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NEW CONDITIONS FOR THE REVERSE ORDER LAWS FOR $\{1,3\}$ AND $\{1,4\}$-GENERALIZED INVERSES*

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Abstract. In this note, the reverse order laws for $\{1,3\}$ and $\{1,4\}$-generalized inverses of matrices are considered. New necessary and sufficient conditions for $(AB)\{1,3\} \subseteq B\{1,3\} \cdot A\{1,3\}$ and $(AB)\{1,4\} \subseteq B\{1,4\} \cdot A\{1,4\}$ are presented.

Key words. Generalized inverses, $\{1,3\}$-Generalized inverses, $\{1,4\}$-Generalized inverses, MP-inverse, Reverse order law.

AMS subject classifications. 15A09.

1. Results. Let A be a complex matrix. We denote by $\mathcal{R}(A)$, $\mathcal{N}(A)$, $\mathrm{r}(A)$ and $\mathrm{nul}(A)$ the range, the null space, the rank, and the nullity of a matrix A, respectively. By P_M, we denote the orthogonal projection ($P = P^2 = P^*$) on the subspace M.

The Moore–Penrose inverse of $A \in \mathbb{C}^{n\times m}$ is the unique matrix $A^\dagger \in \mathbb{C}^{m\times n}$ satisfying the four Penrose equations in [10],

$$(1)\ AA^\dagger A = A, \quad (2)\ A^\dagger AA^\dagger = A^\dagger, \quad (3)\ (AA^\dagger)^* = AA^\dagger, \quad (4)\ (A^\dagger A)^* = A^\dagger A.$$

It is well-known that each matrix A has its Moore–Penrose inverse.

For a subset $K \subseteq \{1,2,3,4\}$, we say that $B \in \mathbb{C}^{m\times n}$ is a K*-inverse* of $A \in \mathbb{C}^{n\times m}$ if B satisfies the Penrose equation (j) for each $j \in K$. We use AK for the collection of all K-inverses of A, and A^K for an unspecified element $X \in AK$.

The reverse order law for the Moore–Penrose inverse was first studied by Greville [5] in the 1960's, giving a necessary and sufficient condition for the reverse order law

$$(AB)^\dagger = B^\dagger A^\dagger, \tag{1.1}$$

for matrices A and B. This was followed (see [4]) by further equivalent conditions for (1.1). Sun and Wei [11] considered the reverse order law for the weighted Moore–

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Penrose inverses of two matrices. Hartwig [6] and Tian [13, 14] studied the reverse order law for the Moore–Penrose inverse of the product of three or more matrices.

The next step was to consider the reverse order law for K-inverses, where $K \subseteq \{1,2,3,4\}$. The cases $K = \{1,3\}$ and $K = \{1,4\}$ were considered by M. Wei and Guo [16] who obtained the equivalent conditions for $B\{1,3\}A\{1,3\} \subseteq (AB)\{1,3\}$, $(AB)\{1,3\} \subseteq B\{1,3\}A\{1,3\}$, and $(AB)\{1,3\} = B\{1,3\}A\{1,3\}$ by applying product singular value decomposition (P-SVD) of matrices. They proved that for $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$, $(AB)\{1,3\} \subseteq B\{1,3\}A\{1,3\}$ if and only if

$$\dim(R(Z_{14})) = \dim(R(Z_{12}, Z_{14}))$$

and

$$0 \leq \min\{p - r_2, m - r_1\} \leq n - r_1 - r_2^2 - \mathrm{r}(Z_{14}),$$

where Z_{12}, Z_{14} and constants r_1, r_2 are described in [16, Theorem 1.1] for P-SVD of matrices A and B.

Later, also in the settings of matrices, Takane et al. [12] discovered using other techniques some new necessary and sufficient conditions for

$$B\{1,3\}A\{1,3\} \subseteq (AB)\{1,3\}.$$

Djordjević [3] considered necessary and sufficient conditions for $B\{1,3\}A\{1,3\} \subseteq (AB)\{1,3\}$ in the case of bounded linear operators on Hilbert spaces. Cvetković-Ilić and Harte [2] offered purely algebraic necessary and sufficient conditions for reverse order law $B\{1,3\}A\{1,3\} \subseteq (AB)\{1,3\}$ for generalized inverses in C*-algebras, extending rank conditions for matrices and range conditions for Hilbert space operators. Liu and Yang [7] derived some necessary and sufficient conditions for all three types of reverse order laws using the method of maximal and minimal rank of matrix expressions. They proved that for $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times k}$, $(AB)\{1,3\} \subseteq B\{1,3\}A\{1,3\}$ if and only if

$$\mathrm{r}(A^*AB, B) + \mathrm{r}(A) = \mathrm{r}(AB) + \min\{\mathrm{r}(A^*, B), \max\{n + \mathrm{r}(A) - m, n + \mathrm{r}(B) - k\}\}.$$

As can be seen from the above, reverse order laws for generalized inverses have been considered in quite a number of papers. However, only [7, 16] were dealing with the particular one

$$(AB)\{1,3\} \subseteq B\{1,3\} \cdot A\{1,3\}.$$

In this note, using some techniques different than those of [7, 16], we give new necessary and sufficient conditions for the inclusions:

$$(AB)\{1,3\} \subseteq B\{1,3\} \cdot A\{1,3\},$$

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$$(AB)\{1,4\} \subseteq B\{1,4\} \cdot A\{1,4\}.$$

It is well known that the sets of $\{1,3\}$ and $\{1,4\}$-generalized inverses of $A \in \mathbb{C}^{n\times m}$ are described by

$$A\{1,3\} = \{A^\dagger + (I - A^\dagger A)Y : Y \in \mathbb{C}^{m\times n}\}$$

and

$$A\{1,4\} = \{A^\dagger + Z(I - AA^\dagger) : Z \in \mathbb{C}^{m\times n}\}.$$

In the following, we state some auxiliary lemmas.

Theorem 1.1. [1] *Let $A \in \mathbb{C}^{n\times m}$, $B \in \mathbb{C}^{p\times k}$, and $C \in \mathbb{C}^{n\times k}$. Then the matrix equation*

$$AXB = C$$

is consistent if and only if, for some $A^{(1)} \in A\{1\}$ and $B^{(1)} \in B\{1\}$,

$$AA^{(1)}CB^{(1)}B = C$$

in which case the general solution is

$$X = A^{(1)}CB^{(1)} + Y - A^{(1)}AYBB^{(1)}$$

for arbitrary $Y \in \mathbb{C}^{m\times p}$.

Corollary 1.2. *Let $B \in \mathbb{C}^{p\times k}$ and $C \in \mathbb{C}^{n\times k}$. Then the matrix equation*

$$XB = C$$

is consistent if and only if $\mathcal{N}(B) \subseteq \mathcal{N}(C)$ or, equivalently, $\mathcal{R}(C^*) \subseteq \mathcal{R}(B^*)$. In that case, the general solution is given by

$$X = CB^{(1)} + Y - YBB^{(1)}$$

for arbitrary $Y \in \mathbb{C}^{n\times p}$ and some $B^{(1)} \in B\{1\}$.

Proof. By Theorem 1.1, we see that the equation $XB = C$ is consistent if and only if $CB^{(1)}B = C$, i.e., $B^*(B^*)^{(1)}C^* = C^*$. Since $B^*(B^*)^{(1)}$ is a projection, we have that $XB = C$ is consistent if and only if $\mathcal{R}(C^*) \subseteq \mathcal{R}(B^*(B^*)^{(1)})$, i.e., $\mathcal{N}(B^{(1)}B) \subseteq \mathcal{N}(C)$. Finally, by $\mathcal{N}(B^{(1)}B) = \mathcal{N}(B)$, we get that $XB = C$ is consistent if and only if $\mathcal{N}(B) \subseteq \mathcal{N}(C)$. □

Lemma 1.3. *Let $A \in \mathbb{C}^{n\times m}$ and let $P \in \mathbb{C}^{m\times m}$ and $Q \in \mathbb{C}^{n\times n}$ be orthogonal projections. Then*

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(*i*) $(AP)^\dagger = P(AP)^\dagger$,
(*ii*) $(QA)^\dagger = (QA)^\dagger Q$.

Proof. (*i*) It is easy to verify that $P(AP)^\dagger$ is Moore–Penrose inverse of AP. The proof for (*ii*) is similar. $\square$

LEMMA 1.4. *Let $A \in \mathbb{C}^{n\times m}$ and $B \in \mathbb{C}^{m\times k}$. Set $S = B^\dagger(I - A^\dagger A)$ and $C = I - A^\dagger A - S^\dagger S$. Then C is an orthogonal projection and*

$$C = (I - A^\dagger A)(I - S^\dagger S) = (I - S^\dagger S)(I - A^\dagger A) = P_{\mathcal{N}(A)\cap\mathcal{N}(B^*)}.$$

Proof. Since $SA^\dagger = 0$ and $AS^* = 0$, we get that $S^\dagger SA^\dagger A = 0$ and $A^\dagger AS^\dagger S = 0$, respectively. Hence, $C = (I-A^\dagger A)(I-S^\dagger S) = (I-S^\dagger S)(I-A^\dagger A)$. Now, it is evident that C is an orthogonal projection. We need to prove that $\mathcal{R}(C) = \mathcal{N}(A) \cap \mathcal{N}(B^*)$. Let $x \in \mathcal{N}(A) \cap \mathcal{N}(B^*)$. Then $Ax = 0$ and $B^*x = 0$ which imply that $Sx = 0$, i.e., $Cx = x$. So $x \in \mathcal{R}(C)$. On the other hand, if we suppose that $x \in \mathcal{R}(C)$, we get that $x = Cx$, i.e., $x = (I - A^\dagger A)(I - S^\dagger S)x = (I - S^\dagger S)(I - A^\dagger A)x$. Clearly, $Ax = A(I - A^\dagger A)(I - S^\dagger S)x = 0$. Also, $B^\dagger x = Sx + B^\dagger A^\dagger Ax = Sx = SCx = 0$. Hence, $B^*x = 0$, so $x \in \mathcal{N}(A) \cap \mathcal{N}(B^*)$. We conclude that $C = P_{\mathcal{N}(A)\cap\mathcal{N}(B^*)}$. $\square$

LEMMA 1.5. *For $A \in \mathbb{C}^{n\times m}$ and $B \in \mathbb{C}^{m\times k}$, let $S = B^\dagger(I - A^\dagger A)$. Then the following conditions are equivalent:*

(*i*) $(I - SS^\dagger)((AB)^\dagger - B^\dagger A^\dagger) = 0$,
(*ii*) $\big(I - SS^\dagger\big)\big((AB)^\dagger A - B^\dagger A^\dagger A\big) = 0$,
(*iii*) $(I - SS^\dagger)((AB)^\dagger A - B^\dagger) = 0$.

Proof. $(i) \Rightarrow (ii):$ Multiplying (*i*) from right by A, we get (*ii*).

$(ii) \Rightarrow (i):$ Multiplying (*ii*) from right by $A^\dagger$ and using the fact that $(AB)^\dagger AA^\dagger = (AA^\dagger AB)^\dagger AA^\dagger = (AB)^\dagger$, which holds by Lemma 1.3, we get that (*i*) holds.

$(i) \Rightarrow (iii):$ Suppose that (*i*) holds, i.e., $(I-SS^\dagger)(AB)^\dagger = (I-SS^\dagger)B^\dagger A^\dagger$. Then

$$\begin{aligned}(I - SS^\dagger)((AB)^\dagger A - B^\dagger) &= (I - SS^\dagger)(B^\dagger A^\dagger A - B^\dagger)\\ &= (I - SS^\dagger)(-S) = 0.\end{aligned}$$

$(iii) \Rightarrow (i):$ Multiplying (*iii*) from right by $A^\dagger$ and using the fact that $(AB)^\dagger AA^\dagger = (AB)^\dagger$, we get that (*i*) holds. $\square$

LEMMA 1.6. *Let $A \in \mathbb{C}^{n\times m}$, $B \in \mathbb{C}^{m\times k}$, $T \in \mathbb{C}^{m\times n}$, and $Z \in \mathbb{C}^{k\times n}$. Set $S = B^\dagger(I - A^\dagger A)$, $C = I - A^\dagger A - S^\dagger S$, $D = (AB)^\dagger - B^\dagger A^\dagger$ and $E = A^\dagger + S^\dagger\Big(D + (B^\dagger B - (AB)^\dagger AB)Z\Big) + CT$. If $(I - SS^\dagger)D = 0$, then*

$$\mathcal{N}(E) = \mathcal{N}(A^*) \cap \mathcal{N}((B^\dagger B - (AB)^\dagger AB)Z) \cap \mathcal{N}(CT). \tag{1.2}$$

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Proof. Denote the right side of (1.2) by F. Suppose that $x \in \mathcal{N}(E)$. Then

$$\Big(A^\dagger + S^\dagger\Big(D + (B^\dagger B - (AB)^\dagger AB)Z\Big) + CT\Big)x = 0. \tag{1.3}$$

Multiplying (1.3) from the left side by $A^\dagger A$, we get $A^\dagger x = 0$, i.e., $x \in \mathcal{N}(A^*)$, since $AS^* = AS^\dagger = 0$ and $AC = 0$. Similarly, multiplying (1.3) by $I - A^\dagger A$ from the left side, we get

$$\Big(S^\dagger\Big(D + (B^\dagger B - (AB)^\dagger AB)Z\Big) + CT\Big)x = 0. \tag{1.4}$$

Since $(I - SS^\dagger)D = 0$, multiplying (1.4) from the left by S gives

$$\Big(D + (B^\dagger B - (AB)^\dagger AB)Z\Big)x = 0, \tag{1.5}$$

by Lemma 1.4 and Lemma 1.5. We already proved that $A^\dagger x = 0$. Hence, by Lemma 1.3, it follows that $Dx = 0$. Now, from (1.5), we get $(B^\dagger B - (AB)^\dagger AB)Zx = 0$. By (1.4), it follows that $CTx = 0$, so $x \in F$.

On the other hand, if $x \in F$, then $x \in \mathcal{N}(A^*)$, so $A^\dagger x = 0$ which implies that $Dx = 0$. Also, $x \in \mathcal{N}((B^\dagger B - (AB)^\dagger AB)Z) \cap \mathcal{N}(CT)$. Hence, $(B^\dagger B - (AB)^\dagger AB)Zx = CTx = 0$, which implies $x \in \mathcal{N}(E)$. $\square$

We now state the main result of our paper.

THEOREM 1.7. *Let $A \in \mathbb{C}^{n\times m}$ and $B \in \mathbb{C}^{m\times k}$. Then the following conditions are equivalent:*

(*i*) $(AB)\{1,3\} \subseteq B\{1,3\} \cdot A\{1,3\}$,
(*ii*) $(I - SS^\dagger)((AB)^\dagger - B^\dagger A^\dagger) = 0$ *and* $\mathrm{r}(C) \geq \min\{n - \mathrm{r}(A), k - \mathrm{r}(B)\}$,

where $S = B^\dagger(I - A^\dagger A)$ and $C = I - A^\dagger A - S^\dagger S$.

Proof. We first remark that by Lemma 1.4, we have that C is an orthogonal projection and

$$C = (I - A^\dagger A)(I - S^\dagger S) = (I - S^\dagger S)(I - A^\dagger A) = P_{\mathcal{N}(A)\cap\mathcal{N}(B^*)}.$$

$(ii) \Rightarrow (i)$: Suppose that (ii) holds. We must prove that for arbitrary $(AB)^{(1,3)}$ there exist $A^{(1,3)}$ and $B^{(1,3)}$ such that $(AB)^{(1,3)} = B^{(1,3)} \cdot A^{(1,3)}$. Thus, given any $Z \in \mathbb{C}^{k\times n}$, we must show that there exist $X \in \mathbb{C}^{m\times n}$ and $Y \in \mathbb{C}^{k\times m}$ such that

$$\begin{aligned}&(AB)^\dagger + \big(I - (AB)^\dagger(AB)\big)Z\\ &= \Big(B^\dagger + (I - B^\dagger B)Y\Big)\Big(A^\dagger + (I - A^\dagger A)X\Big).\end{aligned} \tag{1.6}$$

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Multiplying (1.6) from left first by $B^\dagger B$ and then by $(I-B^\dagger B)$ and using Lemma 1.3, we get that (1.6) implies the following:

$$(AB)^\dagger + \big(B^\dagger B - (AB)^\dagger (AB)\big)Z = B^\dagger A^\dagger + B^\dagger (I - A^\dagger A)X, \tag{1.7}$$

$$\big(I - B^\dagger B\big)Z = (I - B^\dagger B)Y(A^\dagger + (I - A^\dagger A)X). \tag{1.8}$$

If we sum (1.7) and (1.8), we get (1.6). Hence, (1.6) is equivalent to (1.7) and (1.8).

Now, we have to prove that for arbitrary $Z \in \mathbb{C}^{k\times n}$, there exist $X \in \mathbb{C}^{m\times n}$ and $Y \in \mathbb{C}^{k\times m}$ such that (1.7) and (1.8) hold. Since Lemma 1.5 implies that

$$(I - SS^\dagger)((AB)^\dagger - B^\dagger A^\dagger) = 0$$

is equivalent to

$$(I - SS^\dagger)(B^\dagger - (AB)^\dagger A) = 0,$$

we have that for each $Z \in \mathbb{C}^{k\times n}$

$$(I - SS^\dagger)\Big((AB)^\dagger - B^\dagger A^\dagger + \big(B^\dagger B - (AB)^\dagger (AB)\big)Z\Big) = 0,$$

so equation (1.7) is solvable for each $Z \in \mathbb{C}^{k\times n}$. The set of solutions is described by

$$S_Z = \{S^\dagger\Big((AB)^\dagger - B^\dagger A^\dagger + (B^\dagger B - (AB)^\dagger AB)Z\Big) + (I - S^\dagger S)T : T \in \mathbb{C}^{m\times n}\}. \tag{1.9}$$

Now, substituting X given by (1.9) in equation (1.8), we get

$$(I - B^\dagger B)Z = (I - B^\dagger B)Y\Big(A^\dagger + S^\dagger\Big(D + (B^\dagger B - (AB)^\dagger AB)Z\Big) + CT\Big), \tag{1.10}$$

where $D = (AB)^\dagger - B^\dagger A^\dagger$. Thus, to prove (i), it is sufficient to prove that for arbitrary Z, there exist matrices P and T such that

$$(I - B^\dagger B)Z = P\Big(A^\dagger + S^\dagger\Big(D + (B^\dagger B - (AB)^\dagger AB)Z\Big) + CT\Big), \tag{1.11}$$

which is, by Corollary 1.2, equivalent to the fact that for arbitrary Z, there exists a matrix T such that

$$\mathcal{N}(A^\dagger + S^\dagger\big(D + (B^\dagger B - (AB)^\dagger AB)Z\big) + CT) \subseteq \mathcal{N}((I - B^\dagger B)Z). \tag{1.12}$$

Put $E = A^\dagger + S^\dagger\big(D + (B^\dagger B - (AB)^\dagger AB)Z + CT\big)$. By Lemma 1.6, we have

$$\mathcal{N}(E) = \mathcal{N}(A^*) \cap \mathcal{N}((B^\dagger B - (AB)^\dagger AB)Z) \cap \mathcal{N}(CT).$$

Hence, (1.12) is equivalent to

$$\mathcal{N}(A^*) \cap \mathcal{N}((B^\dagger B - (AB)^\dagger AB)Z) \cap \mathcal{N}(CT) \subseteq \mathcal{N}((I - B^\dagger B)Z). \tag{1.13}$$

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If $x \in \mathcal{N}(A^*) \cap \mathcal{N}((B^\dagger B - (AB)^\dagger AB)Z) \cap \mathcal{N}(CT)$, then $B^\dagger BZx = (AB)^\dagger ABZx$, so we conclude that for such x, the condition $x \in \mathcal{N}((I - B^\dagger B)Z)$ is equivalent to the condition $x \in \mathcal{N}((I - (AB)^\dagger AB)Z)$. Now, we get that (1.13) like (1.12) is equivalent to

$$\mathcal{N}(A^*) \cap \mathcal{N}((I - (AB)^\dagger AB)B^\dagger BZ) \cap \mathcal{N}(CT) \subseteq \mathcal{N}((I - (AB)^\dagger AB)Z). \tag{1.14}$$

Set $Q = \mathcal{N}((I - (AB)^\dagger AB)Z)$ and $Q_1 = \mathcal{N}((I - (AB)^\dagger AB)B^\dagger BZ)$. Then $Q = \{x \in \mathbb{C}^n : Zx \in \mathcal{R}(B^*A^*)\}$, $Q_1 = \{x \in \mathbb{C}^n : Zx \in \mathcal{N}(B) \oplus \mathcal{R}(B^*A^*)\}$, and $Q \subseteq Q_1$. So, to prove (i), we must show that for every matrix Z, there exists a linear mapping (matrix) T which maps the set

$$C_1 = \{x \in \mathbb{C}^n : Zx \in \mathcal{N}(B) \oplus \mathcal{R}(B^*A^*), P_{\mathcal{N}(B)}Zx \neq 0\} \cap \mathcal{N}(A^*)$$

to the set

$$C_2 = \{y \in \mathbb{C}^m : P_{\mathcal{R}(C)}y \neq 0\}. \tag{1.15}$$

In the case when $\mathrm{r}(C) \geq n - \mathrm{r}(A)$, there exists a linear T which maps the subspace $\mathcal{N}(A^*)$ injectively into $\mathcal{R}(C)$, thus mapping C_1 to C_2. Now suppose that $\mathrm{r}(C) \geq k - \mathrm{r}(B)$.

Put $Q_1' = Q_1 \cap \mathcal{N}(A^*)$ and $Q' = Q \cap \mathcal{N}(A^*)$. Also, denote by Z_0 the restriction of Z to the subspace Q_1'. Let $T_0 : Q_1' \to \mathcal{R}(C)$ be the mapping defined by $T_0 = M \circ P_{\mathcal{N}(B)} \circ Z_0$, for some injective linear mapping $M : \mathcal{N}(B) \to \mathcal{R}(C)$ and let T be any linear extension of T_0 to the space $\mathbb{C}^n$. Then $T(C_1) = T_0(C_1) \subseteq T_0(Q_1') \subseteq \mathcal{R}(C)$. Let us show that $Tx \neq 0$ on $C_1 = Q_1' \setminus Q'$. If $Tx = 0$ for some $x \in Q_1' \setminus Q'$, it follows that $T_0(x) = 0$, so $(P_{\mathcal{N}(B)} \circ Z_0)(x) = 0$ which implies that $x \in Q'$ which is a contradiction. Hence, $T(C_1) \subseteq \mathcal{R}(C)$ and $T(x) \neq 0$ for every $x \in C_1$ and hence, $T(C_1) \subseteq C_2$.

$(i) \Rightarrow (ii) :$ If (i) holds, then for arbitrary $Z \in \mathbb{C}^{k\times n}$, there exist $X \in \mathbb{C}^{m\times n}$ and $Y \in \mathbb{C}^{k\times m}$ such that (1.6) holds.

Multiplying (1.6) by $B^\dagger B$ from the left, we get that

$$(AB)^\dagger + \big(B^\dagger B - (AB)^\dagger (AB)\big)Z = B^\dagger A^\dagger + B^\dagger (I - A^\dagger A)X.$$

For $Z = 0$, we get that the equation

$$(AB)^\dagger - B^\dagger A^\dagger = B^\dagger (I - A^\dagger A)X$$

is solvable. Hence,

$$\mathcal{R}((AB)^\dagger - B^\dagger A^\dagger) \subseteq \mathcal{R}(S)$$

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which is equivalent to

$$(I - SS^\dagger)((AB)^\dagger - B^\dagger A^\dagger) = 0. \tag{1.16}$$

Using the previous part of the proof, we get that if (1.16) and (i) hold, then for arbitrary Z, there exists a matrix T which maps the set $C_1 = \{x \in \mathbb{C}^n : Zx \in \mathcal{N}(B) \oplus \mathcal{R}(B^*A^*), P_{\mathcal{N}(B)}Zx \neq 0\} \cap \mathcal{N}(A^*)$ to the set $C_2 = \{y \in \mathbb{C}^m : P_{\mathcal{R}(C)}y \neq 0\}$.

We distinguish two cases:

1° If $\dim(\mathcal{N}(A^*)) \leq \dim(\mathcal{N}(B))$, then there exists Z such that $C_1 = \mathcal{N}(A^*) \setminus \{0\}$. It is now easy to see that there exists T which maps C_1 into the set C_2 defined by (1.15) if and only if $\dim \mathcal{N}(A^*) \leq \dim(\mathcal{R}(C))$.

2° Suppose now $\dim(\mathcal{N}(A^*)) > \dim(\mathcal{N}(B))$. Pick a subspace K of $\mathcal{N}(A^*)$ of dimension $\mathrm{nul}(B)$, a linear $Z : \mathbb{C}^k \to \mathbb{C}^n$ mapping K isomorphically to $\mathcal{N}(B)$ and let $T : \mathbb{C}^k \to \mathbb{C}^m$ be as supposed to exist. Clearly, we have $K \setminus \{0\} \subseteq C_1$. Thus, $(P_{\mathcal{R}(C)} \circ T)x \neq 0$ for $x \in K \setminus \{0\}$, i.e., $P_{\mathcal{R}(C)} \circ T$ maps injectively K into $\mathcal{R}(C)$. This proves $\dim(\mathcal{N}(B)) = \dim K \leq \dim(\mathcal{R}(C))$. $\square$

REMARK 1.8.

(1) The first condition from Theorem 1.7 (ii):

$$(I - SS^\dagger)((AB)^\dagger - B^\dagger A^\dagger) = 0$$

has a few equivalent forms which are given in Lemma 1.5. Beside these forms it is equivalent to:

$$(I - SS^\dagger)((AB)^\dagger AB - B^\dagger B) = 0.$$

(2) The second condition from Theorem 1.7 (ii):

$$\mathrm{r}(C) \geq \min\{n - \mathrm{r}(A), k - \mathrm{r}(B)\},$$

can be written as

$$\mathrm{r}(C) \geq \min\{\mathrm{nul}(A^*), \mathrm{nul}(B)\}.$$

(3) Since $C = P_{\mathcal{N}(A) \cap \mathcal{N}(B^*)}$, we have that $\mathrm{r}(C) = \dim(\mathcal{N}(A) \cap \mathcal{N}(B^*))$.

The proof of the part $(i) \Rightarrow (ii)$ of Theorem 1.7 yields the following:

COROLLARY 1.9. *Let $A \in \mathbb{C}^{n\times m}$ and $B \in \mathbb{C}^{m\times k}$. Then the following conditions are equivalent:*

(i^*) $(AB)\{1,3\} \subseteq B\{1,3\} \cdot A\{1,3\}$,

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(ii^*) $(I - SS^\dagger)((AB)^\dagger - B^\dagger A^\dagger) = 0$*, and at least one of the two conditions below holds:*

(a) $\mathrm{r}(C) \geq k - \mathrm{r}(B)$*,* $k - \mathrm{r}(B) < n - \mathrm{r}(A)$*,*

(b) $\mathrm{r}(C) \geq n - \mathrm{r}(A)$*,* $k - \mathrm{r}(B) \geq n - \mathrm{r}(A)$*,*

where $S = B^\dagger(I - A^\dagger A)$ *and* $C = I - A^\dagger A - S^\dagger S$*.*

Proof. $(ii^*) \Rightarrow (i^*)$: If any of the conditions (a) and (b) holds, then we have that $\mathrm{r}(C) \geq \min\{n - \mathrm{r}(A), k - \mathrm{r}(B)\}$, so the condition (ii) from Theorem 1.7 is satisfied, which implies (i^*).

$(i^*) \Rightarrow (ii^*)$: If (i^*) holds, then, by Theorem 1.7, we have that $(I - SS^\dagger)((AB)^\dagger - B^\dagger A^\dagger) = 0$. Using the part $(ii) \Rightarrow (i)$ of the proof of Theorem 1.7, we get that for arbitrary Z, there exists a matrix T which maps the set $C_1 = \{x \in \mathbb{C}^n : Zx \in \mathcal{N}(B) \oplus \mathcal{R}(B^* A^*), P_{\mathcal{N}(B)} Zx \neq 0\} \cap \mathcal{N}(A^*)$ into the set $C_2 = \{y \in \mathbb{C}^m : P_{\mathcal{R}(C)} y \neq 0\}$.

Now, as in the proof of that theorem we distinguish two cases:

a) Suppose that $\dim(\mathcal{N}(A^*)) > \dim(\mathcal{N}(B))$, i.e., $k - \mathrm{r}(B) < n - \mathrm{r}(A)$. Pick a subspace K of $\mathcal{N}(A^*)$ of dimension $\mathrm{nul}(B)$, a linear $Z : \mathbb{C}^k \to \mathbb{C}^n$ mapping K isomorphically to $\mathcal{N}(B)$ and let $T : \mathbb{C}^k \to \mathbb{C}^m$ be as supposed to exist. Clearly, we have $K \setminus \{0\} \subseteq C_1$. Thus, $(P_{\mathcal{R}(C)} \circ T)x \neq 0$ for $x \in K \setminus \{0\}$, i.e., $P_{\mathcal{R}(C)} \circ T$ maps injectively K into $\mathcal{R}(C)$. This proves $\dim(\mathcal{N}(B)) = \dim K \leq \dim(\mathcal{R}(C))$, i.e., $\mathrm{r}(C) \geq k - \mathrm{r}(B)$. ∎

b) If $\dim(\mathcal{N}(A^*)) \leq \dim(\mathcal{N}(B))$, i.e., $k - \mathrm{r}(B) \geq n - \mathrm{r}(A)$, then there exists Z such that $C_1 = \mathcal{N}(A^*) \setminus \{0\}$. Hence, the existence of mapping T which maps C_1 to the set C_2 is equivalent to $\dim \mathcal{N}(A^*) \leq \dim(\mathcal{R}(C))$, i.e., $\mathrm{r}(C) \geq n - \mathrm{r}(A)$.

Now, we conclude that (ii^*) holds. ∎

Corollary 1.10. *Let* $A \in \mathbb{C}^{n\times m}$ *and* $B \in \mathbb{C}^{m\times k}$*. If* $m < n$ *and* $m < k$*, then*

$$(AB)\{1,3\} \subseteq B\{1,3\} \cdot A\{1,3\},$$

cannot be satisfied.

Proof. Since $\mathrm{r}(C) = \dim(\mathcal{N}(A) \cap \mathcal{N}(B^*))$, we have that $\mathrm{r}(C) \leq m - \mathrm{r}(A)$ and $\mathrm{r}(C) \leq m - \mathrm{r}(B)$. If $m < n$ and $m < k$, then neither of the conditions (a) and (b) from Corollary 1.9 can be satisfied. ∎

The case $K = \{1,4\}$ is treated completely analogously and the corresponding result follows by taking adjoints, or by reversal of products:

Theorem 1.11. *Let* $A \in \mathbb{C}^{n\times m}$ *and* $B \in \mathbb{C}^{m\times k}$*. Then the following conditions are equivalent:*

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(i') $(AB)\{1,4\} \subseteq B\{1,4\} \cdot A\{1,4\}$,
(ii') $((AB)^\dagger - B^\dagger A^\dagger)(I - V^\dagger V) = 0$ *and* $\mathrm{r}(D) \geq \min\{n - \mathrm{r}(A), k - \mathrm{r}(B)\}$,

where $V = (I - BB^\dagger)A^\dagger$ *and* $D = I - BB^\dagger - VV^\dagger$.

COROLLARY 1.12. *Let* $A \in \mathbb{C}^{n\times m}$ *and* $B \in \mathbb{C}^{m\times k}$. *Then the following conditions are equivalent:*

(i'') $(AB)\{1,4\} \subseteq B\{1,4\} \cdot A\{1,4\}$,
(ii'') $((AB)^\dagger - B^\dagger A^\dagger)(I - V^\dagger V) = 0$ *and at least one of the two conditions below holds:*
(a') $\mathrm{r}(D) \geq n - \mathrm{r}(A)$, $n - \mathrm{r}(A) < k - \mathrm{r}(B)$
(b') $\mathrm{r}(D) \geq k - \mathrm{r}(B)$, $n - \mathrm{r}(A) \geq k - \mathrm{r}(B)$,

where $V = (I - BB^\dagger)A^\dagger$ *and* $D = I - BB^\dagger - VV^\dagger$.

COROLLARY 1.13. *Let* $A \in \mathbb{C}^{n\times m}$ *and* $B \in \mathbb{C}^{m\times k}$. *If* $m < n$ *and* $m < k$, *then*

$$(AB)\{1,4\} \subseteq B\{1,4\} \cdot A\{1,4\}$$

cannot be satisfied.

It is interesting that in Theorem 1.7 and Theorem 1.11, the matrices C and D are equal. Furthermore, $C = D = P_{\mathcal{N}(A)\cap\mathcal{N}(B^*)}$ and $\mathrm{r}(C) = \mathrm{r}(D) = \dim(\mathcal{N}(A)\cap\mathcal{N}(B^*))$. Also, the second condition from (ii) of Theorem 1.7 and the second condition of (ii') of Theorem 1.11 are exactly the same. So, we have the following result.

THEOREM 1.14. *Let* $A \in \mathbb{C}^{n\times m}$ *and* $B \in \mathbb{C}^{m\times k}$. *Then the following conditions are equivalent:*

(i) $(AB)\{1,3\} \subseteq B\{1,3\} \cdot A\{1,3\}$, $(AB)\{1,4\} \subseteq B\{1,4\} \cdot A\{1,4\}$,
(ii) $(I - SS^\dagger)((AB)^\dagger - B^\dagger A^\dagger) = 0$, $((AB)^\dagger - B^\dagger A^\dagger)(I - V^\dagger V) = 0$, *and* $\mathrm{r}(C) \geq \min\{n - \mathrm{r}(A), k - \mathrm{r}(B)\}$,

where $S = B^\dagger(I - A^\dagger A)$, $V = (I - BB^\dagger)A^\dagger$ *and* $C = I - A^\dagger A - S^\dagger S$.

2. Numerical examples. The following examples illustrate application of the presented results.

EXAMPLE 2.1. Let

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 111 & 0 \\ 10 & -10 & -4 & -4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \\ -1 & -1 & -1 \end{bmatrix}.$$

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We have that $\mathrm{r}(A) = 3$, $\mathrm{r}(B) = 2$, and $\mathrm{r}(C) = 0$. Using the program `Mathematica`, we can easily check that

$$(I - SS^{\dagger})((AB)^{\dagger} - B^{\dagger}A^{\dagger}) \neq 0,$$

while

$$((AB)^{\dagger} - B^{\dagger}A^{\dagger})(I - V^{\dagger}V) = 0, \quad \mathrm{r}(C) \geq n - \mathrm{r}(A) \quad \text{and} \quad k - \mathrm{r}(B) \geq n - \mathrm{r}(A).$$

Hence, $(AB)\{1,3\} \subseteq B\{1,3\} \cdot A\{1,3\}$ is not satisfied while $(AB)\{1,4\} \subseteq B\{1,4\} \cdot A\{1,4\}$.

Example 2.2. Let

$$A = \begin{bmatrix} 5 & 5 & 5 & 5 & 5 & 6 \\ 5 & 1 & 111 & 0 & 2 & 3 \\ 1111 & 23450 & -4 & -4 & 3 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1543 & 1 \\ 1 & 1234 & 213 \\ 1 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}.$$

We have that $\mathrm{r}(A) = \mathrm{r}(B) = 3$ and $\mathrm{r}(C) = 0$. Furthermore, all the conditions in Theorem 1.14 are satisfied and hence,

$$(AB)\{1,3\} \subseteq B\{1,3\} \cdot A\{1,3\} \quad \text{and} \quad (AB)\{1,4\} \subseteq B\{1,4\} \cdot A\{1,4\}.$$

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