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ON MAXIMAL DISTANCES IN A COMMUTING GRAPH[*]

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Abstract. It is shown that matrices over algebraically closed fields that are farthest apart in the commuting graph must be non-derogatory. Rank-one matrices and diagonalizable matrices are also characterized in terms of the commuting graph.

Key words. Commuting graph, Matrix algebra, Algebraically closed field, Centralizer, Distance in graphs.

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1. Introduction and preliminaries. Let $M_n(\mathbb{F})$ be an algebra of $n \times n$ matrices over a field $\mathbb{F}$. Its commuting graph $\Gamma(M_n(\mathbb{F}))$ is a simple graph (i.e., undirected and loopless) with the vertex set consisting of all non-scalar matrices. Two distinct vertices X, Y form an edge $X \!-\! Y$ if the corresponding matrices commute, i.e., if $XY = YX$.

To date, much research has concerned the isomorphisms between commuting graphs (see, e.g., [1, 16]) and the determination of the diameter of commuting graphs of various algebraic structures (see, e.g., [4, 6, 10, 11, 19]). If $n \geq 3$ and $\mathbb{F}$ is algebraically closed field, then the diameter of $\Gamma(M_n(\mathbb{F}))$ is always four, and if $\mathbb{F}$ is not algebraically closed, then either the commuting graph is disconnected or the diameter is between four and six [4]. It is conjectured that the diameter is at most five [4]. Note that for $n = 2$ the commuting graph is always disconnected [5, Remark 8]. In the present paper, we are interested in the commuting graph of matrix algebra $M_n(\mathbb{F})$ over algebraically closed field $\mathbb{F}$ with $n \geq 3$. In particular, we study vertices which are farthest apart in the commuting graph, i.e., at the distance four.

Let us briefly recall some standard definitions and notations. Throughout the paper $\mathbb{F}$ is an algebraically closed field. We make no assumption on the characteristic of $\mathbb{F}$. Further, $M_{m,n}(\mathbb{F})$ is the space of $m \times n$ matrices over $\mathbb{F}$ with a standard basis

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E_{ij}, and $M_n(\mathbb{F}) = M_{n,n}(\mathbb{F})$ is the matrix algebra with identity I_n and the zero matrix 0_n. When it is clear from the context, we omit the subscript. Given an integer $k \geq 2$, we denote by $J_k(\mu) = \mu I_k + \sum_{i=1}^{k-1} E_{i(i+1)} \in M_k(\mathbb{F})$ the upper-triangular elementary Jordan block corresponding to an eigenvalue μ. Let $J_1(\mu) = \mu \in \mathbb{F}$. For convenience, we use J_k to denote $J_k(0)$. Matrix B is a *conjugate* to A if $B = S^{-1}AS$ for some invertible matrix S. The transpose of a matrix A is denoted by A^{T}, and $\mathrm{rk}\, A$ denotes its rank. Given a subset $\Omega \subseteq M_n(\mathbb{F})$, let

$$\mathcal{C}(\Omega) = \{X \in M_n(\mathbb{F}) : AX = XA \text{ for every } A \in \Omega\}.$$

The set $\mathcal{C}(\Omega)$ is called the *centralizer* of Ω. If Ω is a singleton set $\{A\}$, then we set $\mathcal{C}(A) = \mathcal{C}(\{A\})$. Note that by a double centralizer theorem, $\mathcal{C}(\mathcal{C}(A)) = \mathbb{F}[A]$ where $\mathbb{F}[A]$ is the unital subalgebra of $M_n(\mathbb{F})$ generated by A (see [21, Theorem 2, p. 106]).

A centralizer induces two natural relations on $M_n(\mathbb{F})$. One is the equivalence relation, defined by $A \sim B$ if $\mathcal{C}(A) = \mathcal{C}(B)$. We call such two matrices *centralizer-equivalent* (or *$\mathcal{C}$-equivalent*). The other relation is a preorder given by $A \preceq B$ if $\mathcal{C}(A) \subseteq \mathcal{C}(B)$. It was already observed that minimal and maximal matrices in this preorder are of special importance, see for example [8, 9, 20]. Recall that a matrix A is *minimal* if $\mathcal{C}(X) \subseteq \mathcal{C}(A)$ implies $\mathcal{C}(X) = \mathcal{C}(A)$. It was shown in [20, Lemma 3.2] that a matrix A is minimal if and only if it is non-derogatory which means that its Jordan canonical form is equal to $J = J_{n_1}(\lambda_1) \oplus \cdots \oplus J_{n_k}(\lambda_k)$ with $\lambda_i \neq \lambda_j$ for $i \neq j$. In this case,

$$\mathcal{C}(J) = \mathbb{F}[J_{n_1}(\lambda_1)] \oplus \cdots \oplus \mathbb{F}[J_{n_k}(\lambda_k)] = \mathbb{F}[J] \qquad (1.1)$$

(see, for example, [7, Proposition 4.1] or [13, Theorem 3.2.4.2]). So $\mathcal{C}(A) = \mathbb{F}[A]$ if A is non-derogatory. A matrix is *derogatory* if it is not non-derogatory or equivalently, if it is not minimal, which we abbreviate to *non-minimal*. Recall also that a non-scalar matrix A is *maximal* if $\mathcal{C}(A) \subseteq \mathcal{C}(X)$ implies $\mathcal{C}(A) = \mathcal{C}(X)$ or X is a scalar matrix. It is known (see [8, Lemma 4] and also [20, Lemma 3.1]) that a matrix is maximal if and only if it is equal to $\alpha I + \beta P$ or to $\alpha I + N$, where $P^2 = P$ is a non-scalar idempotent, $N \neq 0$ is a square-zero matrix (i.e., $N^2 = 0$), and a scalar β is nonzero. It should be noted that in [8, 20] the description of minimal and maximal matrices was given only for the field of complex numbers, but the arguments can be applied almost unchanged for arbitrary algebraically closed fields.

A *path of length k* in a commuting graph, denoted by $X_0 - X_1 - \cdots - X_k$, is a sequence of $k+1$ distinct vertices $X_0, \ldots, X_k$ such that $X_{i-1} - X_i$ is an edge for $i = 1, \ldots, k$. The *distance* $d(A, B)$ between a pair of distinct vertices A and B is the length of a shortest path between them, and $d(A, B) = 0$ if $A = B$. If there exists no path between them, then we define $d(A, B) = \infty$. The *diameter* of commuting graph is the maximal distance between any pair of vertices in it.

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The main purpose of this paper is to characterize matrices for which the diameter of commuting graph $\Gamma(M_n(\mathbb{F}))$ is attained, that is, to prove the following theorem.

THEOREM 1.1. *Let $n \geq 3$ and let $\mathbb{F}$ be algebraically closed field. Then the following statements are equivalent for a non-scalar matrix $A \in M_n(\mathbb{F})$.*

   (i) A is non-derogatory.
   (ii) A is minimal with respect to the preorder $\preceq$.
   (iii) There exists a matrix $X \in M_n(\mathbb{F})$ such that $d(A, X) = 4$.

Hence, a matrix A is non-derogatory if and only if there exists a matrix X, such that for every $B \in \mathcal{C}(A)$ and every $Y \in \mathcal{C}(X)$ we have $\mathcal{C}(B) \cap \mathcal{C}(Y) = \mathbb{F}I$.

REMARK 1.2. There exist infinitely many non-derogatory matrices which are pairwise at the maximal distance but each of them is at distance two from a fixed rank-one matrix; see Lemma 2.7.

To prove Theorem 1.1, we need the characterization of matrices that are $\mathcal{C}$-equivalent to matrices of rank one. Since these matrices are important, e.g., in theory of preservers [17], we state this characterization as a theorem.

THEOREM 1.3. *Let $n \geq 3$ and let $\mathbb{F}$ be algebraically closed field. Then the following statements are equivalent for a non-scalar matrix $A \in M_n(\mathbb{F})$.*

   (i) $A = \lambda I + R$ for some rank-one matrix $R \in M_n(\mathbb{F})$ and some scalar $\lambda \in \mathbb{F}$.
   (ii) A is $\mathcal{C}$-equivalent to a rank-one matrix.
   (iii) $d(A, X) \leq 2$ for every non-minimal matrix $X \in M_n(\mathbb{F})$.

So, $\mathcal{C}(A) = \mathcal{C}(R)$ for some rank-one matrix R if and only if $\mathcal{C}(A) \cap \mathcal{C}(X)$ contains a non-scalar matrix for each derogatory matrix X.

We will conclude the paper with the characterization of diagonalizable matrices.

THEOREM 1.4. *Let $n \geq 3$ and let $\mathbb{F}$ be algebraically closed field. Then the following statements are equivalent for a non-scalar matrix $A \in M_n(\mathbb{F})$.*

   (i) A is diagonalizable.
   (ii) There exists a minimal $B \in \mathcal{C}(A)$ such that for each path $B \!-\! X \!-\! Y$ in $\Gamma(M_n(\mathbb{F}))$ there exists a minimal matrix M such that $X \!-\! M \!-\! Y$ is a path in $\Gamma(M_n(\mathbb{F}))$.

Recall, if X commutes with a minimal matrix M, then $X \in \mathbb{F}[M]$ by (1.1). Thus, Theorem 1.4 says that a matrix A is diagonalizable if and only if for every non-derogatory $B \in \mathcal{C}(A)$ and arbitrary $X \in \mathcal{C}(B)$ and $Y \in \mathcal{C}(X)$, there exists a non-derogatory matrix M with $X, Y \in \mathbb{F}[M]$.

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2. Proofs. Throughout $\mathbb{F}$ is an algebraically closed field and $n \geq 3$. We start with four lemmas that we need to characterize rank-one matrices.

LEMMA 2.1. *For every non-scalar matrix $A \in M_n(\mathbb{F})$, there exists a rank-one matrix $R \in M_n(\mathbb{F})$ with $d(A, R) \leq 1$.*

Proof. It suffices to show that A commutes with at least one matrix of rank one. Let x and y be eigenvectors of A and of A^{T}, respectively, corresponding to the same eigenvalue λ. Then $R = xy^{\mathrm{T}}$ is a rank-one matrix with $AR = (Ax)y^{\mathrm{T}} = \lambda xy^{\mathrm{T}} = x(A^{\mathrm{T}} y)^{\mathrm{T}} = RA$. $\square$

Using Lemma 2.1, we can give an alternative proof of the result [4, Corollary 7] on the diameter of a commuting graph.

COROLLARY 2.2. *The distance between arbitrary two matrices in the commuting graph is at most four.*

Proof. Let A and B be arbitrary matrices. By Lemma 2.1, there exist rank-one matrices $R_1 = xf^{\mathrm{T}} \in \mathcal{C}(A)$ and $R_3 = yg^{\mathrm{T}} \in \mathcal{C}(B)$. Since $n \geq 3$ we can find a nonzero $z \in \mathbb{F}^n$ with $f^{\mathrm{T}} z = g^{\mathrm{T}} z = 0$ and a nonzero $h \in \mathbb{F}^n$ with $h^{\mathrm{T}} x = h^{\mathrm{T}} y = 0$. Then, for a rank-one matrix $R_2 = zh^{\mathrm{T}}$, we obtain $A - R_1 - R_2 - R_3 - B$. $\square$

LEMMA 2.3. *Let $A = J_{k_1} \oplus J_{k_2} \in M_{k_1+k_2}(\mathbb{F})$ be a nilpotent matrix with two Jordan blocks of sizes k_1, k_2. Then, for each rank-one $R \in M_{k_1+k_2}(\mathbb{F})$, there exists a rank-one matrix $Z \in \mathcal{C}(A) \cap \mathcal{C}(R)$.*

Proof. If $k_1 = k_2 = 1$, then A is a zero matrix and the conclusion is obvious. Otherwise, $k_1 \geq 2$ or $k_2 \geq 2$. Let $k = k_1 + k_2$ and let $e_1, \ldots, e_n$ be the standard basis of column vectors in $\mathbb{F}^n$. For every choice of $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{F}$, the matrix $Z = \lambda_1 \mu_1 E_{1k_1} + \lambda_1 \mu_2 E_{1k} + \lambda_2 \mu_1 E_{(k_1+1)k_1} + \lambda_2 \mu_2 E_{(k_1+1)k} = (\lambda_1 e_1 + \lambda_2 e_{k_1+1})(\mu_1 e_{k_1} + \mu_2 e_k)^t$ is of rank at most one and commutes with A. Let $R \in M_{k_1+k_2}(\mathbb{F})$ be a rank-one matrix. Then $R = ab^{\mathrm{T}}$ for some column vectors $a, b \in \mathbb{F}^k$ and we may choose $(\lambda_1, \lambda_2) \neq (0,0)$ and $(\mu_1, \mu_2) \neq (0,0)$ so that $Za = Z^t b = 0$. With this choice, $ZR = RZ = 0$. $\square$

LEMMA 2.4. *Suppose that a non-scalar matrix $A \in M_n(\mathbb{F})$ is not minimal. Then $d(A, R) \leq 2$ for each rank-one matrix $R \in M_n(\mathbb{F})$.*

Proof. Since A is non-minimal, at least two of its Jordan blocks, $J_{k_1}(\lambda_1)$ and $J_{k_2}(\lambda_2)$, satisfy $\lambda_1 = \lambda_2 = \lambda$. Let $k = k_1 + k_2$. As $\mathcal{C}(A) = \mathcal{C}(A - \lambda I)$ and by using appropriate conjugation, we may assume that $A = J_{k_1} \oplus J_{k_2} \oplus \tilde{A}$ for some matrix $\tilde{A} \in M_{n-k}(\mathbb{F})$. A rank-one matrix R can be written as $R = xy^{\mathrm{T}}$ with $x = x_1 \oplus x_2 \in \mathbb{F}^k \oplus \mathbb{F}^{n-k}$ and $y = y_1 \oplus y_2 \in \mathbb{F}^k \oplus \mathbb{F}^{n-k}$. We define two vectors $\widehat{x}_1, \widehat{y}_1 \in \mathbb{F}^k$ to be $\widehat{x}_1 = x_1$ if $x_1 \neq 0$, and $\widehat{x}_1 = e_1$ if $x_1 = 0$. Similarly, it is defined that $\widehat{y}_1 = y_1$ if $y_1 \neq 0$, and $\widehat{y}_1 = e_1$ if $y_1 = 0$. By Lemma 2.3, there exists a rank-one $\widehat{Z} \in M_k(\mathbb{F})$ which

commutes with $\widehat{x}_1 \widehat{y}_1^{\mathrm{T}}$ and with $J_{k_1} \oplus J_{k_2}$. Since $\widehat{Z}\widehat{x}_1\widehat{y}_1^{\mathrm{T}} = \widehat{x}_1\widehat{y}_1^{\mathrm{T}}\widehat{Z}^{\mathrm{T}} = \widehat{x}_1(\widehat{Z}\widehat{y}_1)^{\mathrm{T}}$, $\widehat{x}_1 \neq 0$ and $\widehat{y}_1 \neq 0$, it follows that there is a scalar λ such that $\widehat{Z}x_1 = \lambda x_1$ and $\widehat{Z}^{\mathrm{T}} y_1 = \lambda y_1$.

Let $Z = \widehat{Z} \oplus \lambda I_{n-k}$. Then $Z \in \mathcal{C}(J_{k_1} \oplus J_{k_2}) \oplus (\mathbb{F}I_{n-k}) \subseteq \mathcal{C}(A)$ is a non-scalar matrix. Clearly, $Zx = \widehat{Z}x_1 \oplus \lambda x_2 = \lambda x$ and $Z^{\mathrm{T}} y = \lambda y$. Hence, Z commutes with $R = xy^{\mathrm{T}}$ and A. $\square$

LEMMA 2.5. *Let $n \geq 4$. Suppose that a non-scalar $A \in M_n(\mathbb{F})$ is either*

 (i) *a maximal matrix with $2 \leq \mathrm{rk}\, A \leq n-2$, or*
 (ii) *a nilpotent matrix with $A^3 = 0$ and $\mathrm{rk}(A^2) = 1$.*

Then there exists a non-scalar and non-minimal matrix X with $d(A, X) \geq 3$.

Proof. Suppose (i) holds. It was already mentioned in the first section that a maximal matrix A is either $A = \alpha I + N$ for some square-zero matrix N or $A = \alpha I + \beta P$ for some idempotent P. Since A is singular, it follows that $A = N$, $A = \beta P$, or $A = \beta(P - I)$.

Let $k = \mathrm{rk}\, A$. If A is a square-zero matrix, then $2 \leq k \leq \frac{n}{2}$. For every ℓ, we define $s_\ell = (1, 1, \ldots, 1)^{\mathrm{T}} \in \mathbb{F}^\ell$ and $z_{2\ell} = (0, 1, 0, 1, \ldots, 0, 1)^{\mathrm{T}} \in \mathbb{F}^{2\ell}$. Also, let $N_{2\ell} = \bigoplus_{i=1}^{\ell} J_2^{\mathrm{T}} \in M_{2\ell}(\mathbb{F})$. Note that $N_{2\ell}^2 = 0$ and $\mathrm{rk}\, N_{2\ell} = \ell$. It is easy to see that a matrix

$$\begin{bmatrix} N_{2k-2} & 0 & z_{2k-2} \\ 0 & 0_{n-2k+1} & s_{n-2k+1} \\ 0 & 0 & 0_1 \end{bmatrix} \tag{2.1}$$

is a square-zero of rank k, hence conjugate to A. So, we can assume without loss of generality that A is already in the form (2.1). Define also a non-minimal matrix $X = J_2 \oplus 0_1 \oplus D$, where D is a diagonal matrix with $n-3$ distinct nonzero diagonal entries.

Let $B \in \mathcal{C}(X) \cap \mathcal{C}(A)$ be arbitrary. We prove that B is a scalar matrix. Since $B \in \mathcal{C}(X)$, it easily follows that $B = B_3 \oplus D_{n-3}$ for some $B_3 = \begin{bmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & 0 \\ 0 & \delta & \epsilon \end{bmatrix}$ and some $D_{n-3} = \mathrm{diag}(\lambda_3, \ldots, \lambda_n)$. By denoting $\lambda_n = \lambda$, we have $Be_n = \lambda e_n$. Also $B \in \mathcal{C}(A)$, i.e., $BA = AB$. Therefore, $BAe_n = ABe_n = \lambda Ae_n$, and $x = Ae_n = (e_2 + e_4 + \cdots + e_{2k-2}) + (e_{2k-1} + \cdots + e_{n-1})$ is an eigenvector of B corresponding to eigenvalue λ. Since $B = B_3 \oplus D_{n-3}$, we obtain $Be_2 = \lambda e_2$ and $D_{n-3} = \mathrm{diag}(\lambda, \mu_5, \lambda, \ldots, \mu_{2k-3}, \lambda, \ldots, \lambda)$. From $Be_2 = \lambda e_2$, it follows $B_3 = \begin{bmatrix} \lambda & 0 & \gamma \\ 0 & \lambda & 0 \\ 0 & 0 & \epsilon \end{bmatrix}$. Also, $\lambda e_{2k-2} = Be_{2k-2} = BAe_{2k-3} = ABe_{2k-3} = \mu_{2k-3}Ae_{2k-3} = \mu_{2k-3}e_{2k-2}$, so $\mu_{2k-3} = \lambda$. Proceeding backwards gives $\mu_{2k-3} = \cdots = \mu_5 = \lambda$, so $B = B_3 \oplus \lambda I_{n-3}$. If $k = 2$, then $x = e_2 + e_3 + \cdots + e_{n-1}$ is an eigenvector of B so $B_3 = \lambda I_3$. If $k \geq 3$, then $\lambda e_4 = Be_4 = BAe_3 = ABe_3 = A(\epsilon e_3 + \gamma e_1) = \epsilon e_4 + \gamma e_2$, so $\gamma = 0$ and $\epsilon = \lambda$, and so $B = \lambda I$. This completes the

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proof that $d(A, X) \geq 3$ if $A^2 = 0$.

If A is a scalar multiple of an idempotent, then we may assume that A is an idempotent of rank k with $\frac{n}{2} \leq k \leq n - 2$, because $\mathcal{C}(\mu A) = \mathcal{C}(A) = \mathcal{C}(I - A)$ for $\mu \neq 0$ and if A is an idempotent then $\mathrm{rk}(I - A) = n - \mathrm{rk}\, A$. Using an appropriate conjugation, we can additionally assume that $A = \begin{bmatrix} I_k & W \\ 0 & 0_{n-k} \end{bmatrix}$, where $k \times (n - k)$ matrix W has ones at positions $(k, 1), (k, n-k)$, and $(i, n-k-i+1)$, $i = 1, \ldots, n-k$, and zeros elsewhere. We also take a non-minimal matrix $X = J_k \oplus 0_1 \oplus I_{n-k-1}$.

Let $B \in \mathcal{C}(X) \cap \mathcal{C}(A)$ be arbitrary. Since $B \in \mathcal{C}(A)$, it is easy to see that $B = \begin{bmatrix} M & MW - WN \\ 0 & N \end{bmatrix}$ for some $M \in M_k(\mathbb{F}), N \in M_{n-k}(\mathbb{F})$. Since $B \in \mathcal{C}(X)$, we deduce that $M = \sum_{i=1}^{k} m_i J_k^{i-1}$ is upper-triangular Toeplitz, $N = \lambda \oplus Y$ with $\lambda \in \mathbb{F}$, $Y = \begin{bmatrix} y_{ij} \end{bmatrix}_{2 \leq i, j \leq n-k} \in M_{n-k-1}(\mathbb{F})$, and $(MW - WN)_{ij} = 0$, except possibly for $i = j = 1$. By equations

$$0 = (MW - WN)_{k,1} = m_1 - \lambda,$$
$$0 = (MW - WN)_{k,n-k} = m_1 - y_{n-k,n-k},$$
$$0 = (MW - WN)_{i,n-k} = m_{k-i+1} \quad \text{for all } i \text{ with } (n-k) \leq i \leq (k-1),$$

it follows that $m_1 = y_{n-k,n-k} = \lambda$ and, if $\frac{n}{2} < k$, $m_2 = m_3 = \cdots = m_{2k-n+1} = 0$. Moreover, if $\frac{n}{2} < k$, then $0 = (MW - WN)_{i,1} = m_{n-k-i+1} + m_{k-i+1}$ for $i = (n-k-1), \ldots, 2$, and since $m_2 = m_3 = \cdots = m_{2k-n+1} = 0$, we recursively obtain $m_{2k-n+2} = \cdots = m_{k-1} = 0$. If $\frac{n}{2} = k$, then $0 = (MW - WN)_{i,1} = m_{k-i+1}$ for $i = 2, 3, \ldots, n-k-1 = k-1$ and again $m_2 = \cdots = m_{k-1} = 0$. Now, equation $0 = (MW - WN)_{1,n-k} = m_1 + m_k - y_{n-k,n-k} = m_k$ completes the proof that $M = \lambda I_k$.

We proceed by $0 = (MW - WN)_{i,n-k-i+1} = m_1 - y_{n-k-i+1,n-k-i+1}$ for $i = 2, \ldots, n-k-1$ and $0 = (MW - WN)_{i,j} = -y_{n-k-i+1,j}$ for $i = 1, 2, \ldots, n-k-1$ and $j = 2, 3, \ldots, n-k$, such that $i + j \neq n-k+1$. It follows that $N = \lambda I_{n-k}$ and $(MW - WN) = 0$. Thus, $B = \lambda I$ and $d(A, X) \geq 3$.

(ii) Let A be a nilpotent matrix such that $A^3 = 0$ and $\mathrm{rk}(A^2) = 1$. We may assume A is already in its Jordan canonical form, i.e.,

$$A = J_3 \oplus \bigoplus_{i=1}^{k} J_2 \oplus 0_{n-3-2k}.$$

The centralizer of A is contained in the set of matrices of the form $B = \begin{bmatrix} T & S_1 \\ S_2 & V \end{bmatrix}$, where $T = t_1 I_3 + t_2 J_3 + t_3 J_3^2 \in M_3(\mathbb{F})$, $V \in M_{n-3}(\mathbb{F})$, and where the first column of

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$S_2 \in M_{n-3,3}(\mathbb{F})$ as well as the last row of $S_1 \in M_{3,n-3}(\mathbb{F})$ contain only zero entries. We define a non-minimal matrix $X = 1 \oplus 0 \oplus J_{n-2}$.

Let $B \in \mathcal{C}(A) \cap \mathcal{C}(X)$ be arbitrary. Since $B \in \mathcal{C}(X)$, its off-diagonal entries in the first row and the first column are all zero. Comparing with the above form for B, we deduce that $T = t_1 I_3$. Moreover, $B \in \mathcal{C}(X)$ also implies that the bottom-right $(n-2) \times (n-2)$ block of B is upper triangular Toeplitz matrix, which is equal to $t'_1 I_{n-2}$ for some $t'_1 \in \mathbb{F}$, by the fact that the third row of S_1 vanishes. Actually, $t_1 = t'_1$ because a 3×3 block T overlaps with $(n-2) \times (n-2)$ bottom right block. Further, $B \in \mathcal{C}(X)$ implies that the only possible off-diagonal nonzero entries in the second row and column are at positions $(2, n)$, and $(3, 2)$. Actually, $B_{32} = T_{32} = 0$, while from $B \in \mathcal{C}(A)$ we deduce that if $B_{2n} \neq 0$, then also $B_{1(n-1)} \neq 0$, which would contradict the fact that the first row of B has zero off-diagonal entries. Hence, $B = t_1 I$ is a scalar matrix, and therefore, $d(A, X) \geq 3$. $\square$

Proof of Theorem 1.3. If (i) holds, then $\mathcal{C}(A) = \mathcal{C}(\lambda I + R) = \mathcal{C}(R)$. Hence, A is $\mathcal{C}$-equivalent to a rank-one matrix R. Inversely, if A is $\mathcal{C}$-equivalent to some rank-one matrix R', i.e., if $\mathcal{C}(A) = \mathcal{C}(R')$, then, by a double centralizer theorem, $A \in \mathcal{C}(\mathcal{C}(A)) = \mathcal{C}(\mathcal{C}(R')) = \mathbb{F}[R'] = \{\lambda I + \mu R' : \lambda, \mu \in \mathbb{F}\}$. Thus, (i) follows with $R = \mu R'$, and hence (i) is equivalent to (ii). Let us prove that (ii) is also equivalent to (iii). Recall that non-minimal matrices are exactly derogatory ones.

If $n = 3$, then the Jordan canonical form of every non-minimal matrix X has at most three Jordan blocks, and two of them must have the same eigenvalue, say λ. Then X is $\mathcal{C}$-equivalent to a rank-one matrix $X - \lambda I$. Hence, by Lemma 2.4, (ii) and (iii) are equivalent for $n = 3$.

It remains to prove the equivalence for $n \geq 4$.

(ii)$\Longrightarrow$(iii). We can assume without loss of generality that $A = R$ is rank-one. Let X be an arbitrary non-minimal matrix. Then, by Lemma 2.4, $d(A, X) \leq 2$.

$\neg$(ii) $\Longrightarrow \neg$(iii). Suppose that A is not $\mathcal{C}$-equivalent to a rank-one matrix. Note that there exists at least one non-invertible maximal matrix $M \succeq A$. In fact, if $A = A_1 \oplus \cdots \oplus A_r$ is a primary decomposition of A (see, e.g., [12]), then, by [7, Proposition 4.1], $\mathbb{F}[A] = \mathbb{F}[A_1] \oplus \cdots \oplus \mathbb{F}[A_r]$. Hence, using the Jordan structure of A_i, we can find a polynomial p such that $M = p(A)$ is a non-scalar idempotent matrix or a non-scalar square-zero matrix. Hence, $1 \leq \text{rk } M \leq n - 1$. Note that $\text{rk } M = n - 1$ implies M is an idempotent, and therefore, it is $\mathcal{C}$-equivalent to a maximal matrix of rank one. Thus, we can assume that $1 \leq \text{rk } M \leq n - 2$.

If there exists a maximal $M \succeq A$ with $2 \leq \text{rk } M \leq n - 2$, then, by Lemma 2.5, there exists a non-scalar and non-minimal matrix X with $d(M, X) \geq 3$. Hence, also $d(A, X) \geq 3$ because $\mathcal{C}(A) \subseteq \mathcal{C}(M)$.

Otherwise, every maximal matrix $M \succeq A$ is $\mathcal{C}$-equivalent to a rank-one idempotent matrix or to a rank-one nilpotent matrix. Note that $M \succeq A$ if $\mathcal{C}(A) \subseteq \mathcal{C}(M)$, which implies $M \in \mathcal{C}(\mathcal{C}(M)) \subseteq \mathcal{C}(\mathcal{C}(A)) = \mathbb{F}[A]$. Therefore, for $n \geq 4$, the primary decomposition of A contains at most two blocks, and in the Jordan structure of each block there are only blocks of size at most three. Moreover, A can have only one Jordan block of maximal size. This implies that (1) A is $\mathcal{C}$-equivalent to a nilpotent matrix whose Jordan structure is equal to $J_3 \oplus \bigoplus_{i=1}^{k} J_2 \oplus 0_{n-3-2k}$, (2) A is $\mathcal{C}$-equivalent to a matrix whose Jordan structure is equal to $1 \oplus J_2 \oplus 0_{n-3}$, or (3) A is $\mathcal{C}$-equivalent to a matrix whose Jordan structure is equal to $1 \oplus J_3 \oplus \bigoplus_{i=1}^{k} J_2 \oplus 0_{n-4-2k}$.

In the case (1), Lemma 2.5 assures that there exists a non-minimal X with $d(A, X) \geq 3$. In the case (2) we have, modulo conjugation, $A = 1 \oplus J_2 \oplus 0_{n-3}$. It is easy to see that $X = J_2 \oplus J_{n-2}$ is non-minimal and $d(A, X) \geq 3$. In case (3) we have, modulo conjugation, $A \preceq A' = 0 \oplus J_3 \oplus \bigoplus_{i=1}^{k} J_2 \oplus 0_{n-4-2k}$. By Lemma 2.5, there exists a non-minimal matrix X with $d(A', X) \geq 3$ and hence, $d(A, X) \geq 3$. $\square$

It was proven in [4, Lemma 2] that for matrices of order $n \geq 3$ the diameter of the commuting graph is at most four (see also Corollary 2.2 above) and that $d(J, J^{\mathrm{T}}) = 4$. These results imply that the diameter of the commuting graph of matrix algebra over algebraically closed field is equal to four. It is well-known that the transpose of a matrix is conjugate to the original (see, for example, [13, p. 134]). Thus, [4, Lemma 2] implies that the maximal distance from J to some of its conjugates is equal to four. Our next lemma will strengthen this result by considering maximal distances between an arbitrary minimal matrix $A \in M_n(\mathbb{F})$ and matrices from conjugation orbit $\{S^{-1}BS : S \text{ invertible}\}$ of another minimal matrix $B \in M_n(\mathbb{F})$. Recall that a matrix is minimal if it is conjugate to $\bigoplus_{i=1}^{k} J_{n_i}(\lambda_i)$, where $\lambda_i \neq \lambda_j$ for $i \neq j$, and $(n_1, n_2, \ldots, n_k)$ is a partition of n. In the following lemma we show that for two arbitrary partitions of n, we can find two minimal matrices at distance four, having their Jordan forms corresponding to the two partitions. One of the matrices is already in its Jordan canonical form, while the other is a matrix which is conjugate to its Jordan canonical form by an invertible matrix with all of its minors nonzero. Such invertible matrix is, for example, a Cauchy matrix $\left[\frac{1}{x_i - y_j}\right]_{ij}$ (see [18]).

LEMMA 2.6. *Let S be a matrix with all of its minors nonzero. For two arbitrary minimal matrices $A = \bigoplus_{i=1}^{k} J_{n_i}(\lambda_i) \in M_n(\mathbb{F})$ and $B = \bigoplus_{i=1}^{l} J_{m_i}(\mu_i) \in M_n(\mathbb{F})$, we have $d(A, S^{-1}BS) = 4$.*

Proof. Assume to the contrary that $d(A, S^{-1}BS) \leq 3$. Since $\mathcal{C}(A) = \mathcal{C}(\alpha A)$ for all nonzero $\alpha \in \mathbb{F}$, we can make every path longer by adding vertices which correspond to scalar multiples of matrices. Hence, there exists a path $A \text{—} X \text{—} Y \text{—} S^{-1}BS$ of length 3 in $\Gamma(M_n(\mathbb{F}))$. We can assume without loss of generality that X and Y are maximal

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matrices. If X is not maximal, then there exists a maximal $X' \succeq X$. Thus, we could consider a path $A - X' - Y - S^{-1}BS$ of length 3, since $A, Y \in \mathcal{C}(X) \subseteq \mathcal{C}(X')$. A similar argument applies to Y.

We now show that no two maximal matrices $X \in \mathcal{C}(A)$ and $Y \in \mathcal{C}(S^{-1}BS)$ commute and thus obtain a contradiction to the assumption $d(A, S^{-1}BS) \leq 3$. Since all maximal matrices are $\mathcal{C}$-equivalent either to a non-scalar square-zero matrix or to a non-scalar idempotent matrix, we will consider the following three cases.

First, let us assume that both X and Y are nonzero square-zero matrices. Note that $A = \bigoplus_{i=1}^{k} J_{n_i}(\lambda_i)$ and $B = \bigoplus_{i=1}^{l} J_{m_i}(\mu_i)$ are minimal and hence, $\lambda_i \neq \lambda_j$ and $\mu_i \neq \mu_j$ for $i \neq j$. Thus, we have that

$$X = T_1 \oplus T_2 \oplus \cdots \oplus T_k \quad \text{and} \quad Y = S^{-1}(T_1' \oplus T_2' \oplus \cdots \oplus T_l')S,$$

where all T_i and T_j' are upper triangular Toeplitz matrices. Since X is a nonzero square-zero matrix, its image $\operatorname{Im} X$ is a linear combination of a subset of vectors of standard basis, i.e., $\operatorname{Im} X = \operatorname{Lin}\{e_{\sigma(1)}, e_{\sigma(2)}, \ldots, e_{\sigma(r)}\}$ for some permutation σ of length n and some integer r, $1 \leq r \leq \frac{n}{2}$. Moreover, since the nonzero X is Toeplitz block-diagonal matrix, there exists a column with exactly one nonzero entry, i.e., there exist indices t and s such that $Xe_t = \alpha e_s \neq 0$. For simplicity, we denote $T = T_1' \oplus T_2' \oplus \cdots \oplus T_l'$. Now, assume $YX = XY$. Then $S^{-1}TSXe_t = YXe_t = XYe_t \in \operatorname{Im} X$, and therefore,

$$\alpha T S e_s \in S(\operatorname{Im} X) = \operatorname{Lin}\{Se_{\sigma(1)}, Se_{\sigma(2)}, \ldots, Se_{\sigma(r)}\}$$

which is clearly possible if and only if the rank of the $n \times r$ matrix

$$M = \left[\frac{1}{\alpha} Se_{\sigma(1)}, \frac{1}{\alpha} Se_{\sigma(2)}, \ldots, \frac{1}{\alpha} Se_{\sigma(r)} \right] \tag{2.2}$$

is the same as the rank of the augmented matrix $[M \,|\, TSe_s]$. However, we will show that this is not the case. Since all minors of S are nonzero, the s-th column of S, Se_s, has no zero entries, and it cannot be annihilated by a nonzero Toeplitz block-diagonal matrix T, i.e., $TSe_s \neq 0$. However $T^2 = 0$, so T has at least $\frac{n}{2}$ zero rows, hence the vector TSe_s has at least $\frac{n}{2}$ zero entries. Recall that $r \leq \frac{n}{2}$ and consequently there exists an $(r+1) \times (r+1)$ submatrix of the augmented matrix having exactly r zeros and one nonzero element in its last column. Using the Laplace expansion along the last column of this $(r+1) \times (r+1)$ submatrix, we observe that its determinant is equal to a multiple of an $r \times r$ minor of the matrix M which, by (2.2), is equal to $(\frac{1}{\alpha})^r$ times an $r \times r$ minor of S. By the assumption every minor of S is nonzero and so $r + 1 = \operatorname{rk}[M \,|\, TSe_s] > \operatorname{rk} M = r$. This implies $TSe_s \notin S(\operatorname{Im} X)$, which is a contradiction.

Second, suppose that a non-scalar idempotent matrix $X \in \mathcal{C}(A)$ commutes with a non-scalar square-zero matrix $Y \in \mathcal{C}(S^{-1}BS)$. We can assume that $r = \operatorname{rk} X \leq$

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$\frac{n}{2}$. Otherwise, we can take $I - X$ instead of X. Hence, $X = \sum_{i=1}^{r} E_{\sigma(i)\sigma(i)}$ and $Y = S^{-1}TS$, where σ is an appropriate permutation of length n and $T = T'_1 \oplus T'_2 \oplus \cdots \oplus T'_l$ is as above. Let $t = \sigma(1)$. As before, if $YX = XY$, then $S^{-1}TSXe_t \in \operatorname{Im} X = \operatorname{Lin}\{e_{\sigma(1)}, e_{\sigma(2)}, \ldots, e_{\sigma(r)}\}$, that is, $TSe_t \in \operatorname{Lin}\{Se_{\sigma(1)}, Se_{\sigma(2)}, \ldots, Se_{\sigma(r)}\}$. We proceed as in the first case to obtain a contradiction.

By the symmetry the only case remaining is the case when X and Y are both non-scalar idempotents. Then $X = \sum_{i=1}^{r} E_{\sigma(i)\sigma(i)}$ and $Y = S^{-1}PS$ for idempotent $P = \sum_{i=1}^{s} E_{\tau(i)\tau(i)}$ and appropriate permutations σ and τ of length n. We can assume that $r \leq \frac{n}{2}$, since otherwise we substitute X by $I - X$, and $s \leq \frac{n}{2}$, since otherwise we substitute Y by $I - Y$. Again, let $t = \sigma(1)$. If $YX = XY$, then $S^{-1}PSXe_t \in \operatorname{Im} X = \operatorname{Lin}\{e_{\sigma(1)}, e_{\sigma(2)}, \ldots, e_{\sigma(r)}\}$ as before, or equivalently $PSe_t \in \operatorname{Lin}\{Se_{\sigma(1)}, Se_{\sigma(2)}, \ldots, Se_{\sigma(r)}\}$. Since PSe_t has $n - s \geq \frac{n}{2}$ zero entries, we obtain a contradiction as in the first case. This shows that $d(A, S^{-1}BS) \geq 4$. But the diameter of commuting graph is equal to four and hence, $d(A, S^{-1}BS) = 4$. □

Now we prove that minimal matrices are the ones which maximize the distance in a commuting graph.

Proof of Theorem 1.1. We already know that (i) and (ii) are equivalent by [20]. Also, $(ii) \implies (iii)$ follows by Lemma 2.6. To prove $(iii) \implies (ii)$ consider a non-minimal matrix A and let X be an arbitrary non-scalar matrix. By Lemma 2.1 there exists a rank-one matrix R with $d(X, R) \leq 1$. By Theorem 1.3 we have $d(A, R) \leq 2$, so by a triangle inequality it follows $d(A, X) \leq 3$. □

We continue by proving the assertions in Remark 1.2.

LEMMA 2.7. *There exist an infinite family of matrices $(X_\alpha)_\alpha \in M_n(\mathbb{F})$ and a rank-one matrix Z such that $d(X_\alpha, X_\beta) = 4$ for $\alpha \neq \beta$ and $d(X_\alpha, Z) = 2$ for all α.*

Proof. Suppose first that $n \geq 4$ and let $A = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ where λ_i are pairwise distinct. Since $|\mathbb{F}| = \infty$, we may choose a nonzero scalar γ with $\gamma^2 \neq 2 - n$ to form an infinite family of rank-one nilpotent matrices $R_\alpha = x(f^{\mathrm{T}} + \alpha g^{\mathrm{T}})$, $\alpha \in \mathbb{F}$, where $x = (1, \ldots, 1, \gamma)^{\mathrm{T}}$, $f = (2 - n, 1, \ldots, 1, 0)^{\mathrm{T}}$ and $g = (2 - n - \gamma^2, 1, \ldots, 1, \gamma)^T$. Note that $R_\alpha R_\beta = 0$ for all $\alpha, \beta \in \mathbb{F}$. Thus, matrices $S_\alpha = I + R_\alpha$ are invertible with $S_\alpha^{-1} = I - R_\alpha$. For every $\alpha \in \mathbb{F}$, let $X_\alpha = S_\alpha A S_\alpha^{-1}$. We claim that $d(X_\alpha, X_\beta) = 4$ for $\alpha \neq \beta$.

Fix $\alpha, \beta \in \mathbb{F}$ and denote $S = S_\alpha^{-1}S_\beta = (I - R_\alpha)(I + R_\beta) = I + \tilde{x}g^{\mathrm{T}}$, where $\tilde{x} = (\beta - \alpha)x$. Since the distance is invariant for simultaneous conjugation, we can replace (X_α, X_β) with (A, SAS^{-1}). Now, to prove $d(A, SAS^{-1}) = 4$, it suffices to show that, non-scalar matrices $D_1 \in \mathcal{C}(A)$ and $SDS^{-1} \in S\mathcal{C}(A)S^{-1}$ do not commute.

Assume to the contrary that D_1 and SDS^{-1} commute. This implies that $D_1(I +$

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$\tilde{x}g^{\mathrm{T}})D(I - \tilde{x}g^{\mathrm{T}}) = (I + \tilde{x}g^{\mathrm{T}})D(I - \tilde{x}g^{\mathrm{T}})D_1$. Since A is minimal (hence, non-derogatory), $\mathcal{C}(A)$ consists of diagonal matrices only, which implies that D_1 and D are diagonal. Since diagonal matrices commute, after expansion and simplification, we get

$$
\begin{aligned}
(D_1\tilde{x})(Dg)^{\mathrm{T}} - (D_1 D\tilde{x})g^{\mathrm{T}} - (g^{\mathrm{T}} D\tilde{x})(D_1\tilde{x})g^{\mathrm{T}} \\
= \tilde{x}\big(D_1 Dg - (g^{\mathrm{T}} D\tilde{x})D_1 g\big)^{\mathrm{T}} - (D\tilde{x})(D_1 g)^{\mathrm{T}}. \quad (2.3)
\end{aligned}
$$

Observe that g and Dg are linearly independent because each entry of g is nonzero and D is a non-scalar diagonal matrix. Hence there exists a vector y such that $g^{\mathrm{T}} y = 0$ and $(Dg)^{\mathrm{T}} y = 1$. Then post-multiplying both sides of equation (2.3) with y gives $D_1\tilde{x} = \mu\tilde{x} + \nu D\tilde{x}$, where $\mu = (D_1 Dg - (g^{\mathrm{T}} D\tilde{x})D_1 g)^{\mathrm{T}} y$ and $\nu = -(D_1 g)^{\mathrm{T}} y$. We infer that $(D_1 - \nu D - \mu I)\tilde{x} = 0$. This implies that $D_1 = \mu I + \nu D$, since $\tilde{x}$ has all its entries nonzero. Hence, equation (2.3) is simplified into

$$
\begin{aligned}
(D\tilde{x})(Dg)^{\mathrm{T}} - (D^2\tilde{x})g^{\mathrm{T}} - (g^{\mathrm{T}} D\tilde{x})(D\tilde{x})g^{\mathrm{T}} \\
= \tilde{x}\big(D^2 g - (g^{\mathrm{T}} D\tilde{x})Dg\big)^{\mathrm{T}} - (D\tilde{x})(Dg)^{\mathrm{T}}. \quad (2.4)
\end{aligned}
$$

If char $\mathbb{F} \neq 2$, we choose a vector z such that $g^{\mathrm{T}} z = 0$ and $(Dg)^{\mathrm{T}} z = 1$. By evaluating both sides of equation (2.4) at z, we obtain that

$$
2D\tilde{x} = ((D^2 g)^{\mathrm{T}} z - (g^{\mathrm{T}} D\tilde{x}))\tilde{x}.
$$

Then $\tilde{x}$ is an eigenvector of a non-scalar diagonal matrix $2D$, which is a contradiction because all entries of $\tilde{x}$ are nonzero. Hence, $d(X_\alpha, X_\beta) = 4$ for each $\alpha \neq \beta$.

If char $\mathbb{F} = 2$, then we choose a vector z such that $g^{\mathrm{T}} z = 1$ and $(Dg)^{\mathrm{T}} z = 0$. Similarly, this simplifies equation (2.4) into $D^2\tilde{x} + (g^{\mathrm{T}} D\tilde{x})D\tilde{x} = \lambda\tilde{x}$, where $\lambda = (D^2 g)^{\mathrm{T}} z$. Arguing as above, $D^2 + (g^{\mathrm{T}} D\tilde{x})D - \lambda I = 0$. Since D is a non-scalar diagonal matrix, which is annihilated by a quadratic polynomial $p(t) = t^2 + (g^{\mathrm{T}} D\tilde{x})t - \lambda = (t - \delta_1)(t - \delta_2)$, it has exactly two distinct eigenvalues, say δ_1 and δ_2 (with multiplicities k and $n - k$, respectively). Hence, $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$, where $d_i \in \{\delta_1, \delta_2\}$ and without loss of generality $d_1 = \delta_1$. Comparing the coefficients of polynomial p in characteristics 2, we obtain

$$
\delta_1 + \delta_2 = (g^{\mathrm{T}} D\tilde{x}) = (\beta - \alpha)\Big((2 - n - \gamma^2)d_1 + \sum_{i=2}^{n-1} d_i + \gamma^2 d_n\Big). \quad (2.5)
$$

Note that $d_1 + d_2 + \cdots + d_n = k\delta_1 + (n - k)\delta_2$. If $d_n = \delta_1$, then (2.5) simplifies into $\delta_1 + \delta_2 = (\beta - \alpha)(n - k)(\delta_1 + \delta_2)$. If $d_n = \delta_2$, then it simplifies into $\delta_1 + \delta_2 = (\beta - \alpha)(n - k + \gamma^2 + 1)(\delta_1 + \delta_2)$. Since D is not a scalar matrix, we can divide by $\delta_1 + \delta_2$ to obtain either $(\beta - \alpha)(n - k) = 1$ or $(\beta - \alpha)(n - k + \gamma^2 + 1) = 1$. Note that

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$n-k \in \{0,1\}$ (mod 2), and therefore, we obtain that $\beta \in \alpha + \{1, \frac{1}{\gamma^2+1}, \frac{1}{\gamma^2+2}\}$. Clearly, we can choose an infinite subset of indices $\mathcal{A} = \{0, \alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \dots\} \subseteq \mathbb{F}$ such that $\alpha - \beta \notin \{1, \frac{1}{\gamma^2+1}, \frac{1}{\gamma^2+2}\}$ for $\alpha, \beta \in \mathcal{A}$ and for this subset, $d(X_\alpha, X_\beta) = 4$.

It remains to find a matrix Z such that $d(X_\alpha, Z) = 2$ for each α. Observe that the rank-one matrix

$$S_\alpha E_{11} S_\alpha^{-1} = (I + x(f + \alpha g)^{\mathrm{T}}) e_1 e_1^{\mathrm{T}} (I - x(f + \alpha g)^{\mathrm{T}})$$

commutes with $X_\alpha = S_\alpha A S_\alpha^{-1}$. Since $n \geq 4$ and $g = x + (1 - n - \gamma^2)e_1$, there exists a nonzero vector w with $w^{\mathrm{T}} e_1 = w^{\mathrm{T}} x = w^{\mathrm{T}} f = w^{\mathrm{T}} g = 0$. Then rank-one matrix $Z = ww^{\mathrm{T}}$ commutes with $S_\alpha E_{11} S_\alpha^{-1}$ and hence, we have path $X_\alpha \!-\! S_\alpha E_{11} S_\alpha^{-1} \!-\! Z$ in $\Gamma(M_n(\mathbb{F}))$ for every α. Actually no shorter path between X_α and Z exists, because otherwise $d(X_\alpha, X_\beta) \leq 3$ for each β, which is a contradiction. Thus, $d(X_\alpha, Z) = 2$ for every α.

Consider now the remaining case $n = 3$. Choose $Z = E_{11}$ and define matrices $R_\alpha = (0, 1, \alpha)^{\mathrm{T}} (0, \alpha, -1)$, $\alpha \in \mathbb{F}$, that form an infinite family of pairwise non-commuting nilpotent rank-one matrices. Notice that each R_α commutes with E_{11}. Moreover, for each $\alpha \in \mathbb{F}$, define a nilpotent matrix $X_\alpha = \begin{bmatrix} 0 & \alpha & -1 \\ 1 & 0 & 0 \\ \alpha & 0 & 0 \end{bmatrix}$ with rank-two. Note that $X_\alpha^2 = R_\alpha$. Since $n = 3$ and X_α is minimal, all non-scalar matrices which commute with X_α are $\mathcal{C}$-equivalent to either X_α or R_α. Therefore, as $d(R_\alpha, R_\beta) = 2$ for $\alpha \neq \beta$, we see that $d(X_\alpha, X_\beta) = 4$ for $\alpha \neq \beta$. □

Diagonalizable matrices can also be classified using the distance in the commuting graph. Before doing that we need two lemmas.

LEMMA 2.8. *Suppose a minimal matrix $B \in M_n(\mathbb{F})$ is diagonalizable and let $B\!-\!X\!-\!Y$ be a path in $\Gamma(M_n(\mathbb{F}))$. Then there exists a minimal matrix $M \in \mathcal{C}(X) \cap \mathcal{C}(Y)$.*

Proof. Assume with no loss of generality that B is diagonal. Since B is minimal and hence non-derogatory, every $X \in \mathcal{C}(B)$ is also diagonal. Using simultaneous conjugation on (B, X) we may further assume that $X = \lambda_1 I_{n_1} \oplus \cdots \oplus \lambda_k I_{n_k}$, with $\lambda_1, \dots, \lambda_k$ pairwise distinct and $n_1, \dots, n_k \geq 1$. Since $Y \in \mathcal{C}(X) = M_{n_1}(\mathbb{F}) \oplus \cdots \oplus M_{n_k}(\mathbb{F})$, it follows that $Y = Y_1 \oplus \cdots \oplus Y_k$ is block-diagonal. Thus, we can find an invertible block-diagonal matrix $S = S_1 \oplus \cdots \oplus S_k$ such that $S^{-1}XS = X$ and $S^{-1}YS = \bigoplus_{i=1}^{k} S_i^{-1} Y_i S_i = \bigoplus_{i=1}^{s} J_{m_i}(\mu_i)$ is in upper-triangular Jordan form, $m_i \geq 1$, $s \geq k$. Then we can choose distinct $\nu_1, \dots, \nu_s \in \mathbb{F}$ such that the matrix $M = S \bigoplus_{i=1}^{s} J_{m_i}(\nu_i) S^{-1}$ is neither equal to X nor Y. Also, since $\nu_1, \dots, \nu_s$ are distinct, M is minimal and it commutes with X and with Y. □

LEMMA 2.9. *Suppose a minimal $B \in M_n(\mathbb{F})$ is not diagonalizable. Then there*

exist non-scalar matrices X, Y forming a path $B \text{---} X \text{---} Y$ in $\Gamma(M_n(\mathbb{F}))$ such that $\mathcal{C}(X) \cap \mathcal{C}(Y)$ contains only non-minimal matrices.

Proof. Without loss of generality assume B is already in its upper-triangular Jordan form, $B = J_{n_1}(\lambda_1) \oplus \cdots \oplus J_{n_k}(\lambda_k)$ with $\lambda_1, \ldots, \lambda_{n_k}$ distinct and $n_1 \geq 2$. Define $X = E_{1n_1} = J_{n_1}^{n_1-1} \oplus 0_{n-n_1} \in \mathcal{C}(B)$, and for fixed $k \in \{1, \ldots, n\} \setminus \{1, n_1\}$ define $Y = E_{1k}$. Clearly, X commutes with Y. Let us show that no minimal $A = S \bigoplus_{j=1}^{s} J_{n_j}(\mu_j) S^{-1}$ commutes with both X and Y. Since $X, Y \in \mathcal{C}(A) = \mathbb{F}[A]$ are of rank one, it follows that $X \in \mathbb{F}S\big(0_{m_{j_1-1}} \oplus J_{n_{j_1}}^{n_{j_1}-1} \oplus 0_{s_{j_1-1}}\big)S^{-1}$ and $Y \in \mathbb{F}S\big(0_{m_{j_2-1}} \oplus J_{n_{j_2}}^{n_{j_2}-1} \oplus 0_{s_{j_2-1}}\big)S^{-1}$ for some j_1, j_2, where $m_{j_i-1} = n_1 + \cdots + n_{j_i-1}$ and $s_{j_i-1} = n - m_{j_i-1} - n_{j_i}$. However, $\operatorname{rk}(X + Y) = \operatorname{rk}(E_{1n_1} + E_{1k}) = 1$ and so $j_1 = j_2$, which implies X and Y must be linearly dependent, giving a contradiction. $\square$

Proof of Theorem 1.4. Suppose A is diagonalizable and assume without loss of generality that A is already diagonal. Choose distinct scalars $\mu_1, \ldots, \mu_n$ to form a minimal matrix $B = \operatorname{diag}(\mu_1, \ldots, \mu_n)$ which clearly commutes with A. Then (ii) follows from Lemma 2.8.

If A is not diagonalizable, then choose a minimal $B \in \mathcal{C}(A)$. Note that such B always exists. For example, if $A = S \bigoplus_{i=1}^{k} J_{n_i}(\lambda_i) S^{-1}$, then $B = S \bigoplus_{i=1}^{k} J_{n_i}(\mu_i) S^{-1} \in \mathcal{C}(A)$ is minimal for distinct scalars $\mu_1, \ldots, \mu_k$. Since $\mathcal{C}(B) = \mathbb{F}[B]$, it follows that $A \in \mathbb{F}[B]$ which implies that B itself is not diagonalizable. It now follows from Lemma 2.9 that there exist X, Y with $B \text{---} X \text{---} Y$, but no minimal matrix commutes with both X and Y. Thus, (ii) does not hold, which proves the theorem. $\square$

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