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ON THE CORE-NILPOTENT DECOMPOSITION OF THRESHOLD GRAPHS*

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Abstract. In this work, we show that the core-nilpotent decomposition of the adjacency matrix $A(G)$ of a threshold graph G can be obtained directly from the graph through its null decomposition. More precisely, we construct two invertible matrices Q and C, defined in terms of the adjacency relations of G, such that $Q^{-1}A(G)Q$ is a block diagonal matrix with two blocks. The first block is C, an $r \times r$ matrix satisfying $\mathrm{rk}(C) = \mathrm{rk}(A(G)) = r$, and the second block is the zero matrix. Consequently, the core-nilpotent decomposition of $A(G)$ can be read directly from the combinatorial structure of G.

Key words. Threshold graphs, Core-nilpotent decomposition, Null decomposition, Rank.

AMS subject classifications. 15A09, 05C38.

1. Introduction.

Let M be an $n \times n$ matrix with rank r (i.e., $\mathrm{rk}(M) = r$). The *index* of M is the smallest nonnegative integer k such that $\mathrm{rk}(M^k) = \mathrm{rk}(M^{k+1})$. Thus,

$$\mathbb{R}^n = \mathcal{R}(M^k) \oplus \mathcal{N}(M^k),$$

i.e., $\mathbb{R}^n = \mathcal{R}(M^k) + \mathcal{N}(M^k)$ and $\mathcal{R}(M^k) \cap \mathcal{N}(M^k) = \mathbf{0}$, where $\mathcal{R}(M)$ and $\mathcal{N}(M)$ are the range and the null space of M, see [1]. Let N be an square matrix. N is said to be *nilpotent* whenever $N^t = \mathbf{0}$ for some positive integer t. Thus, if k is the smallest positive integer such that $N^k = \mathbf{0}$, then k is the index of N.

If M is an $n \times n$ singular matrix of index k such that $\mathrm{rk}(M^k) = r$, then there exists a nonsingular matrix Q such that

$$Q^{-1}MQ = \begin{bmatrix} C & \mathbf{0} \\ \mathbf{0} & N \end{bmatrix}, \tag{1.1}$$

where C is a $r \times r$ nonsingular matrix, and N is nilpotent of index k. When M is nonsingular, $Q = I$ and $C = M$. The block-diagonal matrix in (1.1) is called a *core-nilpotent decomposition* of M.

If M is an $n \times n$ real symmetric matrix with rank r, then M has index 1 and

$$\mathbb{R}^n = \mathcal{R}(M) \oplus \mathcal{N}(M).$$

Thus, for any nonsingular matrix $Q = [X|Y]$, where the columns of the $n \times r$ matrix X and the columns of the $n \times (n-r)$ matrix Y form basis for $\mathcal{R}(M)$ and $\mathcal{N}(M)$, respectively, the following holds:

$$Q^{-1}MQ = \begin{bmatrix} C & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},$$

since, if N is nilpotent of index 1, then $N = \mathbf{0}$.

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The main idea behind the core-nilpotent decomposition of graphs is to find optimal vertex labelings that allow the construction of simple bases for both the range and the null space. Such bases can be obtained by analyzing the adjacency relations among vertices. This approach has already been carried out for trees and unicyclic graphs: first, their null decomposition was studied (see [9, 11]), and later their core–nilpotent decomposition was analyzed (see [10, 12]). In this work, we follow the same line of action to study threshold graphs. We begin by analyzing their null decomposition and subsequently derive their core–nilpotent decomposition.

The null space of threshold graphs has been studied in several works; see [3, 6]. In this paper, we provide an explicit basis for the null space. Using the support of this basis, we obtain a partition of the vertex set. Finally, we derive a formula for the rank of threshold graphs in terms of this partition.

For the core-nilpotent decomposition, we adopt the framework introduced in [10, 12], and ultimately show that the core-nilpotent decomposition of threshold graphs is similar to that of trees, except for a submatrix associated with vertices having the same supported neighborhood.

This paper is organized into six sections. Section 2 covers notation and preliminary results. In Section 3, we provide a simple basis for the null space of the adjacency matrix of any threshold graph. Section 4 is concerned with the null decomposition of threshold graphs. Section 5 provides a simple basis for the range of the adjacency matrix of any threshold graph, which is key for the graph-theoretic interpretation of the core–nilpotent decomposition of threshold graphs. In Section 6, we present a proof of our main result.

2. Notation and preliminaries. Let $G = (V, E)$ be a simple undirected graph of order n, with vertex set $V = \{1, \dots, n\}$ and edge set E, the adjacency matrix $A(G) = (a_{uv})_{n\times n}$ of G is defined as

$$a_{uv} = \begin{cases} 1, & \text{if } \{u, v\} \in E; \\ 0, & \text{otherwise.} \end{cases}$$

The null space of $A(G)$, denoted by $\mathcal{N}(G)$, is the set $\{\mathbf{x} \in \mathbb{R}^n : A(G)\mathbf{x} = \mathbf{0}\}$. The nullity of G is denoted by $\mathrm{nl}(G)$. The rank of $A(G)$ is denoted as $\mathrm{rk}(G)$. For brevity, we use expressions like "the null space of G" or "the nullity of G is $\mathrm{nl}(G)$", etc.

For any $v \in V(G)$, the neighborhood of v, denoted by $N(v)$, is the set $\{u \in V(G) : u \sim v\}$, where $u \sim v$ means that u and v are neighbors: $\{u, v\} \in E(G)$. The neighbor of a subset $W \subseteq V(G)$ is $N(W) = \bigcup_{v\in W} N(v)$. Two vertices $u, v \in V(G)$ are called duplicate vertices if $N(u) = N(v)$. For each $i \in \mathbb{N}$, we use $\mathbf{e}_i$ to denote the vector with a one in the ith position and zeros elsewhere; the order of the vector will always be clear from the context. The vector $\sum_{v\in W} \mathbf{e}_v$ is denoted by $\mathbf{e}_W$. Regarding duplicate vertices, it is known that if v, u are duplicated, then $\mathbf{e}_v - \mathbf{e}_u \in \mathcal{N}(G)$. For more details, we recommend reading Section 5 in [4].

A way to characterize threshold graphs is through an iterative process that starts with an isolated vertex, and where, at each step, either a new isolated vertex is added, or a dominating vertex is added. We can represent a threshold graph G on n vertices by a binary sequence $s = b_1 b_2 \dots b_n$, where if $b_i = 0$, then v_i was added as an isolated vertex, and if $b_i = 1$, then v_i was added as a dominating vertex. This representation has been called a creation sequence in [2] and is the most convenient for our work. From now on, w.l.g. we assume that b_1 is always zero. Consider a connected threshold graph with at least two vertices. Its creation sequence s is given by

$$s = \underbrace{0\cdots0}_{r_1}\underbrace{1\cdots1}_{t_1}\underbrace{0\cdots0}_{r_2}\underbrace{1\cdots1}_{t_2}\cdots\underbrace{0\cdots0}_{r_m}\underbrace{1\cdots1}_{t_m},$$

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where $t_i, r_i > 0$ for all i and $r_1 + t_1 + r_2 + t_2 + \cdots + r_m + t_m = n$. To shorten the notation, we will write $s = 0^{r_1}1^{t_1}0^{r_2}1^{t_2}\cdots 0^{r_m}1^{t_m}$.

3. A basis for the null space of a threshold graph. In this section, we obtain a basis for the null space of a threshold graph. These results will be crucial for the construction of the null decomposition in the next section.

Let G be a connected threshold graph with the creation sequence

$$s = b_1 b_2 \cdots b_n = 0^{r_1}1^{t_1}0^{r_2}1^{t_2}\cdots 0^{r_m}1^{t_m}.$$

We will denote by z_i a subsequence of s formed by r_i zeros, where its zeros are chosen consecutively, that is, $z_i = b_{p_i}b_{p_i+1}b_{p_i+2}\cdots b_{p_i+r_i-1} = 0^{r_i}$. Similarly, we will denote by u_i a subsequence of s composed of t_i ones, where its ones are chosen consecutively, that is, $u_i = b_{q_i}b_{q_i+1}b_{q_i+2}\cdots b_{q_i+t_i-1} = 1^{t_i}$.

The set of vertices generated by z_i and u_i will be denoted by $\mathbb{V}(z_i)$ and $\mathbb{V}(u_i)$, respectively, that is,

$$\mathbb{V}(z_i) = \{p_i, p_i + 1, \ldots, p_i + r_i - 1\},$$

and

$$\mathbb{V}(u_i) = \{q_i, q_i + 1, \ldots, q_i + t_i - 1\}.$$

Through the recursive process of construction of a threshold graph G, we can make the following remark.

REMARK 3.1. *Consider two different vertices u and v such that $u, v \in \mathbb{V}(z_k)$. Note that u and v are duplicated because $N(v) = \bigcup_{i=k}^{m} \mathbb{V}(u_i) = N(u)$.*

The next lemma is a known result and can be found in [7, 8].

LEMMA 3.2. *Let G be a connected threshold graph with the creation sequence $s = 0^{r_1}1^{t_1}0^{r_2}1^{t_2}\cdots 0^{r_m}1^{t_m}$. Then*

$$\mathrm{nl}(G) = \sum_{i=1}^{m}(r_i - 1).$$

The following theorem is a reformulation of a known basis for the null space of threshold graphs due to Andelić, Koledin, and Stanić [5], expressed in terms of the creation sequence. Since this result is well known, we omit its proof.

THEOREM 3.3. *Let G be a connected threshold graph with the creation sequence*

$$s = 0^{r_1}1^{t_1}0^{r_2}1^{t_2}\cdots 0^{r_m}1^{t_m}.$$

Let $z_i = b_{p_i}b_{p_i+1}b_{p_i+2}\cdots b_{p_i+r_i-1}$ be a subsequence of s. Consider $\mathcal{B}_i$ a set of vectors given by:

- $\mathcal{B}_i = \{\mathbf{e}_{p_i} - \mathbf{e}_{p_i+j} : j = 1, 2, \ldots, r_i - 1\}$, *when* $r_i \geq 2$;
- $\mathcal{B}_i = \emptyset$, *when* $r_i = 1$;

Then, $\mathcal{B} = \bigcup_{i=1}^{m} \mathcal{B}_i$ is a basis of $\mathcal{N}(G)$.

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Example 3.4. *Consider*

$$s = 010001100001 = 010^3 1^2 0^4 1 = z_1 u_1 z_2 u_2 z_3 u_3.$$

This sequence determines the graph G in Fig. 1. Note that

- $z_1 = b_1 = 0$ *and* $u_1 = b_2 = 1$*;*
- $z_2 = b_3 b_4 b_5 = 000 = 0^3$ *and* $u_2 = b_6 b_7 = 11 = 1^2$*;*
- $z_3 = b_8 b_9 b_{10} b_{11} = 0000 = 0^4$ *and* $u_3 = b_{12} = 1$.

The set of vertices generated by each of these subsequences are

$$\mathbb{V}(z_1) = \{1\},\ \mathbb{V}(u_1) = \{2\},\ \mathbb{V}(z_2) = \{3,4,5\},\ \mathbb{V}(u_2) = \{6,7\},\ \mathbb{V}(z_3) = \{8,9,10,11\} \text{ and } \mathbb{V}(u_3) = \{12\}.$$

Theorem 3.3 allows us to obtain a basis of $\mathcal{N}(G)$. Note that the creation sequence s of G has three zero sub-sequences: $z_1 = 0^{r_1}$, $z_2 = 0^{r_2}$, and $z_3 = 0^{r_3}$, where $r_2, r_3 \geq 2$. Thus, the basis of $\mathcal{N}(G)$ is formed by the vectors:

$$\mathbf{e}_3 - \mathbf{e}_4 = \begin{bmatrix} 0\\0\\1\\-1\\0\\0\\0\\0\\0\\0\\0\\0 \end{bmatrix}, \mathbf{e}_3 - \mathbf{e}_5 = \begin{bmatrix} 0\\0\\1\\0\\-1\\0\\0\\0\\0\\0\\0\\0 \end{bmatrix}, \mathbf{e}_8 - \mathbf{e}_9 = \begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\1\\-1\\0\\0\\0 \end{bmatrix}, \mathbf{e}_8 - \mathbf{e}_{10} = \begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\1\\0\\-1\\0\\0 \end{bmatrix}, \mathbf{e}_8 - \mathbf{e}_{11} = \begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\1\\0\\0\\-1\\0 \end{bmatrix}.$$

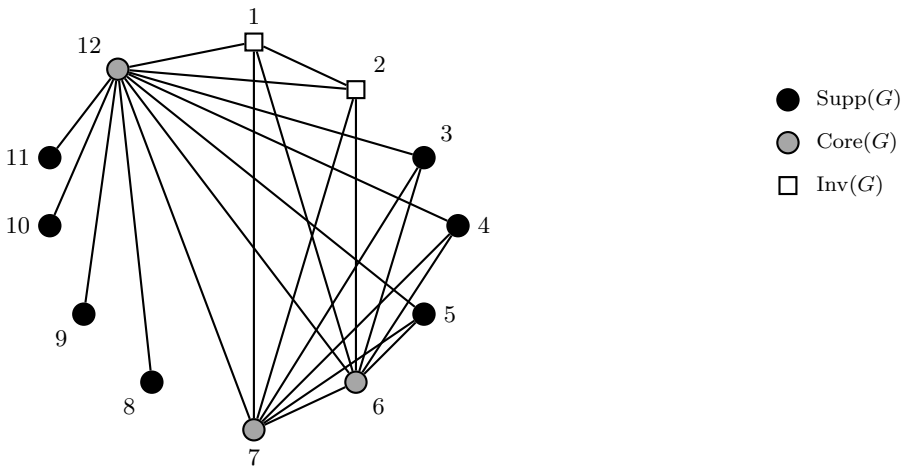


Figure 1: Threshold graph G with creation sequence $s = 010^3 1^2 0^4 1$.

4. Null decomposition of threshold graphs. In this section, we present the null decomposition of threshold graph, more precisely we compute its null decomposition from its creation sequence.

Definition 4.1. *Let G be a graph. We define the following subsets of $V(G)$:*

- $\mathrm{Supp}(G) = \{v \in V(G) : \exists\, \mathbf{x} \in \mathcal{N}(G) \text{ such that } \mathbf{x}_v \neq 0\}$*;*

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- $\mathrm{Core}(G) = N(\mathrm{Supp}(G))$;
- $\mathrm{Inv}(G) = V(G) - (\mathrm{Supp}(G) \cup \mathrm{Core}(G))$.

Following the terminology introduced in [9], we refer to the vertices in $\mathrm{Supp}(G)$, $\mathrm{Core}(G)$, *and* $\mathrm{Inv}(G)$ *as support, core, and invertible vertices, respectively. The cardinalities of* $\mathrm{Supp}(G)$, $\mathrm{Core}(G)$, *and* $\mathrm{Inv}(G)$ *are denoted by* $\mathrm{supp}(G)$, $\mathrm{core}(G)$, *and* $\mathrm{inv}(G)$, *respectively.*

The following result shows that in order to compute the support of G, it is enough to analyze the coordinates of the vectors of a basis of null space.

LEMMA 4.2 ([9]). *Let G be a graph and $\mathcal{B} = \{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ a basis of $\mathcal{N}(G)$, then*

$$\mathrm{Supp}(G) = \{v \in V(G) : \exists\, \mathbf{x}_i \in \mathcal{B} \text{ such that } (\mathbf{x}_i)_v \neq 0\}.$$

In the next propositions, we compute $\mathrm{Supp}(G)$, $\mathrm{Core}(G)$, and $\mathrm{Inv}(G)$ of a threshold graph G using its sub-sequences z_i's and u_i's.

PROPOSITION 4.3. *Let G be a connected threshold graph with the creation sequence*

$$s = 0^{r_1}1^{t_1}0^{r_2}1^{t_2}\cdots 0^{r_m}1^{t_m}.$$

If k is the minimum index such that $r_k \geq 2$, then

(i) $\mathrm{Supp}(G) = \bigcup_{r_i \geq 2} \mathbb{V}(z_i)$;

(ii) $\mathrm{Core}(G) = \bigcup_{i=k}^{m} \mathbb{V}(u_i)$;

(iii) $\mathrm{Inv}(G) = \left(\bigcup_{i=1}^{k-1} \mathbb{V}(u_i)\right) \cup \left(\bigcup_{r_i=1} \mathbb{V}(z_i)\right)$;

(iv) $\mathrm{Supp}(G)$ *is a independent set.*

Proof. (*i*) The result follows from observing the nonzero entries of the basis of $\mathcal{N}(G)$ provided by Theorem 3.3. We conclude that the only nonzero entries correspond to the vertices generated by the sub-sequences z_i such that $r_i \geq 2$.

(*ii*) From item (*i*), we know that the support vertices are those generated by the sub-sequences z_i such that $r_i \geq 2$. Through the iterative construction process of G, we find that the only neighbors of the support are the vertices generated by sub-sequences u_i such that $i \geq k$.

(*iii*) Simply use items (*i*) and (*ii*).

(*iv*) Simply use item (*i*). □

EXAMPLE 4.4. *Example 3.4 shows a basis for the null space of the graph G in Fig. 1. Note that*

$$\mathrm{Supp}(G) = \{3, 4, 5\}, \quad \mathrm{Core}(G) = \{6, 7, 12\} \quad \textit{and} \quad \mathrm{Inv}(G) = \{1, 2\}.$$

5. A basis for the range of threshold graphs. In this section, we obtain a basis for the range of threshold graphs. This basis will be fundamental for obtaining the core-nilpotent decomposition of threshold graph in the next section. Next, we will define some combinatorial matrices associated to threshold graphs, similar to those in [10, 12].

DEFINITION 5.1. *Let G be a threshold graph and $v \in \mathrm{Core}(G)$. We call v-flower of G to the set $N(v) \cap \mathrm{Supp}(G)$. The v-flower is denoted by $\mathcal{F}_G(v)$.*

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Definition 5.2. *Let G be a graph. Define a relation R on* $\mathrm{Core}(G)$ *by*

$$uRv \quad \text{if and only if} \quad \mathcal{F}_G(u) = \mathcal{F}_G(v).$$

It is straightforward to verify that R is an equivalence relation on $\mathrm{Core}(G)$. *For $v \in \mathrm{Core}(G)$, let*

$$[v] := \{w \in \mathrm{Core}(G) : wRv\}.$$

We define

$$TW(G) := \{[v] : v \in \mathrm{Core}(G)\}.$$

For $[v] \in TW(G)$, define $\mathcal{F}_G([v]) := \mathcal{F}_G(v)$.

By Proposition 4.3, we have the following corollary that exhibits some properties of v-flowers.

Corollary 5.3. *Let G be a connected threshold graph with creation sequence $0^{r_1}1^{t_1}0^{r_2}1^{t_2}\cdots 0^{r_m}1^{t_m}$. If $v \in \mathrm{Core}(G) \cap \mathbb{V}(u_j)$ and $w \in \mathrm{Core}(G) \cap \mathbb{V}(u_k)$ such that $j \leq k$, then*

(i) $\mathcal{F}_G(v) = \bigcup_{\substack{r_i \geq 2 \\ i \leq j}} \mathbb{V}(z_i)$;
(ii) $\mathcal{F}_G(v) = \mathcal{F}_G(w)$ *if and only if* $j = k$ *or* $r_{j+1} = r_{j+2} = \cdots = r_k = 1$;
(iii) $|TW(G)| = \sum_{r_i \geq 2} 1$.

The next proposition provides a formula for the rank of threshold graphs.

Proposition 5.4. *Let G be a connected threshold graph. Then,*

$$\mathrm{rk}(G) = \mathrm{core}(G) + \mathrm{inv}(G) + |TW(G)|.$$

Proof. Consider $0^{r_1}1^{t_1}0^{r_2}1^{t_2}\cdots 0^{r_m}1^{t_m}$ as the sequence of creation of G. By Lemma 3.2, we have that

$$\mathrm{nl}(G) = \sum_{i=1}^{m}(r_i - 1).$$

Using Proposition 4.3, Corollary 5.3, and the rank-nullity theorem, we obtain

$$\begin{aligned}
\mathrm{rk}(G) &= |V(G)| - \mathrm{nl}(G) \\
&= \sum_{i=1}^{m}(t_i + r_i) - \sum_{i=1}^{m}(r_i - 1) \\
&= \sum_{i=1}^{m}(t_i + 1) \\
&= \left(\sum_{i=1}^{m} t_i\right) + \left(\sum_{i=1}^{m} 1\right) \\
&= \left(\sum_{i=1}^{m} t_i\right) + \left(\sum_{r_i \geq 2} 1\right) + \left(\sum_{r_i = 1} 1\right) \\
&= \mathrm{core}(G) + |TW(G)| + \mathrm{inv}(G).
\end{aligned}$$

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A null labeling of a threshold graph G is a labeling of G with the positive integers $\{1, 2, \ldots, |V(G)|\}$, where the core vertices are labeled first, then the invertible vertices, and finally the supported vertices of G. Let G be a null-labeled threshold graph. Let $\mathcal{B}_R(G)$ be the following set of vectors

$$\mathcal{B}_R(G) := \{\mathbf{e}_v : v \in \mathrm{Core}(G)\} \cup \{\mathbf{e}_u : u \in \mathrm{Inv}(G)\} \cup \left\{\mathbf{e}_{\mathcal{F}_G([v])} : [v] \in TW(G)\right\}.$$

Next theorem guarantees that the set $\mathcal{B}_R(G)$ is a basis for the range of $A(G)$.

THEOREM 5.5. *Let G be a null-labeled threshold graph. Then, $\mathcal{B}_R(G)$ is a basis of the range of $A(G)$.*

Proof. Let A_u be the column of $A(G)$ associated with the vertex u. If $u \in \mathrm{Supp}(G)$, then

$$A_u = \mathbf{e}_{N(u)} = \sum_{v \in \mathrm{Core}(G) \cap N(u)} \mathbf{e}_v \,.$$

If $u \in \mathrm{Core}(G)$, then

$$\begin{aligned} A_u = \mathbf{e}_{N(u)} &= \sum_{v \in \mathrm{Supp}(G) \cap N(u)} \mathbf{e}_v + \sum_{v \in \mathrm{Core}(G) \cap N(u)} \mathbf{e}_v + \sum_{v \in \mathrm{Inv}(G) \cap N(u)} \mathbf{e}_v \\ &= \mathbf{e}_{\mathcal{F}_G([u])} + \sum_{v \in \mathrm{Core}(G) \cap N(u)} \mathbf{e}_v + \sum_{w \in \mathrm{Inv}(G) \cap N(u)} \mathbf{e}_w. \end{aligned}$$

If $u \in \mathrm{Inv}(G)$, then

$$A_u = \mathbf{e}_{N(u)} = \sum_{v \in \mathrm{Core}(G) \cap N(u)} \mathbf{e}_v + \sum_{w \in \mathrm{Inv}(G) \cap N(u)} \mathbf{e}_w \,.$$

Therefore, the linear space spanned by $\mathcal{B}_R(G)$ contains the range of $A(G)$. Note that $\mathcal{B}_R(G)$ is obviously a set of linearly independent vectors. Moreover, by Proposition 5.4, we have that $|\mathcal{B}_R(G)| = \mathrm{core}(G) + |TW(G)| + \mathrm{inv}(G) = \mathrm{rk}(G)$. Therefore, $\mathcal{B}_R(G)$ is a basis of the range of $A(G)$. □

EXAMPLE 5.6. *Figure 2 shows a null-labeled threshold graph G. Note that* $\mathrm{Core}(G) = \{1, 2, \ldots, 6\}$, $\mathrm{Inv}(G) = \{7, 8, 9, 10, 11\}$, *and* $\mathrm{Supp}(G) = \{12, 13, \ldots, 18\}$. *We have that*

- $\mathcal{F}_G(1) = \{12, 13\} = \mathcal{F}_G(2)$;
- $\mathcal{F}_G(3) = \{12, 13, 14, 15, 16\} = \mathcal{F}_G(4) = \mathcal{F}_G(5)$;
- $\mathcal{F}_G(3) = \mathrm{Supp}(G)$.

Thus, $[1] = [2] = \{1, 2\}$, $[3] = [4] = [5] = \{3, 4, 5\}$ *and* $[6] = \{6\}$. *Therefore,* $TW(G) = \{[1], [3], [6]\}$. *Thus, a basis for range of G is*

$$\begin{aligned} \mathcal{B}_R(G) &= \left\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{11}, \mathbf{e}_{\mathcal{F}_G([1])}, \mathbf{e}_{\mathcal{F}_G([3])}, \mathbf{e}_{\mathcal{F}_G([6])}\right\} \\ &= \left\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{11}, \mathbf{e}_{\{12,13\}}, \mathbf{e}_{\{12,13,\ldots,16\}}, \mathbf{e}_{\{12,13,\ldots,18\}}\right\}. \end{aligned}$$

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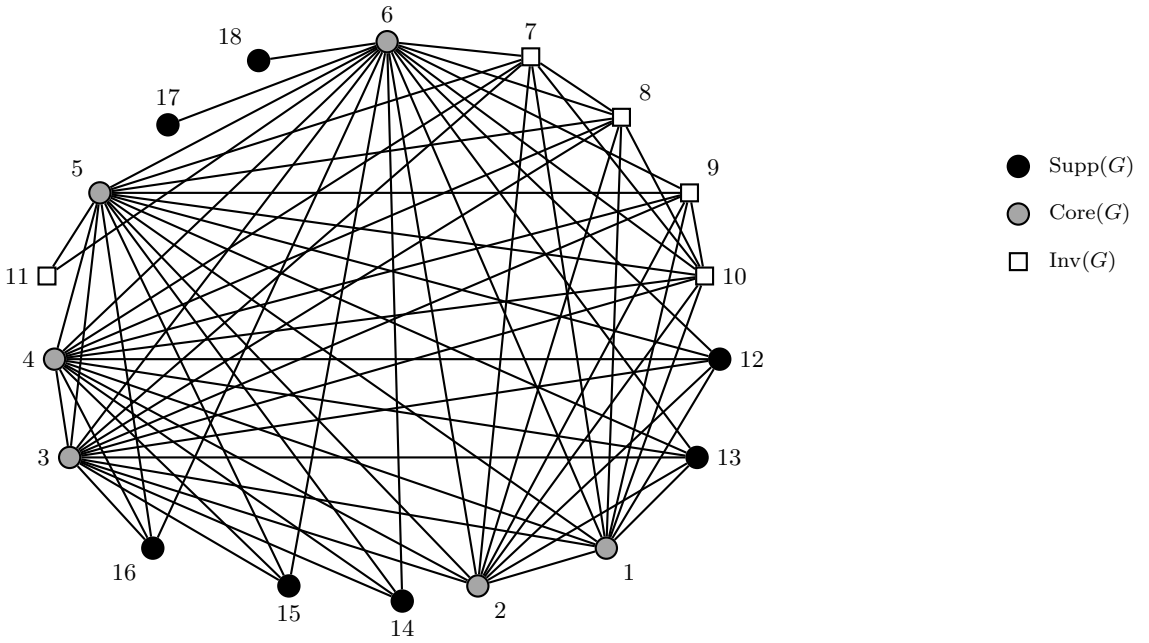


Figure 2: Null-labeled threshold graph G with creation sequence $s = 01010^2 1^2 0^3 1^2 010^2 1$.

6. Core-nilpotent decomposition of threshold graphs. Let G be a null-labeled threshold graph. Note that, by Proposition 4.3, we have that

$$A(G) = \begin{bmatrix} A_{CC}(G) & A_{CN}(G) & A_{CS}(G) \\ A_{NC}(G) & A_{NN}(G) & \\ A_{SC}(G) & & \end{bmatrix},$$

where, throughout this section, A_{HK} denotes the adjacency matrix between the vertex sets H and K, with $H, K \in \{S, C, N\}$ ($S := \mathrm{Supp}(G)$, $C := \mathrm{Core}(G)$, and $N := \mathrm{Inv}(G)$), and all omitted blocks are zero blocks.

We define the following matrices, which will be useful in obtaining the core-nilpotent decomposition of threshold graphs. Given $u \in \mathrm{Supp}(G)$ and $[v] \in TW(G)$, let $A_{ST}(G)$ be a $\mathrm{supp}(G) \times |TW(G)|$ matrix given by

$$A_{ST}(G)_{u,[v]} = \begin{cases} 1, & \text{if } u \sim v, \\ 0, & \text{otherwise,} \end{cases}$$

i.e., if $v \in TW(G)$, then the v-column of $A_{ST}(G)$ is the vector $\mathbf{e}_{\mathcal{F}_G(v)}$. We define the matrix $F(G) := A_{CS}(G)A_{ST}(G)$. Note that for $u, v \in \mathrm{Core}(G)$

$$F(G)_{uv} = (A_{CS}(G)A_{ST}(G))_{uv} = \begin{cases} |\mathcal{F}_G(v)|, & \text{if } u \in [v], \\ |\mathcal{F}_G(u) \cap \mathcal{F}_G(v)|, & \text{otherwise.} \end{cases}$$

Let $\mathcal{B}_{\mathrm{null}}$ be a basis of the null space of a null-labeled threshold graph G. In Section 3, it was proved that the $|V(G)| \times \mathrm{nl}(G)$ matrix whose columns are the vectors of $\mathcal{B}_{\mathrm{null}}$ has the following form:

$$\begin{bmatrix} 0_1 \\ N(G) \end{bmatrix},$$

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where 0_1 is a $(\text{core}(G) + \text{inv}(G)) \times \text{nl}(G)$ zero matrix and $N(G)$ is a $\text{supp}(G) \times \text{nl}(G)$ matrix. By $I_{\text{core}(G)}$ and $I_{\text{inv}(G)}$, we denote identities matrices of sizes $\text{core}(G) \times \text{core}(G)$ and $\text{inv}(G) \times \text{inv}(G)$, respectively.

Let $Q(G)$ be a $|V(G)| \times |V(G)|$ matrix given by

$$Q(G) := \begin{bmatrix} I_{\text{core}(G)} & & & \\ & I_{\text{inv}(G)} & & \\ & & A_{ST}(G) & N(G) \end{bmatrix}.$$

The first $\text{core}(G) + \text{inv}(G) + |TW(G)|$ columns of $Q(G)$ are the columns of $\mathcal{B}_R(G)$, i.e., form a basis for the range of $A(G)$, and the last $\text{nl}(G)$ columns of $Q(G)$ form a basis of the null space of $A(G)$.

Given $[u] \in TW(G)$ and $v \in \text{Core}(G)$, let $I_{TW}(G)$ be a $|TW(G)| \times \text{core}(G)$ matrix given by

$$I_{TW}(G)_{[u],v} := \begin{cases} 1, & \text{if } v \in [u], \\ 0, & \text{otherwise.} \end{cases}$$

LEMMA 6.1. *Let G be a threshold graph. Then,*

$$A_{ST}(G) I_{TW}(G) = A_{SC}(G).$$

Proof. Given $u \in \text{Supp}(G)$ and $v, w \in \text{Core}(G)$. Note that $A_{ST}(G)_{u,[v]} = 1$ if $u \sim v$ and 0 otherwise. Moreover, $I_{TW}(G)_{[v],w} = 1$ if $w \in [v]$ and 0 otherwise. Thus, if $w \in [v]$ and $u \in N(v)$, then $w \sim u$, since $\mathcal{F}_G(v) = \mathcal{F}_G(w)$. Therefore,

$$\begin{aligned} (A_{ST}(G) I_{TW}(G))_{uw} &= \begin{cases} 1, & \text{if } u \sim w, \\ 0, & \text{otherwise,} \end{cases} \\ &= (A_{SC}(G))_{uw}. \end{aligned}$$

∎

Let G be a null-labeled threshold graph. Let $K(G)$ be a $\text{rk}(G) \times \text{rk}(G)$ matrix given by

$$K(G) = \begin{bmatrix} A_{CC}(G) & A_{CN}(G) & F(G) \\ A_{NC}(G) & A_{NN}(G) & \\ I_{TW}(G) & & \end{bmatrix}.$$

THEOREM 6.2. *Let G be a null-labeled threshold graph. Then,*

$$Q(G)^{-1} A(G) Q(G) = \begin{bmatrix} K(G) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Proof. By Lemma 6.1, we have that

$$A(G)Q(G) = \begin{bmatrix} A_{CC}(G) & A_{CN}(G) & F(G) & \mathbf{0} \\ A_{NC}(G) & A_{NN}(G) & \mathbf{0} & \mathbf{0} \\ A_{SC}(G) & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} = Q(G) \begin{bmatrix} K(G) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

∎

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EXAMPLE 6.3. *For G in Fig. 2, we have that*

$$A(G) = \begin{bmatrix} A_{CC}(G) & A_{CN}(G) & A_{CS}(G) \\ A_{NC}(G) & A_{NN}(G) & \mathbf{0} \\ A_{SC}(G) & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$= \left[\begin{array}{cccccc|cccccc|cccccc} 0&1&1&1&1&1&1&1&1&1&0&1&1&0&0&0&0&0 \\ 1&0&1&1&1&1&1&1&1&1&0&1&1&0&0&0&0&0 \\ 1&1&0&1&1&1&1&1&1&1&0&1&1&1&1&1&0&0 \\ 1&1&1&0&1&1&1&1&1&1&0&1&1&1&1&1&0&0 \\ 1&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1&0&0 \\ 1&1&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1&1 \\ \hline 1&1&1&1&1&1&0&1&0&1&0&0&0&0&0&0&0&0 \\ 1&1&1&1&1&1&1&0&0&1&0&0&0&0&0&0&0&0 \\ 1&1&1&1&1&1&0&0&0&1&0&0&0&0&0&0&0&0 \\ 1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ \hline 1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \end{array}\right].$$

$$Q(G) = \begin{bmatrix} I_{\mathrm{core}(G)} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{\mathrm{inv}(G)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A_{ST}(G) & N(G) \end{bmatrix}$$

$$= \left[\begin{array}{cccccc|cccccc|ccc|cccc} 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ \hline 0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0 \\ \hline 0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1&1&1&-1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&-1&-1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&-1 \end{array}\right].$$

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and

$$Q(G)^{-1}A(G)Q(G) = \begin{bmatrix} K(G) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$= \begin{bmatrix} A_{CC}(G) & A_{CN}(G) & F(G) & \mathbf{0} \\ A_{NC}(G) & A_{NN}(G) & \mathbf{0} & \mathbf{0} \\ I_{TW}(G) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$= \left[\begin{array}{cccccc|cccccc|ccc|cccc} 0&1&1&1&1&1&1&1&1&1&0&2&2&2&0&0&0&0 \\ 1&0&1&1&1&1&1&1&1&1&0&2&2&2&0&0&0&0 \\ 1&1&0&1&1&1&1&1&1&1&0&2&5&5&0&0&0&0 \\ 1&1&1&0&1&1&1&1&1&1&0&2&5&5&0&0&0&0 \\ 1&1&1&1&0&1&1&1&1&1&1&2&5&5&0&0&0&0 \\ 1&1&1&1&1&0&1&1&1&1&1&2&5&7&0&0&0&0 \\ \hline 1&1&1&1&1&1&0&1&0&1&0&0&0&0&0&0&0&0 \\ 1&1&1&1&1&1&1&0&0&1&0&0&0&0&0&0&0&0 \\ 1&1&1&1&1&1&0&0&0&1&0&0&0&0&0&0&0&0 \\ 1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ \hline 1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ \hline 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{array}\right].$$

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