4.2. Computational complexity. We analyze the computational complexity of our Trimmed Sylvester approach and compare it with existing methods for several polynomial matrices: the generalized Sylvester matrix approach [5] and the staircase algorithm [19].

Complexity of the Trimmed Sylvester Method. To compute the upper triangular form of the Trimmed Sylvester matrix $T_{\sum_{j=1}^{w} c_j - 1}$, we follow the block-by-block reduction algorithm BlockQR, consistent with [5, Section 4.1], where the structure of blocks and computational stages in Trimmed Sylvester matrix $T_{\sum_{j=1}^{w} c_j - 1}$ follow the same pattern as generalized Sylvester matrix A_{dw-1} in [5]. The crucial modification is replacing the total number of blocks dw (product of maximum degree and number of columns) with $\sum_{j=1}^{w} c_j$ (sum of individual column degrees).

For large values, the cost of applying Givens rotation matrices of size 2×2 to a matrix of size $2 \times x$ dominates the cost of constructing the rotators themselves. The total matrix multiplication cost also dominates the transformation of T_0 to upper triangular form. Consequently, the total computational cost to transform $T_{\sum_{j=1}^{w} c_j - 1}$ to an upper triangular matrix is approximately bounded above by $\mathcal{O}\left(3g\left(\sum_{j=1}^{w} c_j\right)^3\right)$. This is an upper bound on the complexity because we have not removed the redundant zero columns in the Trimmed Sylvester matrix.

In the generalized Sylvester matrix approach, one constructs matrices of size $dwg \times dw(1+w)$ at the $(dw-1)$-th stage. The computational cost for transforming this to an upper triangular form is approximately $\mathcal{O}\left(3g(dw)^3\right)$ [5, Section 4.1].

The recent state-space realization approach for GCRD computation proposed in [19] uses the staircase algorithm, whose complexity is cubic in the state-space dimension, which is $(d+1)w$ (see [19, Section 4]), yielding a complexity of approximately $\mathcal{O}\left((d+1)^3 w^3\right)$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 874-895, November 2026.

When column degrees vary a lot (i.e., $\sum_{j=1}^{w} c_j \ll dw$), for instance, if most columns have low degree and only a few have high degree, using Trimmed Sylvester matrices can reduce computations significantly. Table 1 compares the estimated computational cost of the three methods for various sizes and column degrees. Here, "Trimmed Sylvester" refers to our proposed method, "Gen. Sylvester" denotes the generalized Sylvester matrix approach from [5], and "Staircase" indicates the staircase algorithm from [19].

TABLE 1
Complexity comparison: Trimmed Sylvester, Generalized Sylvester and Staircase algorithm.

Size	**Col. Degrees**	**Trimmed Sylvester**	**Gen. Sylvester**	**Staircase**
4×3	$(1, 1, 10)$	2.07×10^4	3.24×10^5	3.59×10^4
6×5	$(2, 2, 3, 3, 40)$	2.25×10^6	1.44×10^8	8.62×10^6
10×7	$(1, 2, 4, 8, 16, 32, 64)$	6.15×10^7	2.70×10^9	9.42×10^7
12×10	$(1, 2, 3, 4, 5, 6, 7, 8, 9, 100)$	1.10×10^8	3.60×10^{10}	1.03×10^9
18×12	$(\underbrace{1, \ldots, 1}_{11}, 150)$	2.25×10^8	3.15×10^{11}	5.95×10^9
30×20	$(\underbrace{1, \ldots, 1}_{19}, 300)$	2.92×10^9	1.94×10^{13}	2.18×10^{11}

As shown in Table 1, for the computation of square GCRDs, when $\sum c_j \ll dw$, the Trimmed Sylvester approach offers significant computational savings over the generalized Sylvester matrix method and the staircase algorithm. The staircase algorithm is advantageous when column degrees are uniform and high.

5. Examples. We now demonstrate the effectiveness of the Trimmed Sylvester approach through detailed numerical examples.

EXAMPLE 5.1. *Consider a set of polynomial matrices,*

$$B_1(s) = \begin{bmatrix} 1 & 3s+3 & 4s^3+5s^2+6s+5 \\ 2 & 3s+3 & 5s^3+3s^2+2s+4 \end{bmatrix},$$
$$B_2(s) = \begin{bmatrix} 1 & 4s+4 & 5s^3+8s^2+6s+3 \\ 3 & 4s+4 & 7s^3+8s^2+8s+7 \end{bmatrix}, \text{ and}$$
$$B_3(s) = \begin{bmatrix} 2 & 3s+3 & 5s^3+9s^2+7s+3 \end{bmatrix}.$$

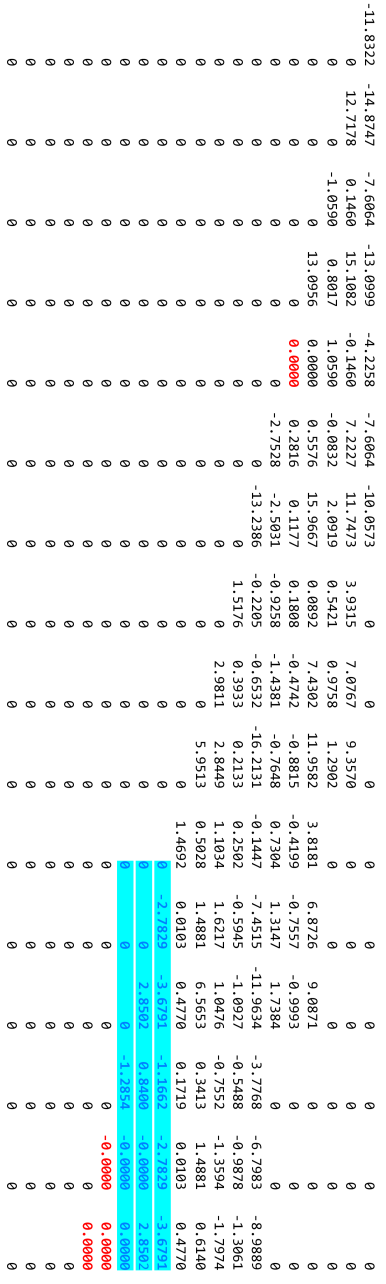
We construct the polynomial matrix $P(s)$ as

$$P(s) = \begin{bmatrix} 1 & 3s+3 & 4s^3+5s^2+6s+5 \\ 2 & 3s+3 & 5s^3+3s^2+2s+4 \\ 1 & 4s+4 & 5s^3+8s^2+6s+3 \\ 3 & 4s+4 & 7s^3+8s^2+8s+7 \\ 2 & 3s+3 & 5s^3+9s^2+7s+3 \end{bmatrix}.$$

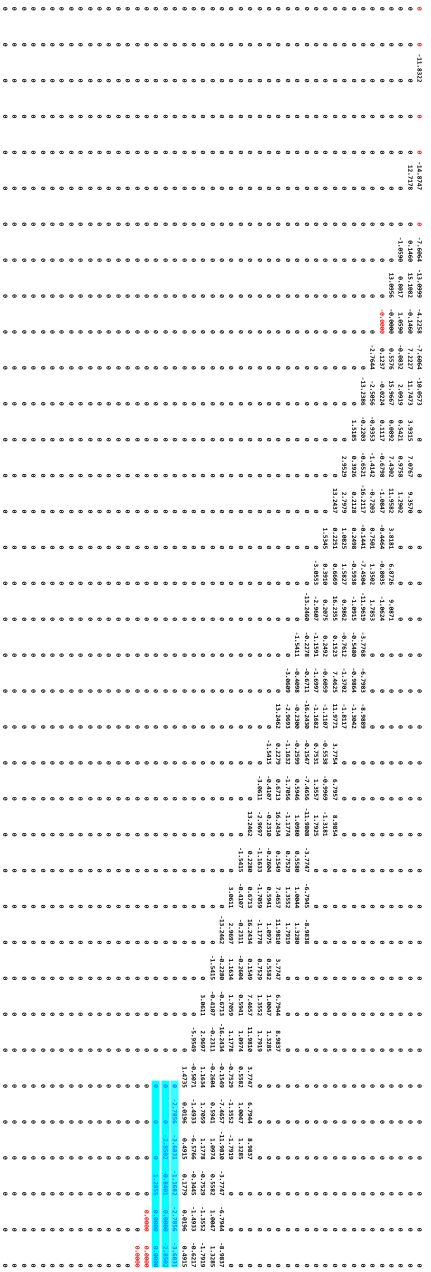
The highest column-degree coefficient matrix P_{hc} is given by

$$P_{hc} = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 3 & 5 \\ 1 & 4 & 5 \\ 3 & 4 & 7 \\ 2 & 3 & 5 \end{bmatrix},$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 874-895, November 2026.



(a) Trimmed Sylvester: $Q_1^\top T_3 = R_1$. Here, $\dim\ker(T_3) = 3 = \deg(\det G(s)) + \sigma_\infty^{(\mathbf{c})}(P)$, with $\deg(\det G(s)) = 2$, $\sigma_\infty^{(\mathbf{c})}(P) = 1$ and $\mathfrak{L}_{P(s)} = 0+1+3-3 = 1$.



(b) Generalized Sylvester: $Q_2^\top A_8 = R_2$. Here, $\dim\ker(A_8) = 8 = \deg(\det G(s)) + \sigma_\infty(P)$, with $\deg(\det G(s)) = 2$, $\sigma_\infty(P) = 6$ and $\mathfrak{L}_{P(s)} = 3\times 3 - 8 = 1$.

FIG. 1. *Upper-triangular forms of the Trimmed and generalized Sylvester matrices in Example 5.1. From Remark 4.6, the highlighted block contains the coefficients of a GCRD. The red zero rows below it give* $\deg\det G(s)$*, and the red zeros above it count the infinite eigenvalues. Together they reflect the rank deficiency of the matrices.*

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 874-895, November 2026.

and $\operatorname{rank}(P_{hc}) = 2 < 3$*, so* $P(s)$ *is not column-reduced. The matrix has* $g = 5$ *rows,* $w = 3$ *columns, degree* $d = 3$*, and column-degree vector* $\mathbf{c} = (0, 1, 3)$*.*

Let the EEQR decomposition of $T_{\sum_{j=1}^{3} c_j - 1} = T_3 \in \mathbb{R}^{20\times 16}$ *be* $Q_1^\top T_3 = R_1$*, where* R_1 *is upper triangular. A GCRD is constructed from the last* $w = 3$ *nonzero block rows of* R_1 *using Theorem 4.5. It is given by*

$$G_1(s) = \begin{bmatrix} -1.1662 & -2.7829s - 2.7829 & -3.6791s - 3.6791 \\ 0.8400 & 0 & 2.8502s + 2.8502 \\ -1.2854 & 0 & 0 \end{bmatrix}.$$

Using the generalized Sylvester matrices, we proceed to stage $dw = 9$*, so that* $A_{dw-1} = A_8 \in \mathbb{R}^{45\times 36}$*. Let the EEQR decomposition*[3] *of* $A_{dw-1} = A_8 \in \mathbb{R}^{45\times 36}$ *be* $Q_2^\top A_8 = R_2$*, where* R_2 *is upper triangular. A GCRD is constructed from the last* $w = 3$ *nonzero block rows of* R_2 *using [5, Theorem 3.9]. It is given by*

$$G_2(s) = \begin{bmatrix} -1.1682 & -2.7856s - 2.7856 & -3.6831s - 3.6831 \\ -0.8401 & 0 & -2.8502s - 2.8502 \\ 1.2855 & 0 & 0 \end{bmatrix}.$$

The invariants computed directly from the upper triangular matrices R_1 *and* R_2 *are shown in Figure 1. The difference between the two rank deficiencies is five, which is exactly equal to the number of columns removed from the generalized Sylvester matrix to form the Trimmed Sylvester matrix at each stage. The Smith form is given by* $P(s) \sim G_1(s) \sim G_2(s) \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & s+1 & 0 \\ 0 & 0 & s+1 \end{bmatrix}$.

Example 5.2. *Consider a set of polynomial matrices*

$$B_1(s) = \begin{bmatrix} 3 & 3s+2 & 4s+6 & s^2+2s+3 & 3s^4-4s^3+s^2-42s+100 \\ 5 & 2s+5 & s+2 & 2s^2+4s+5 & s^4+2s^3-6s^2-19s+47 \end{bmatrix},$$

$$B_2(s) = \begin{bmatrix} 7 & 4s+1 & 5s+5 & 5s^2+3s+6 & 5s^4-9s^3+s^2-39s+105 \\ 2 & 5s+6 & 2s+1 & 4s^2+2s+4 & 2s^4-s^3-2s^2-33s+76 \end{bmatrix}, \text{ and}$$

$$B_3(s) = \begin{bmatrix} 1 & s+3 & 3s+4 & 3s^2+s+7 & 4s^4-3s^3-9s^2-22s+65 \\ 6 & 3s+7 & 2s+3 & 2s^2+5s+2 & 3s^4-4s^3+s^2-34s+75 \\ 4 & 2s+4 & 4s+7 & s^2+4s+1 & 6s^4-11s^3+0s^2-35s+100 \end{bmatrix}.$$

We construct the polynomial matrix $P(s)$ *as*

$$P(s) = \begin{bmatrix} 3 & 3s+2 & 4s+6 & s^2+2s+3 & 3s^4-4s^3+s^2-42s+100 \\ 5 & 2s+5 & s+2 & 2s^2+4s+5 & s^4+2s^3-6s^2-19s+47 \\ 7 & 4s+1 & 5s+5 & 5s^2+3s+6 & 5s^4-9s^3+s^2-39s+105 \\ 2 & 5s+6 & 2s+1 & 4s^2+2s+4 & 2s^4-s^3-2s^2-33s+76 \\ 1 & s+3 & 3s+4 & 3s^2+s+7 & 4s^4-3s^3-9s^2-22s+65 \\ 6 & 3s+7 & 2s+3 & 2s^2+5s+2 & 3s^4-4s^3+s^2-34s+75 \\ 4 & 2s+4 & 4s+7 & s^2+4s+1 & 6s^4-11s^3+0s^2-35s+100 \end{bmatrix}.$$

[3] To obtain $Q_2^\top A_8 = R_2$, remove the redundant zero columns, apply *EEQR*, and reinsert the zero columns in their original positions.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 874-895, November 2026.

The column degrees are $(0,1,1,2,4)$*, and the highest column-degree coefficient matrix* P_{hc} *is given by*

$$P_{hc} = \begin{bmatrix} 3 & 3 & 4 & 1 & 3 \\ 5 & 2 & 1 & 2 & 1 \\ 7 & 4 & 5 & 5 & 5 \\ 2 & 5 & 2 & 4 & 2 \\ 1 & 1 & 3 & 3 & 4 \\ 6 & 3 & 2 & 2 & 3 \\ 4 & 2 & 4 & 1 & 6 \end{bmatrix},$$

and $\operatorname{rank}(P_{hc}) = 5$*, i.e., full rank.*

Since $P(s)$ *is column-reduced, Corollary 3.12 applies; we construct the Trimmed Sylvester matrix* $T_{\sum_{j=1}^{w} c_j - 1} = T_7 \in \mathbb{R}^{56\times 48}$*, which is much smaller than the generalized Sylvester matrix* $A_{dw-1} = A_{19} \in \mathbb{R}^{140\times 120}$*, with* $\dim(\ker(T_7)) = 1 = \deg(\det G(s))$*. A GCRD constructed using Theorem 4.5 is given by*

$$G(s) = \begin{bmatrix} 1.5344 & 1.7477 & 2.6347 & 1.2256 & 47.9675 - 20.4752s \\ 4.5381 & 0.6213 & -0.1072 & 0.5172 & 0.4069 \\ 0 & -5.7992 & 1.1623 & -0.4236 & -3.4745 \\ 0 & 0 & -3.0588 & 0.8428 & -6.1176 \\ 0 & 0 & 0 & 5.3670 & 0 \end{bmatrix}.$$

6. Conclusion. This paper proposes an efficient algorithm for computation of a GCRD of several polynomial matrices using Trimmed Sylvester matrices. The use of Trimmed Sylvester matrices allows computation of a GCRD at a much earlier stage than generalized Sylvester matrices. Furthermore, these matrices systematically remove redundant zero columns present in generalized Sylvester matrices.

The key contributions of this work are **(a)** a novel relationship is established between the rank deficiency of a Trimmed Sylvester matrix and the degree of the determinant of a GCRD, **(b)** an algorithm based on *Effectively Eliminating QR* (*EEQR*) decomposition is proposed that computes the coefficients of a GCRD and helps compute the structural invariants related to it, and **(c)** significant computational advantages are demonstrated by using Trimmed Sylvester matrices, particularly when the column degrees of the polynomial matrices vary a lot.

Acknowledgments. The authors sincerely thank the anonymous reviewer for their valuable suggestions, which significantly improved the content of the paper.

REFERENCES

[1] B.D.O. Anderson and E.I. Jury. Generalized Bezoutian and Sylvester matrices in multivariable linear control. *IEEE Trans. Autom. Control*, 21:551–556, 1976.
[2] L.M. Anguas, F.M. Dopico, R. Hollister, and D.S. Mackey. Van Dooren's index sum theorem and rational matrices with prescribed structural data. *SIAM J. Matrix Analy. Appl.*, 40:720–738, 2019.
[3] P.J. Antsaklis and A.N. Michel. *Linear Systems*. Birkhäuser Boston, 2006.
[4] A. Beniwal, A.S.A. Dilip, and S.R. Khare. On computing integer invariants of intersection of several linear time-invariant behaviors. In: *2026 European Control Conference (ECC)*. IEEE, 196–201, 2026.
[5] A. Beniwal, T. Saha, and S.R. Khare. Computation of an exact GCRD of several polynomial matrices: QR decomposition approach. *Linear Algebra Appl.*, 710:471–506, 2025.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 874-895, November 2026.

[6] R. Bitmead, S.-Y. Kung, B. Anderson, and T. Kailath. Greatest common divisor via generalized Sylvester and Bezout matrices. *IEEE Trans. Autom. Control*, 23:1043–1047, 1978.
[7] F. De Terán, F.M. Dopico, and D.S. Mackey. Spectral equivalence of matrix polynomials and the index sum theorem. *Linear Algebra Appl.*, 459:264–333, 2014.
[8] F. De Terán, F.M. Dopico, and P. Van Dooren. Matrix polynomials with completely prescribed eigenstructure. *SIAM J. Matrix Anal. Appl.*, 36:302–328, 2015.
[9] S. Delvaux and M. Van Barel. Rank structures preserved by the QR-algorithm: The singular case. *J. Computat. Appl. Math.*, 189:157–178, 2006. Proceedings of The 11th International Congress on Computational and Applied Mathematics.
[10] F.M. Dopico and P. Van Dooren. Robustness and perturbations of minimal bases II: The case with given row degrees. *Linear Algebra Appl.*, 576:268–300, 2019.
[11] E. Emre and L. Silverman. New criteria and system theoretic interpretations for relatively prime polynomial matrices. *IEEE Trans. Autom. Control*, 22:239–242, 1977.
[12] A. Fazzi, N. Guglielmi, and I. Markovsky. Computing common factors of matrix polynomials with applications in system and control theory. In: *2019 IEEE 58th Conference on Decision and Control*, 7721–7726, 2019.
[13] A. Fazzi, N. Guglielmi, and I. Markovsky. Generalized algorithms for the approximate matrix polynomial GCD of reducing data uncertainties with application to MIMO system and control. *J. Computat. Appl. Math.*, 393:113499, 2021.
[14] S. Gangopadhyay and K.B. Datta. Coprime matrix fraction description via sylvester's resultant matrix. *IETE J. Res.*, 40:129–133, 1994.
[15] I. Gohberg, P. Lancaster, and L. Rodman. *Matrix Polynomials*. Academic Press, New York, 1982.
[16] T. Kailath. *Linear Systems*. Prentice-Hall Englewood Cliffs, NJ, 1980.
[17] S. Kung, T. Kailath, and M. Morf. A generalized resultant matrix for polynomial matrices. In: *1976 IEEE Conference on Decision and Control including the 15th Symposium on Adaptive Processes*, 892–895, 1976.
[18] C.C. MacDuffee. *The Theory of Matrices*. Springer-Verlag Berlin, Heidelberg, 1933.
[19] V. Noferini and P. Van Dooren. Revisiting the matrix polynomial greatest common divisor. *SIAM J. Matrix Anal. Appl.*, 44:1164–1188, 2023.
[20] T. Saha and S.R. Khare. Computation of the nearest non-prime polynomial matrix: Structured low-rank approximation approach. *Linear Algebra Appl.*, 616:84–124, 2021.
[21] P. Van Dooren and F.M. Dopico. Robustness and perturbations of minimal bases. *Linear Algebra Appl.*, 542:246–281, 2018.
[22] S. Wang and E. Davison. A minimization algorithm for the design of linear multivariable systems. *IEEE Trans. Autom. Control*, 18:220–225, 1973.
[23] W.A. Wolovich. *Linear Multivariable Systems*. Springer-Verlag, New York, 1974.